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ALGEBRAIC NUMBERS AND DENSITY MODULO 1, II

Roman Urban

ABSTRACT. This is a companion paper to [8]. In [8], using ideas of Berend [3] and Kra [6], it was proved that the sets of the form

$$\{\lambda_1^n\mu_1^m\xi_1 + \lambda_2^n\mu_2^m\xi_2 : n, m \geq 1\},$$

where $\xi_1, \xi_2 \in \mathbb{R}$, λ_1, μ_1 and λ_2, μ_2 are two pairs of multiplicatively independent real algebraic numbers satisfying certain technical conditions, including that $\mu_i \in \mathbb{Q}(\lambda_i)$, $i = 1, 2$, are dense modulo $1/\kappa$, for some $\kappa \geq 1$.

In this paper we extend the result from [8], showing that the condition $\mu_i \in \mathbb{Q}(\lambda_i)$ can be removed by imposing appropriate conditions on the norms of conjugates of λ_i, μ_i and the degree of the algebraic numbers $\lambda_i^n\mu_i^m$.

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Dedicated to the memory of Professor Edmund Hlawka

1. Introduction

This is a companion paper to [8]. We extend the result from [8] showing that the condition $\mu_i \in \mathbb{Q}(\lambda_i)$ can be removed by imposing appropriate conditions on the norms of conjugates of λ_i, μ_i and the degree of the algebraic numbers $\lambda_i^n\mu_i^m$. In order to do that we define in § 3.1 the semigroup $\mathbf{\Sigma}$ in a different way than it was done in [8]. Having this new semigroup $\mathbf{\Sigma}$ the main steps of the proof of our result remain essentially the same as in [8]. However, some modifications are required. For example, we have to deal with the higher dimensional dynamical

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systems than that of [8], and hence the proof becomes much more technical and complicated.

Let K be a real algebraic number field, and let K^* denote its multiplicative group. Recall that two numbers $\lambda, \mu \in K^*$ are called *multiplicatively dependant* if there exist integers m and n, not both of which are 0 with $\lambda^m = \mu^n$. Equivalently, they are both rational powers of the same element $\beta \in K^*$. We say that λ and μ are *multiplicatively independent* if they are not multiplicatively dependent.

Denote $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$.

Theorem 1.1. *Let λ_1, μ_1 and λ_2, μ_2 be two distinct pairs of multiplicatively independent real algebraic numbers of degree 2, with absolute values greater than 1, such that the absolute values of their conjugates, $\tilde{\lambda}_1, \tilde{\mu}_1, \tilde{\lambda}_2, \tilde{\mu}_2$, are also greater than 1. Assume that*

(i) *for every $n, m \in \mathbb{N}$, $\deg_{\mathbb{Q}}(\lambda_i^n \mu_i^m) = 4$.*

Let $p_1, p_2, \ldots, p_s \geq 2$ be the primes appearing in the denominators of coefficients of the minimal polynomials $P_{\lambda_i}, P_{\mu_i}, \in \mathbb{Q}[x]$ of λ_i and μ_i, $i = 1, 2$. We set

$$S = \{\infty, p_1, p_2, \ldots, p_s\}.$$

Assume further that the following conditions are satisfied[1]*:*

(ii) $|\lambda_i|_\infty > |\tilde{\lambda}_i|_\infty > 1$ *and* $|\mu_i|_\infty > |\tilde{\mu}_i|_\infty > 1$, $i = 1, 2$,

(iii) *there exist $(\alpha, \beta), (\alpha', \beta') \in \mathbb{N}_0^2 \setminus \{(0, 0)\}$, and two positive integers k, l, $k \neq l$ such that*

$$\min\left(\min_{p \in S\setminus\{\infty\}} |\lambda_2\mu_2|_p^\alpha |\lambda_2^k \mu_2^l|_p^\beta, |\tilde{\lambda}_2\tilde{\mu}_2|_\infty^\alpha |\tilde{\lambda}_2^k \tilde{\mu}_2^l|_\infty^\beta\right) > \max\left(\max_{p \in S\setminus\{\infty\}} |\lambda_1\mu_1|_p^\alpha |\lambda_1^k \mu_1^l|_p^\beta, |\lambda_1\mu_1|_\infty^\alpha |\lambda_1^k \mu_1^l|_\infty^\beta\right)$$

and

$$\min\left(\min_{p \in S\setminus\{\infty\}} |\lambda_1\mu_1|_p^{\alpha'} |\lambda_1^k \mu_1^l|_p^{\beta'}, |\tilde{\lambda}_1\tilde{\mu}_1|_\infty^{\alpha'} |\tilde{\lambda}_1^k \tilde{\mu}_1^l|_\infty^{\beta'}\right) > \max\left(\max_{p \in S\setminus\{\infty\}} |\lambda_2\mu_2|_p^{\alpha'} |\lambda_2^k \mu_2^l|_p^{\beta'}, |\lambda_2\mu_2|_\infty^{\alpha'} |\lambda_2^k \mu_2^l|_\infty^{\beta'}\right),$$

(iv) $|\lambda_i|_p = |\tilde{\lambda}_i|_p > 1$ *and* $|\mu_i|_p = |\tilde{\mu}_i|_p > 1$, *for* $p \in S \setminus \{\infty\}$.

Then for any pair of real numbers ξ_1, ξ_2, with at least one ξ_i non-zero, there exists a natural number κ such that the set

$$\{\lambda_1^n \mu_1^m \kappa \xi_1 + \lambda_2^n \mu_2^m \kappa \xi_2 : n, m \in \mathbb{N}\} \tag{1.2}$$

is dense modulo 1.

[1]Here $|\cdot|_p$ stands for the *p-adic norm* in the field $\mathbb{Q}_p(\lambda_1, \mu_1, \lambda_2, \mu_2)$, whereas $|\cdot|_\infty$ denotes the usual absolute value in $\mathbb{R}$.

As a result we are able to consider more general expressions containing algebraic numbers than that considered in [8]. For example, our Theorem 1.1 implies that the following double-sequence

$$\left(7\sqrt{2}+\frac{1}{2\cdot 3\cdot 5\cdot 7}\right)^{n}\left(\frac{7^2}{\sqrt{5}}+\frac{1}{2^3\cdot 3^2\cdot 5^2\cdot 7^2}\right)^{m}+ \left(7^5\sqrt{3}+\frac{1}{2^{11}\cdot 3^{11}\cdot 5^{11}\cdot 7^{11}}\right)^{n}\left(\frac{7}{\sqrt{7}}+\frac{1}{2\cdot 3\cdot 5\cdot 7}\right)^{m}, \quad n,m\in\mathbb{N} \quad (1.3)$$

is dense modulo $1/\kappa$ for some $\kappa \geq 1$ (see Corollary 2.9).

Remark. We should remark here that Theorem 1.1 is not a generalization of [8, Theorem 1.5]. These two theorems are of different kind. In particular, it is not true that if the algebraic numbers λ_i, μ_i, $i = 1, 2$, satisfy the set of conditions required in [8] then they satisfy all of the assumptions required here. For instance, in the example on p. 647 in [8], illustrating [8, Theorem 1.5], one has $\deg_{\mathbb{Q}} \lambda_i\mu_i = 2$, $i = 1, 2$. Hence, the condition (i) of Theorem 1.1 is not satisfied.

Remark. Although conditions given in (iii) seem to be very complicated they have in fact a simple dynamical meaning. Namely, the various norms may be understood as the speeds of contraction and/or expansion along coordinate axes in appropriate product of p-adic vector spaces (see Lemma 4.2).

Remark. In the case when all the numbers λ_i, μ_i are algebraic integers we have $S = \{\infty\}$ and Theorem 1.1 has much simpler formulation as all p-adic norms disappear. As an example consider the following expression

$$\left(\sqrt{7}+1\right)^{n}\left(3\sqrt{3}+1\right)^{m}\xi_1+\left(100\sqrt{5}+3\right)^{n}\left(2\sqrt{2}+1\right)^{m}\xi_2. \quad (1.4)$$

It follows from Corollary 2.9 that the condition (iii) of Theorem 1.1 is satisfied and hence there is $\kappa \geq 1$ such that (1.4) is dense modulo $1/\kappa$.

Remark. A direct proof of Theorem 1.1 in case $S = \{\infty\}$ would be simpler. In this case the semigroup $\mathbf{\Sigma}$ (constructed in § 3) acts on $\mathbb{T}^4 \times \mathbb{T}^4$ instead of on the product of two solenoids. For the corresponding result concerning the action of a commutative semigroup of continuous endomorphisms on the d-dimensional torus see [1] (cf. [7]).

Structure of the paper

In § 2 we state and prove some auxiliary results which are useful for deciding whether the given algebraic numbers λ_i, μ_i satisfy the assumptions of Theorem 1.1. In § 3 we consider two commutative semigroups Σ_1 and Σ_2 of continuous endomorphisms of Ω_a^4 and study the closed invariant sets for the corresponding action of the diagonal subgroup of $\Sigma_1 \times \Sigma_2$ on the product $\Omega_a^4 \times \Omega_a^4$. In § 4 we prove Theorem 1.1.

