Uniform Distribution Theory **4** (2009), no.2, 39–45

uniform distribution theory

ON AFFINE MAPS AND AN ARITHMETIC LIMIT ON COMPACT GROUPS

Radhakrishnan Nair

ABSTRACT. Let G be a compact connected metric abelian group equipped with its normalised Haar measure. Let $Tx = a + A(x)$ be a continuous surjective affine map of G such that $\gamma \equiv 1$ is the only character of G satisfying $\gamma \circ A^n = \gamma$ for some positive integer n. Then if ϕ is a polynomial with real coefficients mapping the natural numbers to themselves, $(p_l)_{l=1}^{\infty}$ is the sequence of rational primes and f is in $L^p(G)$ for $p > 1$, we prove that

$$\lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} f\left(T^{\phi(p_n)}x\right) = \int_G f(g)dg$$

almost everywhere with respect to Haar measure on G.

Communicated by Pierre Liardet

1. Introduction

Let G be a compact metric abelian group equipped with its normalised Haar measure. We call a map $T : G \to G$ affine if there exists a continuous surjective endomorphism $A : G \to G$ and $a \in G$ such that $Tx = a + A(x)$. In this paper we prove the following theorem.

Theorem. *Let G be a compact connected metric abelian group equipped with its normalised Haar measure. Let $Tx = a + A(x)$ be a continuous surjective affine map of G such that $\gamma \equiv 1$ is the only character of G with $\gamma \circ A^n = \gamma$ for some positive integer n. Suppose that ϕ is a polynomial with real coefficients mapping the natural numbers to themselves, that $(p_l)_{l=1}^{\infty}$ is the sequence of rational primes*

2000 Mathematics Subject Classification: 11K06, 11K55, 11L20, 22D40, 28D05, 28D15.
Keywords: Polynomials in primes, ergodic transformation, affine maps on groups, compact groups.

and that f is in $L^p(G)$ for $p > 1$. Then

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} f\left(T^{\phi(p_n)}x\right) = \int_G f(g)dg \tag{1}$$

almost everywhere with respect to Haar measure on G.

For a measure space (X, β, μ), a map $T : X \to X$ is said to be measure preserving if $\mu(T^{-1}A) = \mu(A)$ for all $A \in \beta$ and we call (X, β, μ, T) a dynamical system. Further (X, β, μ, T) is said to be ergodic if $T^{-1}A = A$ implies that $\mu(A)$ or $\mu(X\backslash A)$ is 0. We say it is weak mixing if the product dynamical system $(X \times X, \beta \times \beta, \mu \times \mu, T \times T)$ is also ergodic. By Birkhoff's theorem, the limit

$$\overline{f}(x) := \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} f(T^n x)$$

exists. Under our condition on the characters of G, the map T is ergodic and so being T invariant $\overline{f}(x) = \int_G f(g)dg$ Haar almost everywhere [Wa]. The ergodicity of T in and of it of itself is not enough to ensure (1) is true. For instance, as it is shown in [AN], there exists an ergodic transformation for which a much more complicated limit is identified. If we set $G = \mathbf{T}^d$ (the d-dimensional torus) with $d \geq 1$ and we set $Tx = A(x)$, where A is the endomorphism of $\mathbf{T}^d$ defined via a $d \times d$ integer entry matrix also denoted A, our character condition on T reduces to none of the eigenvalues of A being roots of unity [Wa]. Our Theorem has the following Corollary, which may be viewed as a refined variant of E. Borel's famous normal number theorem.

Corollary. *Suppose that A is a $d \times d$ integer entry matrix none of whose eigenvalues are roots of unity and that $f \in L^p(\mathbf{T}^d)$ with $p > 1$. Then for ϕ and $(p_n)_{n=1}^{\infty}$ as in our Theorem*

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} f\left(A^{\phi(p_n)}x\right) = \int_{\mathbf{T}^d} f(t)dt \tag{2}$$

almost everywhere with respect to Haar measure on $\mathbf{T}^d$.

These results can be set in a more general framework. To any strictly increasing sequence of natural numbers $(a_k)_{k=1}^{\infty}$, dynamical system (X, β, μ, T), fixed real number $p \geq 1$ and $f \in L^p(X, \beta, \mu)$ we associate the sequence of averages

$$A_N f(x) = \frac{1}{N} \sum_{k=1}^{N} f(T^{a_k}x) \qquad (N = 1, 2, \ldots).$