2. Examples

Lemma 2.1. *Let $\lambda, \mu > 1$ be multiplicatively independent real algebraic numbers.*

(i) *Then for every positive integers $k \neq l$ the numbers $\lambda\mu$ and $\lambda^k\mu^l$ are multiplicatively independent.*

(ii) *Suppose that for some positive integers $k \neq l$, $\lambda\mu$ and $\lambda^k\mu^l$ are multiplicatively independent. Then λ and μ are multiplicatively independent.*

(iii) *Suppose that $\deg_{\mathbb{Q}} \lambda = \deg_{\mathbb{Q}} \mu = 2$ and $\deg_{\mathbb{Q}}(\lambda\mu) = 4$. Then we have $\mathbb{Q}(\lambda, \mu) = \mathbb{Q}(\lambda\mu)$.*

Proof. (i) and (ii) are obvious. We prove (iii). Since $\deg_{\mathbb{Q}} \lambda = \deg_{\mathbb{Q}} \mu = 2$,

$$[\mathbb{Q}(\lambda,\mu) : \mathbb{Q}] = [\mathbb{Q}(\lambda,\mu), \mathbb{Q}(\lambda)][\mathbb{Q}(\lambda), \mathbb{Q}] = [\mathbb{Q}(\lambda,\mu), \mathbb{Q}(\lambda)] \cdot 2 \leq 4.$$

By the assumption

$$\deg_{\mathbb{Q}}(\lambda\mu) = 4, \quad \text{and so} \quad [\mathbb{Q}(\lambda\mu) : \mathbb{Q}] = 4.$$

Since $\mathbb{Q}(\lambda\mu) \subset \mathbb{Q}(\lambda,\mu)$, it follows that $[\mathbb{Q}(\lambda,\mu) : \mathbb{Q}] = 4$. □

The following lemmas will be used to check if a given pair of algebraic numbers is multiplicatively independent.

Lemma 2.2. *Let $p, q > 1$ be the square-free numbers and $a, b \in \mathbb{Q}$. If $p \neq q$, then $\sqrt{p} + a$ and $\sqrt{q} + b$ are multiplicatively independent.*

Proof. Suppose that $\frac{\log(\sqrt{p}+a)}{\log(\sqrt{q}+b)} = w$ is rational and denote $w = \frac{r}{s}$. Then $(\sqrt{p}+a)^s = (\sqrt{q}+b)^r$. Since $\mathbb{Q}(\sqrt{p}) \cap \mathbb{Q}(\sqrt{q}) = \mathbb{Q}$, it follows that $(\sqrt{p}+a)^s$ and $(\sqrt{q}+b)^r$ are rational. Consider $v = (\sqrt{p}+a)^s \in \mathbb{Q}$. Let σ be the automorphism of $\mathbb{Q}(\sqrt{p})$ sending $\sqrt{p} \mapsto -\sqrt{p}$. Then $v = \sigma(v) = (-\sqrt{p}+a)^s$ and consequently $u = \frac{\sqrt{p}+a}{-\sqrt{p}+a}$ is an sth root of unity. Since $u \in \mathbb{Q}(\sqrt{p})$, we must have $u = 1$ or $u = -1$, and we see that $u = -1$ and $a = 0$. Repeating the same argument with $v = (\sqrt{q}+b)^r$ we get that also $b = 0$. Therefore we have $q^w = p$. This is a contradiction. □

The following is an easy generalization of the previous result.

Lemma 2.3. *Let $p, q > 1$ be the square-free numbers and $a, b, c, d \in \mathbb{Q}$. If $p \neq q$, then $c\sqrt{p} + a$ and $d\sqrt{q} + b$ are multiplicatively independent.*

P r o o f. Using the same argument as in the proof of the previous lemma we get,

$$(d\sqrt{q})^w = c\sqrt{p},$$

where $w = \frac{r}{s} \in \mathbb{Q}$, and consequently

$$d^{2r}q^r = c^{2s}p^s.$$

Since at least one of r and s is odd we get a contradiction. □

More generally, in [3, Proposition 4.2] it is proved that for λ and μ which are *effectively given* complex algebraic numbers it is possible effectively to decide whether or not they are multiplicatively independent.

To decide if the condition (i) of Theorem 1.1 is satisfied we need to recall a result from [4]. Let k be a field. An algebraic number β over k is called *torsion-free* if $\tilde{\beta}/\beta$ is not a root of unity for any $\tilde{\beta} \neq \beta$, where $\tilde{\beta}$ and β are conjugate over k.

Theorem 2.4 (Dubickas, [4, Theorem 2]). *Suppose that α is an algebraic number of degree d over a field k of characteristic zero, and let K be a normal closure of $k(\alpha)$ over k. If β is torsion-free and $L = k(\beta)$ is a normal extension of k of degree l and $L \cap K = k$, then*

$$\deg_k(\alpha\beta) = dl.$$

Remark 2.5. It follows from Theorem 2.4 that if both λ_i's and the μ_i's are of degree 2 and are not square roots of rational numbers and $\lambda_i \notin \mathbb{Q}(\mu_i)$, then

$$\deg_{\mathbb{Q}}(\lambda_i^n \mu_i^m) = 4.$$

In the following lemmas we give conditions on λ_i, μ_i that are easy to check and guarantee that (iii) of Theorm 1.1 holds.

Let S be as in Theorem 1.1. We denote $S^* = S \setminus \{\infty\}$. The following two lemmas are easy.

Lemma 2.6. *Let λ_i, μ_i, $i = 1, 2$, be real algebraic numbers of degree 2. Suppose that there exist positive integers $k \neq l$ such that*

$$\min_{p\in S^*} |\lambda_2\mu_2|_p > \max_{p\in S} |\lambda_1\mu_1|_p,$$
$$|\tilde{\lambda}_2\tilde{\mu}_2|_\infty > \max_{p\in S} |\lambda_1\mu_1|_p$$

and

$$\begin{aligned} \max_{p\in S} |\lambda_2^k\mu_2^l|_p &< \min_{p\in S^*} |\lambda_1^k\mu_1^l|_p, \\ \max_{p\in S} |\lambda_2^k\mu_2^l|_p &< |\tilde{\lambda}_1\tilde{\mu}_1|_\infty. \end{aligned} \tag{2.7}$$

Then there are $(\alpha, \beta), (\alpha', \beta') \in \mathbb{N}_0^2 \setminus \{(0, 0)\}$ such that the inequalities (iii) of Theorem 1.1 hold.

Lemma 2.8. *Let λ_i, μ_i, $i = 1, 2$, be real algebraic numbers of degree 2. Suppose that either*

$$\max_{p\in S} |\lambda_2|_p < \min_{p\in S^*} |\lambda_1|_p,$$

$$\max_{p\in S^*} |\lambda_2|_p < \min_{p\in S^*} |\tilde{\lambda}_1|_p,$$

$$|\lambda_2|_\infty < |\tilde{\lambda}_1|_\infty$$

or

$$\min_{p\in S^*} |\mu_1|_p > \max_{p\in S} |\mu_2|_p,$$

$$|\tilde{\mu}_1|_\infty > \max_{p\in S} |\mu_2|_p.$$

Then there exist positive integers $k \neq l$ such that (2.7) *holds.*

To sum up we have the following

Corollary 2.9. *Let λ_1, μ_1 and λ_2, μ_2 be two distinct pairs of multiplicatively independent algebraic numbers of degree 2, with absolute values greater than 1, such that the absolute values of their conjugates, $\tilde{\lambda}_1, \tilde{\mu}_1, \tilde{\lambda}_2, \tilde{\mu}_2$, are also greater than 1. Assume that the following conditions are satisfied:*

- *for every $n, m \in \mathbb{N}$, $\deg_{\mathbb{Q}}(\lambda_i^n \mu_i^m) = 4$,*
- *$|\lambda_i|_\infty > |\tilde{\lambda}_i|_\infty > 1$ and $|\mu_i|_\infty > |\tilde{\mu}_i|_\infty > 1$, $i = 1, 2$,*
- *$|\lambda_i|_p = |\tilde{\lambda}_i|_p > 1$ and $|\mu_i|_p = |\tilde{\mu}_i|_p > 1$, for $p \in S^* = S \setminus \{\infty\}$,*
- *$\min_{p\in S^*} |\lambda_2\mu_2|_p > \max_{p\in S} |\lambda_1\mu_1|_p$ and $|\tilde{\lambda}_2\tilde{\mu}_2|_\infty > \max_{p\in S} |\lambda_1\mu_1|_p$,*
- $$\max_{p\in S} |\lambda_2|_p < \min_{p\in S^*} |\lambda_1|_p \quad \text{and} \quad |\lambda_2|_\infty < |\tilde{\lambda}_1|_\infty$$
  or
  $$\min_{p\in S^*} |\mu_1|_p > \max_{p\in S} |\mu_2|_p \quad \text{and} \quad |\tilde{\mu}_1|_\infty > \max_{p\in S} |\mu_2|_p. \tag{2.10}$$

Then for any pair of real numbers ξ_1, ξ_2, with at least one ξ_i non-zero, there exists a natural number κ such that

$$\{\lambda_1^n \mu_1^m \kappa\xi_1 + \lambda_2^n \mu_2^m \kappa\xi_2 : n, m \in \mathbb{N}\}$$

is dense modulo 1.

3. The product semigroup $\mathbf{\Sigma}$

In this section we construct a semigroup $\mathbf{\Sigma}$ acting on the product of an appropriately chosen solenoid Ω_a and study the properties of this action. The semigroup $\mathbf{\Sigma}$ will play an important role in the proof of Theorem 1.1. The idea of such a construction goes back to [3].

3.1. Definition of Σ

Let λ_1, μ_1 and λ_2, μ_2 be two distinct pairs of multiplicatively independent real algebraic numbers of degree 2, satisfying assumptions of Theorem 1.1. For positive integers $k \neq l$, define,

$$s_0 = \lambda_1 \mu_1, \quad r_0 = \lambda_2 \mu_2 \tag{3.1}$$

and

$$s_1 = \lambda_1^k \mu_1^l, \quad r_1 = \lambda_2^k \mu_2^l. \tag{3.2}$$

By Lemma 2.1, s_0, s_1 and r_0, r_1 are multiplicatively independent and $\lambda_i^k \mu_i^l \in \mathbb{Q}(\lambda_i \mu_i)$. Therefore, we have that $s_1 \in \mathbb{Q}(s_0)$ and $r_1 \in \mathbb{Q}(r_0)$. Hence we can express the elements s_1, r_1 as polynomials with rational coefficients in s_0 and r_0, respectively,

$$s_1 = g(s_0), \quad r_1 = h(r_0), \tag{3.3}$$

where $g, h \in \mathbb{Q}[x]$. Let $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. For $(\alpha, \beta) \in \mathbb{N}_0^2$,

$$\begin{aligned} s_0^\alpha s_1^\beta &= \lambda_1^{\alpha + k\beta} \mu_1^{\alpha + l\beta}, \\ r_0^\alpha r_1^\beta &= \lambda_2^{\alpha + k\beta} \mu_2^{\alpha + l\beta}. \end{aligned} \tag{3.4}$$

By the assumption (i) of Theorem 1.1,

$$\deg_{\mathbb{Q}} s_0 = \deg_{\mathbb{Q}} s_1 = 4 \tag{3.5}$$

and

$$\deg_{\mathbb{Q}} r_0 = \deg_{\mathbb{Q}} r_1 = 4. \tag{3.6}$$

Let $\lambda > 1$ be a real algebraic number of degree d with minimal (monic) polynomial $P_\lambda \in \mathbb{Q}[x]$,

$$P_\lambda(x) = x^d + c_{d-1} x^{d-1} + \ldots + c_1 x + c_0.$$

With λ we associate the following *companion matrix* of P_λ,

$$\sigma_\lambda = \begin{pmatrix} 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 1 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & 0 & \ldots & 1 \\ -c_0 & -c_1 & -c_2 & \ldots & -c_{d-1} \end{pmatrix}. \tag{3.7}$$

Let σ_{s_0} and σ_{r_0} be the companion matrices associated with s_0 and r_0, respectively. By (3.5) and (3.6), $\sigma_{s_0}, \sigma_{r_0} \in \mathrm{GL}(4, \mathbb{Q})$. We put

$$\tau_{s_1} = g(\sigma_{s_0}) \quad \text{and} \quad \tau_{r_1} = h(\sigma_{r_0}), \tag{3.8}$$

where the polynomials g and h are defined in (3.3).

Let $\mathbf{\Sigma}$ be the semigroup generated by $\begin{pmatrix}\sigma_{s_0} & 0\\ 0 & \sigma_{r_0}\end{pmatrix}$ and $\begin{pmatrix}\tau_{s_1} & 0\\ 0 & \tau_{r_1}\end{pmatrix}$, i.e,

$$\mathbf{\Sigma} = \left\langle \begin{pmatrix}\sigma_{s_0} & 0\\ 0 & \sigma_{r_0}\end{pmatrix}, \begin{pmatrix}\tau_{s_1} & 0\\ 0 & \tau_{r_1}\end{pmatrix} \right\rangle.$$

By (3.8) the semigroup $\mathbf{\Sigma}$ is commutative. A general element of the semigroup $\mathbf{\Sigma}$ is denoted by

$$\mathfrak{s}^{(\alpha,\beta)} = \begin{pmatrix}\sigma_{s_0}^{\alpha}\tau_{s_1}^{\beta} & 0\\ 0 & \sigma_{r_0}^{\alpha}\tau_{r_1}^{\beta}\end{pmatrix} \in \mathrm{GL}(8,\mathbb{Q}),\ (\alpha,\beta)\in\mathbb{N}_0^2\setminus\{(0,0)\}.$$

Let

$$\Sigma_1 = \left\{\mathfrak{s}_1^{(\alpha,\beta)} := \sigma_{s_0}^{\alpha}\tau_{s_1}^{\beta} : (\alpha,\beta)\in\mathbb{N}_0^2\setminus\{(0,0)\}\right\},$$
$$\Sigma_2 = \left\{\mathfrak{s}_2^{(\alpha,\beta)} := \sigma_{r_0}^{\alpha}\tau_{r_1}^{\beta} : (\alpha,\beta)\in\mathbb{N}_0^2\setminus\{(0,0)\}\right\}.$$

Clearly, $\Sigma_i, i=1,2$ are finitely generated commutative subsemigroups of $\mathrm{GL}(4,\mathbb{Q})$:

$$\Sigma_1 = \left\langle \mathfrak{s}_1^{(1,0)}, \mathfrak{s}_1^{(0,1)}\right\rangle = \langle \sigma_{s_0}, \tau_{s_1}\rangle,$$
$$\Sigma_2 = \left\langle \mathfrak{s}_2^{(1,0)}, \mathfrak{s}_2^{(0,1)}\right\rangle = \langle \sigma_{r_0}, \tau_{r_1}\rangle.$$

The semigroups Σ_i, $i = 1, 2$, act on the solenoids[2] $\Omega_{a_i}^4$, where a_1 (resp., a_2) is the product of primes appearing in the denominators of the entries of σ_{s_0} and τ_{s_1}, (resp., σ_{r_0} and τ_{r_1}). Equivalently, a_1 is the product of primes appearing in the denominators of the coefficients of the minimal polynomials P_{λ_1} and P_{μ_1} of λ_1,μ_1 (resp., the minimal polynomials P_{λ_2} and P_{μ_2} of λ_2,μ_2). Hence, Σ_1 (resp., Σ_2) is a finitely generated semigroup of continuous endomorphisms of a compact Abelian group $\Omega_{a_1}^4$ (resp., $\Omega_{a_2}^4$). Therefore, the set $\mathbf{\Sigma}$ forms an Abelian semigroup of endomorphisms of $\Omega_{a_1}^4 \times \Omega_{a_2}^4$, with the action given by

$$\mathfrak{s}^{(\alpha,\beta)}(\mathbf{x},\mathbf{y})^t = \left(\mathfrak{s}_1^{(\alpha,\beta)}\mathbf{x}, \mathfrak{s}_2^{(\alpha,\beta)}\mathbf{y}\right)^t,\ \ (\mathbf{x},\mathbf{y})^t \in \Omega_{a_1}^4 \times \Omega_{a_2}^4.$$

Let

$$a_1 = p_{i_1}p_{i_2}\dots p_{i_{s_1}} \quad \text{and} \quad a_2 = p_{j_1}p_{j_2}\dots p_{j_{s_2}}.$$

We denote

$$S_1 = \{p_{i_1}, p_{i_2}, \dots, p_{i_{s_1}}\} \cup \{\infty\}$$

and

[2]Let $a = p_1p_2\dots p_s$, where p_i are different primes. Consider $\mathbb{Z}[1/a]$ as a topological group with the discrete topology. The dual group $\widehat{\mathbb{Z}[1/a]}$ of $\mathbb{Z}[1/a]$ is called an *a-adic solenoid* and we denote it by Ω_a. The compact abelian group Ω_a^d may be considered as a quotient group of the additive group $\mathbb{R}^d \times \mathbb{Q}_{p_1}^d \times \dots \times \mathbb{Q}_{p_s}^d$ by a discrete subgroup $B = \left\{(b, \underbrace{-b, \cdots, -b}_{s}) : b \in \mathbb{Z}[1/a]^d\right\}$. That is, $\Omega_a^d = \mathbb{R}^d \times \mathbb{Q}_{p_1}^d \times \dots \times \mathbb{Q}_{p_s}^d / B$. (For more details on solenoids see [5].)

$$S_2 = \{p_{j_1}, p_{j_2}, \ldots, p_{j_{s_2}}\} \cup \{\infty\}.$$

Observe that the set S of primes defined in the statement of Theorem 1.1 is equal to

$$S = S_1 \cup S_2. \tag{3.9}$$

Remark 3.10. Let a be a product of different primes from the set $S \setminus \{\infty\}$. Then the semigroups Σ_i, $i = 1, 2$, act on Ω_a^4 and $\mathbf{\Sigma}$ acts on $\Omega_a^4 \times \Omega_a^4$.

3.2. Projected semigroups Σ_1 and Σ_2

From now on the semigroups Σ_i act on Ω_a^4 and $\mathbf{\Sigma}$ acts on $\Omega_a^4 \times \Omega_a^4$, where a is defined in Remark 3.10.

We say that the semigroup Σ of continuous endomorphisms of a compact group G has the *ID-property*, or that Σ is an *ID-semigroup*, if the only infinite closed Σ-invariant [3] subset of G is G itself. [4]

Our aim in this subsection is to prove that Σ_1 and Σ_2 are ID-semigroups. We will use the following theorem which gives necessary and sufficient conditions in arithmetical terms for a commutative semigroup Σ of endomorphisms of Ω_a^d to have the ID-property.

Theorem 3.11 (Berend, [2, Theorem II.1]). *A commutative semigroup Σ of continuous endomorphisms of Ω_a^d has the ID-property if and only if the following hold:*

(i) *There exists an endomorphism $\sigma \in \Sigma$ such that the characteristic polynomial f_{σ^n} of σ^n is irreducible over $\mathbb{Q}$ for every positive integer n.*

(ii) *For every common eigenvector v of Σ there exists an endomorphism $\sigma_v \in \Sigma$ whose eigenvalue in the direction of v is of norm greater than* 1.

(iii) *Σ contains a pair of multiplicatively independent endomorphisms.*[5]

Let us explain in more details how to understand the statement of the condition (ii). Note that the semigroup Σ acts in a natural way on the "covering space" $\mathbb{R}^d \times \mathbb{Q}_{p_1}^d \times \cdots \times \mathbb{Q}_{p_s}^d$. It is proved in [2] that the condition (i) implies that the roots $\lambda_{1,\sigma}, \ldots \lambda_{d,\sigma}$ of σ are distinct and that there exists a basis $v^{(i)} \in \mathbb{Q}(\lambda_{i,\sigma})^d$, $i = 1, \ldots, d$, in which Σ has a diagonal form. Let K_j be the splitting field of the characteristic polynomial f_σ of σ over $\mathbb{Q}_{p_j}$, $j = 0, \ldots, s$, and let $v^{1,j}, \ldots, v^{d,j}$ be a basis of K_j^d corresponding to $v^{(i)}$, $i = 1, \ldots, d$. The vectors $v^{i,j}$, $i = 1, \ldots, d$,

[3]Recall that a subset $A \subset G$ is said to be Σ-*invariant* if $\Sigma A \subset A$.

[4]ID stands for *infinite invariant is dense.*

[5]We say, as we do in the case of real numbers, that two endomorphisms σ and τ are *rationally dependent* if there exist integers m and n, not simultaneously equal to 0, such that $\sigma^m = \tau^n$. Otherwise, we say that σ and τ are *rationally independent.*

$j = 0, \ldots, s$, are the common eigenvectors of Σ. Denote by $\lambda_{i,j,\tau}$, $i = 1, \ldots, d$, the eigenvalues of any $\tau \in \Sigma$, considered as a linear map of K_j^d with respect to the basis $v^{1,j}, \ldots, v^{d,j}$. Then the condition (ii) says that for every $1 \leq i \leq d$ and $0 \leq j \leq s$ there exists a $\sigma_{i,j} \in \Sigma$ such that $|\lambda_{i,j,\sigma_{i,j}}|_{p_j} > 1$.

Lemma 3.12. *Σ_1 is a commutative ID-semigroup.*

Proof. Commutativity of Σ_1 follows from (3.3) and definition (3.8) of τ_{s_1}. Now we have to check conditions of Theorem 3.11.

Let us start with (iii). Since s_0 and s_1 are multiplicatively independent it follows that σ_{s_0} and σ_{s_1} are multiplicatively independent.

(ii) For a given matrix A we denote by $\mathrm{Spect}(A)$ the set of eigenvalues of A. By the assumption, for $p \in S$, p-adic absolute values of λ_1, μ_1 and their conjugates are greater than 1. Hence, by (3.1) and (3.2), it follows that

$$\min\{|\rho|_p : \rho \in \mathrm{Spect}(\sigma_{s_0}) \cup \mathrm{Spect}(\tau_{s_1}),\ p \in S_1\} > 1.$$

(i) Since, for every n, $\deg_{\mathbb{Q}}(s_0^n) = 4$ it follows that the characteristic polynomial $f_{\sigma_{s_0^n}}$ of $\sigma_{s_0^n}$ is irreducible over $\mathbb{Q}$ for every $n \in \mathbb{N}$. □

Lemma 3.13. *Σ_2 is a commutative ID-semigroup.*

Proof. Analogous to the proof of Lemma 3.12. □

For a given subset $A \subset \Omega_a^4 \times \Omega_a^4$ and $\omega_1, \omega_2 \in \Omega_a^4$, we define

$$\begin{aligned} A_{\omega_1} &= \{\omega_2 \in \Omega_a^4 : (\omega_1, \omega_2) \in A\} \subset \Omega_a^4, \\ A_{\omega_2} &= \{\omega_1 \in \Omega_a^4 : (\omega_1, \omega_2) \in A\} \subset \Omega_a^4. \end{aligned} \tag{3.14}$$

The following lemma follows immediately from Lemmas 3.12 and 3.13.

Lemma 3.15. *Let A be a non-empty, $\mathfrak{s}^{(1,0)}$- and $\mathfrak{s}^{(0,1)}$-invariant closed subset of $\Omega_a^4 \times \Omega_a^4$. Then*

(i) *the set $P_2 = \{\omega_2 \in \Omega_a^4 : A_{\omega_2} \neq \emptyset\}$ is either the whole Ω_a^4 or is a finite set of torsion elements in Ω_a^4,*

(ii) *the set $P_1 = \{\omega_1 \in \Omega_a^4 : A_{\omega_1} \neq \emptyset\}$ is either the whole Ω_a^4 or is a finite set of torsion elements in Ω_a^4.*

For a given positive integer q we denote by $\Omega_a^d(q)$ the subgroup of Ω_a^d consisting of all elements whose order divides q. It is known (see [2, Lemma II.13]) that for every $q \in \mathbb{N}$, the subgroup

$$\Omega_a^d(q) \simeq \mathbb{Z}[1/a]^d / q\mathbb{Z}[1/a]^d \simeq (\mathbb{Z}[1/a]/q\mathbb{Z}[1/a])^d$$

is finite.

Lemma 3.16 ([8, Lemma 3.2]). *Let σ be a $d \times d$-invertible matrix with entries from the ring $\mathbb{Z}[1/a]$. Let $r \in \Omega_a^d(q) \subset \Omega_a^d$, $q = q_1^{\alpha_1} q_2^{\alpha_2} \dots q_m^{\alpha_m}$, where q_i are different primes and $\alpha_i \in \mathbb{N}$, be a torsion element. Assume that for $1 \leq i \leq m$, $|\det \sigma|_{q_i} = 1$. Then there exists a $k \in \mathbb{N}$ such that*

$$\sigma^k r = r \text{ in } \Omega_a^d.$$

Lemma 3.17. *Let A be a non-empty, $\mathfrak{s}^{(1,0)}$- and $\mathfrak{s}^{(0,1)}$-invariant closed subset of $\Omega_a^4 \times \Omega_a^4$. Then A contains a torsion element.*

Remark. Although Lemma 3.17 can be deduced from appropriate analogues of [8, Lemma 3.3 and Lemma 3.4] and their proofs, we decided to present here a detailed proof as it is one of the crucial results.

P r o o f. By Lemma 3.15 the set $P_2 = \{\omega_2 \in \Omega_a^4 : A_{\omega_2} \neq \emptyset\}$ is either the whole Ω_a^4 or is a finite set of torsion elements in Ω_a^4. There are two cases. The first one, when P_2 contains a torsion element ω_2 of degree $q = q_1^{\alpha_1} \dots q_r^{\alpha_r}$ such that

$$\left|\det \mathfrak{s}_1^{(1,0)}\right|_{q_j} = \left|\det \mathfrak{s}_1^{(0,1)}\right|_{q_j} = 1, \qquad j = 1, \dots, r,$$

(in particular this holds if $P_2 = \Omega_a^4$). Then, by Lemma 3.16, we can find k_1 and $k_2 \in \mathbb{N}$ such that $(\mathfrak{s}_1^{(1,0)})^{k_1} \omega_2 = \omega_2$ and $(\mathfrak{s}_1^{(0,1)})^{k_2} \omega_2 = \omega_2$. Thus we see that A_{ω_2} is a non-empty, closed, $(\mathfrak{s}_1^{(1,0)})^{k_1}$- and $(\mathfrak{s}_1^{(0,1)})^{k_2}$-invariant subset of Ω_a^4. Clearly, the semigroup $\langle (\mathfrak{s}_1^{(1,0)})^{k_1}, (\mathfrak{s}_1^{(0,1)})^{k_2} \rangle$ is an ID-semigroup of endomorphisms of Ω_a^4 (see the proof of Lemma 3.12). Hence, A_{ω_2} is either a finite set of torsion elements or is the whole Ω_a^4. Thus, the lemma follows in this case.

Consider the second case. If P_2 is a finite set of torsion elements but there is no torsion element of degree $q = q_1^{\alpha_1} \dots q_r^{\alpha_r}$ such that

$$\left|\det \mathfrak{s}_1^{(1,0)}\right|_{q_j} = \left|\det \mathfrak{s}_1^{(0,1)}\right|_{q_j} = 1, \qquad j = 1, \dots, r,$$

then we pick up arbitrary torsion element ω_2 from P_2, and instead of A we consider the set A' which is obtained from A by multiplying the second coordinate by the order of ω_2. Then the set P_2' corresponding to A' contains 0. Since A_{ω_2} was non-empty, the set

$$A_0' = \{\omega_1 \in \Omega_a^4 : (\omega_1, 0) \in A'\} \subset \Omega_a^4$$

is also non-empty. Moreover, since A' is $\mathfrak{s}^{(1,0)}$- and $\mathfrak{s}^{(0,1)}$-invariant it follows that A_0' is $\mathfrak{s}_1^{(1,0)}$- and $\mathfrak{s}_1^{(0,1)}$-invariant. Finally, it is clear that A_0' is closed. Since the semigroup $\langle (\mathfrak{s}_1^{(1,0)}), (\mathfrak{s}_1^{(0,1)}) \rangle$ is an ID-semigroup of endomorphisms of Ω_a^4 (Lemma 3.12), it follows that A_0' is either a finite set of torsion elements or is the whole Ω_a^4. Thus, we have proved that A' contains a torsion element. But then also A must contain a torsion element and the lemma follows. □

Remark. Notice that we do not have to use Lemma 3.16 in the proof of Lemma 3.17. In fact, in both cases we can proceed as in the second case. We included the reasoning with Lemma 3.16 since it shows a nice property that for the torsion element ω_2 satisfying assumptions of Lemma 3.16, the set A_{ω_2} is either a finite set of torsion elements or is the whole Ω_a^4.

Lemma 3.18. *Let A be a closed, $\mathfrak{s}^{(1,0)}$- and $\mathfrak{s}^{(0,1)}$-invariant subset of $\Omega_a^4 \times \Omega_a^4$. If all torsion elements of A are isolated in A, then A is finite.*

Proof. The same as the proof of [8, Lemma 3.4]. □

4. Proof of Theorem 1.1

Let a be the product of different primes appearing in a_1 and a_2, that is

$$a = \prod_{p \in S \setminus \{\infty\}} p,$$

where S is defined in (3.9). As we observed there

$$S = \{\infty, p_1, p_2, \ldots, p_s\}$$

with p_i's as in Theorem 1.1.

For $\omega = (\omega_1, \omega_2) \in \Omega_a^4 \times \Omega_a^4$ consider the orbit $\mathbf{\Sigma}\omega$ of the point ω under the action of the semigroup $\mathbf{\Sigma} = \langle \mathfrak{s}^{(1,0)}, \mathfrak{s}^{(0,1)} \rangle$,

$$\mathbf{\Sigma}\omega = \left\{ \left(\mathfrak{s}_1^{(\alpha,\beta)} \omega_1, \mathfrak{s}_2^{(\alpha,\beta)} \omega_2 \right) \in \Omega_a^4 \times \Omega_a^4 : (\alpha, \beta) \in \mathbb{N}_0^2 \setminus \{(0,0)\} \right\}. \tag{4.1}$$

Clearly, $\mathbf{\Sigma}\omega$ is $\mathbf{\Sigma}$-invariant.

Consider a general element $\mathfrak{s}^{(\alpha,\beta)}$ of the semigroup $\mathbf{\Sigma}$,

$$\mathfrak{s}^{(\alpha,\beta)} = \begin{pmatrix} \sigma_{s_0}^{\alpha} \tau_{s_1}^{\beta} & 0 \\ 0 & \sigma_{r_0}^{\alpha} \tau_{r_1}^{\beta} \end{pmatrix}, \quad \text{for some } (\alpha, \beta) \in \mathbb{N}_0^2 \setminus \{(0,0)\}.$$

Denote the diagonal elements of the matrix $\mathfrak{s}^{(\alpha,\beta)}$, which are 4×4-nonsingular matrices belonging to $\mathrm{M}(4, \mathbb{Z}[1/a])$, by $\mathfrak{s}_1^{(\alpha,\beta)}$ and $\mathfrak{s}_2^{(\alpha,\beta)}$. By definition

$$\mathfrak{s}_1^{(\alpha,\beta)} = \sigma_{s_0}^{\alpha} \tau_{s_1}^{\beta} \in \mathrm{M}(4, \mathbb{Z}[1/a_1])$$

and

$$\mathfrak{s}_2^{(\alpha,\beta)} = \sigma_{r_0}^{\alpha} \tau_{r_1}^{\beta} \in \mathrm{M}(4, \mathbb{Z}[1/a_2]).$$

We denote the conjugates of s_0 and s_1 as follows

$$s_0 = s_{0,1} = \lambda_1 \mu_1, \qquad s_{0,2} = \tilde{\lambda}_1 \mu_1, \qquad s_{0,3} = \lambda_1 \tilde{\mu}_1, \qquad s_{0,4} = \tilde{\lambda}_1 \tilde{\mu}_1,$$

$$s_1 = s_{1,1} = \lambda_1^k \mu_1^l, \qquad s_{1,2} = \tilde{\lambda}_1^k \mu_1^l, \qquad s_{1,3} = \lambda_1^k \tilde{\mu}_1^l, \qquad s_{1,4} = \tilde{\lambda}_1^k \tilde{\mu}_1^l,$$

and similarly for r_0 and r_1,

$$r_0 = r_{0,1} = \lambda_2\mu_2, \qquad r_{0,2} = \tilde{\lambda}_2\mu_2, \qquad r_{0,3} = \lambda_2\tilde{\mu}_2, \qquad r_{0,4} = \tilde{\lambda}_2\tilde{\mu}_2,$$
$$r_1 = r_{1,1} = \lambda_2^k\mu_2^l, \qquad r_{1,2} = \tilde{\lambda}_2^k\mu_2^l, \qquad r_{1,3} = \lambda_2^k\tilde{\mu}_2^l, \qquad r_{1,4} = \tilde{\lambda}_2^k\tilde{\mu}_2^l.$$

For $p \in S$, let $\rho_{p,1} \geq \rho'_{p,1}$ ($\rho_{p,2} \geq \rho'_{p,2}$, resp.) denote the p-adic norms of the maximum and minimum of the eigenvalues of the matrix $\mathfrak{s}_1^{(\alpha,\beta)}$ (resp., $\mathfrak{s}_2^{(\alpha,\beta)}$). We have

$$\rho_{p,1} = \max\Big\{|s_{0,i}|_p^\alpha |s_{1,j}|_p^\beta : i,j = 1,2,3,4\Big\},$$
$$\rho'_{p,1} = \min\Big\{|s_{0,i}|_p^\alpha |s_{1,j}|_p^\beta : i,j = 1,2,3,4\Big\}$$

and

$$\rho_{p,2} = \max\Big\{|r_{0,i}|_p^\alpha |r_{1,j}|_p^\beta : i,j = 1,2,3,4\Big\},$$
$$\rho'_{p,2} = \min\Big\{|r_{0,i}|_p^\alpha |r_{1,j}|_p^\beta : i,j = 1,2,3,4\Big\}.$$

It is easy to check that we have the following

Lemma 4.2. *Under the assumptions of Theorem 1.1, there exists (α_0, β_0) such that $\mathfrak{s}^{(\alpha_0,\beta_0)}$ satisfies*

$$\min_{p\in S} \rho'_{p,2} > \max_{p\in S} \rho_{p,1} > 1. \tag{4.3}$$

Moreover, there exists (α'_0, β'_0) such that $\mathfrak{s}^{(\alpha'_0,\beta'_0)}$ satisfies

$$\min_{p\in S} \rho'_{p,1} > \max_{p\in S} \rho_{p,2} > 1. \tag{4.4}$$

We introduce the following notation. Let

$$V_1 \times V_2 := \prod_{j=0}^{s} \mathbb{Q}_{p_j}^4 \times \prod_{j=0}^{s} \mathbb{Q}_{p_j}^4.$$

For $i = 1, 2$ we write $V_i = (V_{i,0}, V_{i,1}, \ldots, V_{i,s})$, where $V_{i,j} = \mathbb{Q}_{p_j}^4$. By $\mathbb{Q}_a^4$ we denote the *"covering space"* of Ω_a^4, i.e.,

$$\mathbb{Q}_a^4 = \prod_{j=0}^{s} \mathbb{Q}_{p_j}^4.$$

If the spaces $\mathbb{Q}_{p_j}^4$, $j = 0, \ldots, s$, are equipped with norms $\|\cdot\|_{p_j}$, then, for

$$\mathbf{z} = (z_0, z_1, \ldots, z_s) \in \mathbb{Q}_a^4,$$

we put

$$\|\mathbf{z}\|_{\mathbb{Q}_a^4} = \max_{0\leq j\leq s} \|z_j\|_{p_j}.$$

The space $\mathbb{Q}_a^4$ becomes a metric space with the distance

$$d_{\mathbb{Q}_a^4}(\mathbf{z},\mathbf{w}) = \|\mathbf{z}-\mathbf{w}\|_{\mathbb{Q}_a^4}.$$

Let

$$\pi : \mathbb{Q}_a^4 \to \Omega_a^4 = \mathbb{Q}_a^4/B$$

be the canonical projection, i.e., $\pi(\mathbf{z}) = \mathbf{z}+B$, where

$$B = \Big\{\big(b, \overbrace{-b,\ldots,-b}^{s}\big) : b \in \mathbb{Z}[1/a]^4\Big\},$$

is a closed discrete subgroup of $\mathbb{Q}_a^4$.

If $\mathbf{b} = (b,-b,\ldots,-b) \in B$, we denote its coordinates by b_j, $0 \le j \le s$, i.e.,

$$b_0 = b \quad \text{and} \quad b_j = -b \;\; \text{for} \;\; j = 1,\ldots,s.$$

It is easy to check that the following function,

$$\begin{aligned} d_{\Omega_a^4}(\mathbf{z}+B,\mathbf{w}+B) &= \inf_{\mathbf{h},\mathbf{b}\in B} d_{\mathbb{Q}_a^4}(\mathbf{z}-\mathbf{h},\mathbf{w}-\mathbf{b}) \\ &= \inf_{\mathbf{b}\in B} \|\mathbf{z}-\mathbf{w}-\mathbf{b}\|_{\mathbb{Q}_a^4} \\ &= \inf_{\mathbf{b}\in B} \max_{0\le j\le s} \|z_j - w_j - b_j\|_{p_j}, \end{aligned} \tag{4.5}$$

defines the metric on Ω_a^4.

The vector $(1,s_0,s_0^2,s_0^3)^t$ is an eigenvector of the matrix $\mathfrak{s}_1^{(1,0)}$ with an eigenvalue s_0, that is

$$\mathfrak{s}_1^{(1,0)}\left(1,s_0,s_0^2,s_0^3\right)^t = s_0\left(1,s_0,s_0^2,s_0^3\right)^t \in \mathbb{R}^4.$$

Since Σ_1 is a commutative semigroup it follows that

$$v = \big(1,s_0,s_0^2,s_0^3,\overbrace{0,\ldots,0}^{4s}\big)^t \in \mathbb{R}^4 \times \mathbb{Q}_{p_1}^4 \times \cdots \times \mathbb{Q}_{p_s}^4$$

is a common eigenvector of Σ_1 acting on $\mathbb{R}^4 \times \mathbb{Q}_{p_1}^4 \times \cdots \times \mathbb{Q}_{p_s}^4$. In particular,

$$\mathfrak{s}_1^{(1,0)} v = s_0 v \quad \text{and} \quad \mathfrak{s}_1^{(0,1)} v = \tau_{s_1} v = g(\sigma_{s_0})v = g(s_0)v = s_1 v. \tag{4.6}$$

Similarly, the vector $(1,r_0,r_0^2,r_0^3)^t$ is an eigenvector of the matrix σ_{r_0} with an eigenvalue r_0. Since $\tau_{r_1} = h(\sigma_{r_0})$ for $h \in \mathbb{Q}[x]$ with $r_1 = h(r_0)$, we get that for the vector

$$w = \big(1,r_0,r_0^2,r_0^3,\overbrace{0,\ldots,0}^{4s}\big)^t \in \mathbb{R}^4 \times \mathbb{Q}_{p_1}^4 \times \cdots \times \mathbb{Q}_{p_s}^4$$

we have

$$\sigma_{r_0}^{\alpha}\tau_{r_1}^{\beta} w = r_0^{\alpha} r_1^{\beta} w. \tag{4.7}$$

Let ξ_1 and ξ_2 be two non-zero real numbers. We set

$$\omega_0 = (v\xi_1, w\xi_2) \in \prod_{p\in S} \mathbb{Q}_p^4 \times \prod_{p\in S} \mathbb{Q}_p^4. \tag{4.8}$$

By (4.6) and (4.7) we have,

$$\mathfrak{s}^{(\alpha,\beta)}\pi(\omega_0) = \pi\Big(s_0^\alpha s_1^\beta \xi_1, s_0^{\alpha+1} s_1^\beta \xi_1, s_0^{\alpha+2} s_1^\beta \xi_1, s_0^{\alpha+3} s_1^\beta \xi_1, \overbrace{0,\dots,0}^{4s}, r_0^\alpha r_1^\beta \xi_2, r_0^{\alpha+1} r_1^\beta \xi_2, r_0^{\alpha+2} r_1^\beta \xi_2, r_0^{\alpha+3} r_1^\beta \xi_2, \overbrace{0,\dots,0}^{4s}\Big). \tag{4.9}$$

We define a homomorphism $\chi_d : \Omega_a^d \to \mathbb{T}^d$. Let

$$\chi_1 : \Omega_a = \mathbb{R} \times \mathbb{Q}_{p_1} \times \cdots \times \mathbb{Q}_{p_s}/B \to \mathbb{T},$$

be given by [6]

$$\chi_1\big((x_0, x_1, \dots, x_s) + B\big) = e^{2\pi i x_0} e^{2\pi i \{x_1\}_{p_1}} \cdots e^{2\pi i \{x_s\}_{p_s}}.$$

Since $x \mapsto e^{2\pi i \{x\}_p}$ is a homomorphism from $\mathbb{Q}_p$ to the 1-torus $\mathbb{T} = \mathbb{R}/\mathbb{Z}$, it is easy to check that the map χ_1 is well defined, i.e., for every $r \in \mathbb{Z}[1/a]$, we have

$$\chi_1\big((x_0 + r, x_1 - r, \dots, x_s - r) + B\big) = \chi_1\big((x_0, x_1, \dots, x_s) + B\big).$$

Now, we extend the map χ_1 to Ω_a^d, $d > 1$. For $j = 0, \dots, s$, we denote

$$x^j = \left(x_1^j, \dots, x_d^j\right) \in \mathbb{Q}_{p_j}^d.$$

Now we define a homomorphism $\chi_d : \Omega_a^d \to \mathbb{T}^d$ by the formula

$$\chi_d\Big(\big(x^0, x^1, \dots, x^s\big) + B^d\Big) = \Big(\chi_1\Big(\big(x_1^0, x_1^1, \dots, x_1^s\big) + B\Big), \dots, \chi_1\Big(\big(x_d^0, x_d^1, \dots, x_d^s\big) + B\Big)\Big). \tag{4.10}$$

Lemma 4.11. *Let $\mathbf{\Omega}$ be the set of accumulation points of the $\mathbf{\Sigma}$-orbit of $\pi(\omega_0)$. If $(0,0) \in \mathbf{\Omega}$ then one of the following holds:*

(1) *the point $(0,0)$ is isolated in $\mathbf{\Omega}$,*

(2) *the set $\mathbf{\Omega}$ contains at least one of the following sets*

$$\begin{aligned} T_1 &= \Omega_a^4 \times \{0\}, \\ T_2 &= \{0\} \times \Omega_a^4. \end{aligned} \tag{4.12}$$

[6] Every $x \in \mathbb{Q}_p$ can be uniquely expressed as a convergent, in $|\cdot|_p$-norm, sum (*Hensel representation*), $x = \sum_{k=t}^{\infty} x_k p^k$, for some $t \in \mathbb{Z}$ and $x_k \in \{0, 1, \dots, p-1\}$. The *fractional part* of $x \in \mathbb{Q}_p$, denoted by $\{x\}_p$, is 0 if the number t in the Hensel representation is greater than or equal to 0, and equal to $\sum_{k<0} x_k p^k$, if $t < 0$.

Remark 4.13. Let F be a close infinite $\mathbf{\Sigma}$-invariant subset of $\Omega_a^4 \times \Omega_a^4$, and let F^{ac} denote the set of accumulation points of F. It will be clear from the proof that Lemma 4.11 is valid for F^{ac} in place of $\mathbf{\Omega}$.

We postpone the proof of Lemma 4.11 to the next subsection and continue with the proof of Theorem 1.1.

First we note that immediately from Lemma 4.11 we get the following

Corollary 4.14. *If* $(0,0) \in \mathbf{\Omega}$ *then one of the following holds:*

(1) *the point* $(0,0)$ *is isolated in* $\mathbf{\Omega}$,

(2) *the set* $\mathbf{\Omega}^* = \{\omega_1 + \omega_2 : \omega = (\omega_1, \omega_2) \in \mathbf{\Omega}\}$ *is equal to* Ω_a^4.

Proof of Theorem 1.1. We can assume that both ξ_1 and ξ_2 are non-zero; if one of them is zero then Theorem 1.1 follows by result of [3]. Consider the set $\mathbf{\Omega}$. Assume first that $(0,0)$ is not isolated in $\mathbf{\Omega}$. Notice that by (3.4) it follows that the first and the $4s+5$'th coordinate of (4.9) are equal to

$$\lambda_1^{\alpha+\beta k}\mu_1^{\alpha+\beta l}\xi_1 \quad \text{and} \quad \lambda_2^{\alpha+\beta k}\mu_1^{\alpha+\beta l}\xi_2.$$

Hence, density modulo 1 of the set

$$\left\{\lambda_1^{\alpha+\beta k}\mu_1^{\alpha+\beta l}\xi_1 + \lambda_2^{\alpha+\beta k}\mu_1^{\alpha+\beta l}\xi_2 : (\alpha,\beta) \in \mathbb{N}_0^2 \setminus \{(0,0)\}\right\} \tag{4.15}$$

follows from Corollary 4.14 if we take the image of the set $\mathbf{\Omega}^* = \Omega_a^4$ by the map χ_4 (4.10) composed with the projection on the first coordinate in $\mathbb{T}^4$. But (4.15) is a subset of (1.2). (Note that in this case $\kappa = 1$.)

Next suppose that $(0,0)$ is isolated. By Lemma 3.17 and Lemma 3.18 there is a non-isolated torsion element $(q_2, q_2)^t \in \mathbf{\Omega}$. Proceeding as in the proof of [8, Theorem 1.5], that is multiplying by the matrix $\kappa\mathrm{Id}$, where $\kappa q_1 = \kappa q_2 = 0$, and using Remark 4.13 we get that $\kappa\mathbf{\Omega}^* = \Omega_a^4$. Hence, the theorem is proved. □

4.1. Proof of Lemma 4.11

The proof follows the main steps of [8], where the product of the 2-dimensional solenoids was considered. In the proof we will need the following lemma. Let k be a local field (in our case $k = \mathbb{R}$ or the finite extension of $\mathbb{Q}_p$), equipped with an *absolute value* $|\cdot|$ ($|\cdot| = |\cdot|_\infty$ or $|\cdot|_p$, resp.), K be an algebraic closure of k. The unique extension of $|\cdot|$ to the absolute value on K will also be denoted by $|\cdot|$.

Lemma 4.16. *Let* $A \in \mathrm{GL}(2,k)$. *Suppose that* A *has two different eigenvalues* $\eta_1, \eta_2 \in K$, *such that* $|\eta_1| > |\eta_2| > 1$. *Then there exists a norm* $\|\cdot\|$ *in* k^2 *such that*

(i) $\|A\| = |\eta_1|$ *and* $\|A^{-1}\| = \frac{1}{|\eta_2|}$,

(ii) $\|Av\| \geq |\eta_2|\|v\|$ *and* $\|A^{-1}v\| \geq \frac{1}{|\eta_1|}\|v\|$.

P r o o f. See for example [8, Lemma 4.6]. □

P r o o f o f L e m m a 4.11. Let pr_i $(i = 1, 2)$ be the projection from the product $\Omega_a^4 \times \Omega_a^4$ onto its first and second "coordinate", respectively. By Lemma 3.12 the semigroup $\Sigma_1 = \langle \mathfrak{s}_1^{(1,0)}, \mathfrak{s}_1^{(0,1)} \rangle$ is an ID-semigroup. Since s_0 is irrational $\mathrm{pr}_1(\pi(v))$ is not a torsion point. Hence, we obtain that for every $\omega_1 \in \Omega_a^4$ there exist sequences $\{\alpha_k\}$ and $\{\beta_k\}$, tending to infinity, such that

$$\mathfrak{s}_1^{(\alpha_k,\beta_k)} \mathrm{pr}_1(\pi(v)) = \left(\mathfrak{s}_1^{(1,0)}\right)^{\alpha_k} \left(\mathfrak{s}_1^{(0,1)}\right)^{\beta_k} \mathrm{pr}_1(\pi(v)) \to \omega_1,$$

as $k \to \infty$. Since Ω_a^4 is compact, we can assume, choosing a subsequence, that there exists $\omega_2 \in \Omega_a^4$ such that $\mathfrak{s}_2^{(\alpha'_k,\beta'_k)} \mathrm{pr}_2(\pi(w)) \to \omega_2$. Therefore, for every $\omega_1 \in \Omega_a^4$ there exists $\omega_2 \in \Omega_a^4$ so that $(\omega_1, \omega_2) \in \mathbf{\Omega}$. In particular, we see that $\mathbf{\Omega}$ is infinite.

Clearly, $\mathbf{\Omega}$ is a non-empty, $\mathfrak{s}^{(1,0)}$- and $\mathfrak{s}^{(0,1)}$-invariant closed subset of $\Omega_a^4 \times \Omega_a^4$. By Lemma 3.15 the intersection of $\mathbf{\Omega}$ with the "axes" T_1 and T_2 either is empty, contains finitely many torsion elements, or equals T_i, $i = 1, 2$. Assume that for $i = 1, 2$,

$$\mathbf{\Omega} \cap T_i \text{ is empty or a finite set of torsion elements.} \tag{4.17}$$

We will show that this assumption leads to contradiction if $(0, 0)$ is not isolated in $\mathbf{\Omega}$.

Since $(0, 0) \in \Omega_a^4 \times \Omega_a^4$ is not isolated in $\mathbf{\Omega}$ there exists a sequence $\{(\mathbf{x}_n + B, \mathbf{y}_n + B)\} \subset \mathbf{\Omega}$ tending to $(0, 0)$. By (4.17) it follows that $\mathbf{x}_n + B \neq 0$ and $\mathbf{y}_n + B \neq 0$. Without loss of generality, choosing an appropriate representative from $\mathbf{x}_n + B$ ($\mathbf{y}_n + B$, resp.), we can assume that $\mathbf{x}_n \neq 0$, $\|\mathbf{x}_n\|_{\mathbb{Q}_a^4} \to 0$, and $\mathbf{y}_n \neq 0$, $\|\mathbf{y}_n\|_{\mathbb{Q}_a^4} \to 0$. Choosing an appropriate subsequence, we can assume that

$$\lim_{n\to\infty} \frac{d^2_{\Omega_a^4}(\mathbf{y}_n + B, 0)}{d^1_{\Omega_a^4}(\mathbf{x}_n + B, 0)} = c \in [0, +\infty]. \tag{4.18}$$

Remark. Since all norms on finite dimensional vector space are equivalent, the metrics $d^i_{\Omega_a^4}$'s defined with the use of different norms on the covering space, are equivalent. In particular, if the limit (4.18) is non-zero (infinite, resp.) for one metric it is non-zero (infinite, resp.) for all equivalent metrics.

Consider the case when $c \neq 0$ or the limit in (4.18) is infinite. By (4.3) of Lemma 4.2 there is $(\alpha_0, \beta_0) \in \mathbb{N}_0^2 \setminus \{(0, 0)\}$ such that the element $\mathfrak{s}^{(\alpha_0,\beta_0)}$ satisfies

$$\min_{p\in S} \rho'_{p,2} > \max_{p\in S} \rho_{p,1} > 1. \tag{4.19}$$

In what follows we fix such an element $\mathfrak{s}^{(\alpha_0,\beta_0)}$. By Lemma 4.16 we have norms $\|\cdot\|_{p_j,i}$ in $V_{i,j}$ such that, for every $y \in V_{2,j}$,

$$\rho_{p_j,2}\|y\|_{p_j,2} \geq \left\|\mathfrak{s}_2^{(\alpha,\beta)}y\right\|_{p_j,2} \geq \rho'_{p_j,2}\|y\|_{p_j,2}, \tag{4.20}$$

and for every x in $V_{1,j}$ we have,

$$\left\|\mathfrak{s}_1^{(\alpha,\beta)}x\right\|_{p_j,1} \leq \rho_{p_j,1}\|x\|_{p_j,1}. \tag{4.21}$$

We consider the product $\Omega_a^4 \times \Omega_a^4$ endowed with $(d^1_{\Omega_a^4}, d^2_{\Omega_a^4})$, where $d^i_{\Omega_a^4}$'s are defined by (4.5) with the use of the norms defined in (4.20) for $i=2$, and (4.21) for $i=1$.

Lemma 4.22. *Let $\mathfrak{s}^{(\alpha_0,\beta_0)}$ be as above. Then there exist constants $\rho > 1$ and $1 > \gamma > 0$ such that for every $\mathbf{y}+B \in \Omega_a^4$ satisfying $d^2_{\Omega_a^4}(\mathbf{y}+B,0) < \frac{\gamma}{2}$, we have*

$$d^2_{\Omega_a^4}\left(\mathfrak{s}_2^{(\alpha_0,\beta_0)}(\mathbf{y}+B),0\right) \geq \rho d^2_{\Omega_a^4}(\mathbf{y}+B,0).$$

In particular, iterating the above inequality, it follows that $\mathfrak{s}_2^{(\alpha_0,\beta_0)}$ is expansive, *i.e., there exists an open ball U (of radius $\gamma/2$) around 0 such that for every*

$$0 \neq \mathbf{y}+B \in U,$$

there exists l such that

$$\left(\mathfrak{s}_2^{(\alpha_0,\beta_0)}\right)^l \mathbf{y}+B \notin U.$$

Proof. Let $\varepsilon = \inf_{\mathbf{b}\in B\setminus\{0\}}\|\mathbf{b}\|_{\mathbb{Q}_a^4}$, $\gamma = \frac{\varepsilon}{\max_{p\in S}\rho_{p,2}}$, and $\rho = \min_{p\in S}\rho'_{p,2}$. Changing the representative $\mathbf{y}$, if necessary, we can assume that

$$\|\mathbf{y}\|_{\mathbb{Q}_a^4} < \gamma/2. \tag{4.23}$$

For simplicity, we denote the matrix $\mathfrak{s}_2^{(\alpha_0,\beta_0)}$ by A. By (4.20) and (4.21) we get,

$$\begin{aligned} d^2_{\Omega_a^4}\big(A(\mathbf{y}+B),0\big) &= \inf_{\mathbf{b}\in B}\max_{0\leq j\leq s}\|Ay_j - b_j\|_{p_j,2} \\ &= \inf_{\mathbf{b}\in B}\max_{0\leq j\leq s}\left\|A(y_j - A^{-1}b_j)\right\|_{p_j,2} \\ &\geq \inf_{\mathbf{b}\in B}\max_{0\leq j\leq s}\rho'_{p_j,2}\left\|y_j - A^{-1}b_j\right\|_{p_j,2} \\ &\geq \inf_{\mathbf{b}\in B}\max_{0\leq j\leq s}(\min_{p\in S}\rho'_{p,2})\left\|y_j - A^{-1}b_j\right\|_{p_j,2} \\ &= \rho\inf_{\mathbf{b}\in B}\left\|\mathbf{y} - A^{-1}\mathbf{b}\right\|_{\mathbb{Q}_a^4}. \end{aligned} \tag{4.24}$$

It follows from Lemma 4.16 and (4.20) that for every non-zero element

$$\mathbf{b} = (b,-b,\ldots,-b) \in B,$$

we have

$$\begin{aligned}\|A^{-1}\mathbf{b}\|_{\mathbb{Q}_a^4} &= \max_{0\le j\le s}\|A^{-1}b\|_{p_j,2} \ge \max_{0\le j\le s}\frac{1}{\rho_{p_j,2}}\|b\|_{p_j,2} \\ &\ge \frac{1}{\max_{p\in S}\rho_{p,2}}\|\mathbf{b}\|_{\mathbb{Q}_a^4} \ge \frac{1}{\max_{p\in S}\rho_{p,2}}\varepsilon = \gamma.\end{aligned} \tag{4.25}$$

By (4.23) and (4.25) we get

$$\inf_{\mathbf{b}\in B}\|\mathbf{y}-A^{-1}\mathbf{b}\|_{\mathbb{Q}_a^4} = \inf_{\mathbf{b}\in B}\|\mathbf{y}-\mathbf{b}\|_{\mathbb{Q}_a^4} = \|\mathbf{y}\|_{\mathbb{Q}_a^4} = d_{\Omega_a^4}(\mathbf{y}+B,0). \tag{4.26}$$

Now (4.26) and (4.24) imply the conclusion. □

For every $l\in\mathbb{N}$,

$$\begin{aligned} d^1_{\Omega_a^4}\left(\left(\mathfrak{s}_1^{(\alpha_0,\beta_0)}\right)^l(\mathbf{x}_n+B),0\right) &= \inf_{\mathbf{b}\in B}\max_{0\le j\le s}\left\|\left(\mathfrak{s}_1^{(\alpha_0,\beta_0)}\right)^l(\mathbf{x}_n)_j - b_j\right\|_{p_j,1} \\ &= \inf_{\mathbf{b}\in B}\max_{0\le j\le s}\left\|\left(\mathfrak{s}_1^{(\alpha_0,\beta_0)}\right)^l\left((\mathbf{x}_n)_j - \left(\mathfrak{s}_1^{(\alpha_0,\beta_0)}\right)^{-l} b_j\right)\right\|_{p_j,1} \\ &\le \inf_{\mathbf{b}\in B}\max_{0\le j\le s}\left(\rho_{p_j,1}\right)^l\left\|(\mathbf{x}_n)_j - \left(\mathfrak{s}_1^{(\alpha_0,\beta_0)}\right)^{-l} b_j\right\|_{p_j,1} \\ &\le \left(\max_{p\in S}\rho_{p,1}\right)^l\inf_{\mathbf{b}\in B}\left\|\mathbf{x}_n - \left(\mathfrak{s}_1^{(\alpha_0,\beta_0)}\right)^{-l}\mathbf{b}\right\|_{\mathbb{Q}_a^4} \\ &\le \left(\max_{p\in S}\rho_{p,1}\right)^l\|\mathbf{x}_n\|_{\mathbb{Q}_a^4}. \end{aligned}$$

Since $\|\mathbf{x}_n\|_{\mathbb{Q}_a^4}\to 0$, we can assume that $\|\mathbf{x}_n\|_{\mathbb{Q}_a^4} < \frac{1}{2}\inf_{\mathbf{b}\in B\setminus\{0\}}\|\mathbf{b}\|_{\mathbb{Q}_a^4}$. Then

$$\|\mathbf{x}_n\|_{\mathbb{Q}_a^4} = d^1_{\Omega_a^4}(\mathbf{x}_n+B,0),$$

and consequently we have

$$d^1_{\Omega_a^4}\left(\left(\mathfrak{s}_1^{(\alpha_0,\beta_0)}\right)^l(\mathbf{x}_n+B),0\right) \le \left(\max_{p\in S}\rho_{p,1}\right)^l d^1_{\Omega_a^4}\left(\mathbf{x}_n+B,0\right). \tag{4.27}$$

Similarly,

$$d^2_{\Omega_a^4}\left(\left(\mathfrak{s}_2^{(\alpha_0,\beta_0)}\right)^l(\mathbf{y}_n+B),0\right) \le \left(\max_{p\in S}\rho_{p,2}\right)^l d^2_{\Omega_a^4}(\mathbf{y}_n+B,0). \tag{4.28}$$

Fix r such that $1/(\max_{p\in S}\rho_{p,2})^r \le \gamma/2$. Since $\|\mathbf{y}_n\|_{\mathbb{Q}_a^4}\to 0$ we may assume that

$$\|\mathbf{y}_n\|_{\mathbb{Q}_a^4} \le \varepsilon/2 \qquad \text{and} \qquad d^2_{\Omega_a^4}(\mathbf{y}_n+B,0) \le 1/\left(\max_{p\in S}\rho_{p,2}\right)^r.$$

By Lemma 4.22 and (4.28), for each n, there exists the smallest number $l_n \in \mathbb{N}$ such that

$$d^2_{\Omega^4_a}\left(\left(\mathfrak{s}_2^{(\alpha_0,\beta_0)}\right)^{l_n}(\mathbf{y}_n+B),0\right) \geq \rho^{l_n} d^2_{\Omega^4_a}(\mathbf{y}_n+B,0), \quad \text{where} \quad \rho = \min_{p\in S}\rho'_{p,2},$$

and

$$\left(\max_{p\in S}\rho_{p,2}\right)^{-(r+1)} \leq d^2_{\Omega^4_a}\left(\left(\mathfrak{s}_2^{(\alpha_0,\beta_0)}\right)^{l_n}(\mathbf{y}_n+B),0\right) \leq \left(\max_{p\in S}\rho_{p,2}\right)^{-r}. \tag{4.29}$$

Since Ω^4_a is compact, we can choose a subsequence n_k such that

$$\lim_{k\to\infty}\left(\left(\mathfrak{s}_2^{(\alpha_0,\beta_0)}\right)^{l_{n_k}}\mathbf{y}_{n_k}+B\right) = \mathbf{y}+B \neq 0$$

and

$$\left(\max_{p\in S}\rho_{p,2}\right)^{-(r+1)} \leq d^2_{\Omega^4_a}(\mathbf{y}+B,0) \leq \left(\max_{p\in S}\rho_{p,2}\right)^{-r}. \tag{4.30}$$

By (4.29) and (4.27) we have

$$\frac{d^2_{\Omega^4_a}\left(\left(\mathfrak{s}_2^{(\alpha_0,\beta_0)}\right)^{l_{n_k}}\mathbf{y}_{n_k}+B,0\right)}{d^1_{\Omega^4_a}\left(\left(\mathfrak{s}_1^{(\alpha_0,\beta_0)}\right)^{l_{n_k}}\mathbf{x}_{n_k}+B,0\right)} \geq \left(\frac{\min_{p\in S}\rho'_{p,2}}{\max_{p\in S}\rho_{p,1}}\right)^{l_{n_k}} \frac{d^2_{\Omega^4_a}(\mathbf{y}_n+B,0)}{d^1_{\Omega^4_a}(\mathbf{x}_n+B,0)}.$$

Now (4.30) and (4.19) together with our assumption that the limit in (4.18) is non-zero or $+\infty$ imply that

$$\lim_{k\to\infty}\frac{d^2_{\Omega^4_a}\left(\left(\mathfrak{s}_2^{(\alpha_0,\beta_0)}\right)^{l_{n_k}}\mathbf{y}_{n_k}+B,0\right)}{d^1_{\Omega^4_a}\left(\left(\mathfrak{s}_1^{(\alpha_0,\beta_0)}\right)^{l_{n_k}}\mathbf{x}_{n_k}+B,0\right)} = +\infty.$$

This and (4.30) imply that

$$d^1_{\Omega^4_a}\left(\left(\mathfrak{s}_1^{(\alpha_0,\beta_0)}\right)^{l_{n_k}}\mathbf{x}_{n_k}+B,0\right) \to 0 \qquad \text{as} \quad k\to\infty.$$

Thus, we have a sequence of points

$$\left\{\left(\left(\mathfrak{s}_1^{(\alpha_0,\beta_0)}\right)^{l_{n_k}}\mathbf{x}_{n_k}+B, \left(\mathfrak{s}_2^{(\alpha_0,\beta_0)}\right)^{l_{n_k}}\mathbf{y}_{n_k}+B\right)\right\} \subset \mathbf{\Omega}$$

such that

$$\left(\left(\mathfrak{s}_1^{(\alpha_0,\beta_0)}\right)^{l_{n_k}}\mathbf{x}_{n_k}+B,\left(\mathfrak{s}_2^{(\alpha_0,\beta_0)}\right)^{l_{n_k}}\mathbf{y}_{n_k}+B\right) \to (B\mathbf{y}+B) \in \{0\}\times\Omega^4_a,$$

with

$$\mathbf{y}+B\neq 0 \qquad \text{and} \qquad \left(\max_{p\in S}\rho_{p,2}\right)^{-(r+1)} \leq d^2_{\Omega^4_a}(\mathbf{y}+B,0) \leq \left(\max_{p\in S}\rho_{p,2}\right)^{-r}.$$

Repeating this construction for the sequence of natural numbers $r \to \infty$ we get a sequence of different points in $(\{0\} \times \Omega_a^4) \cap \mathbf{\Omega}$ tending to zero. This contradicts (4.17).

If the limit in (4.18) is zero, then we proceed analogously to the previous case changing the role of coordinates in $\Omega_a^4 \times \Omega_a^4$. By Lemma 4.2 there exists an element $\mathfrak{s}^{(\alpha_0',\beta_0')} \in \mathbf{\Sigma}$ such that $\min_{p\in S} \rho'_{p,1} > \max_{p\in S} \rho_{p,2} > 1$. By Lemma 4.16, there exist norms $\|\cdot\|_{p_j,i}$ in $V_{i,j}$ such that, for every $x \in V_{1,j}$,

$$\rho_{p_j,1}\|x\|_{p_j,1} \geq \left\| \mathfrak{s}_1^{(\alpha_0',\beta_0')} x \right\|_{p_j,1} \geq \rho'_{p_j,1}\|x\|_{p_j,1},$$

and for every y in $V_{2,j}$ we have,

$$\left\| \mathfrak{s}_2^{(\alpha_0',\beta_0')} y \right\|_{p_j,2} \leq \rho_{p_j,2}\|y\|_{p_j,2}.$$

Now we consider

$$d^i_{\Omega_a^4}, \ i = 1, 2,$$

defined by (4.5) with the use of $\|\cdot\|_{p_j,i}$ defined above, and we prove the analogue of Lemma 4.22 saying that $\mathfrak{s}_1^{(\alpha_0',\beta_0')}$ is expansive. This allows us to construct a sequence of points

$$(\mathbf{y}_r + B, B) \in \Omega_a^4 \times \{0\},$$

with

$$\mathbf{y}_r + B \neq 0, \ (\max_{p\in S} \rho_{p,1})^{-(r+1)} \leq d^1_{\Omega_a^4}(\mathbf{y}_r + B, 0) \leq (\max_{p\in S} \rho_{p,1})^{-r}.$$

This again contradicts (4.17). □

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Roman Urban
Institute of Mathematics
Wroclaw University
Plac Grunwaldzki 2/4
PL-50–384 Wroclaw
POLAND
E-mail: urban@math.uni.wroc.pl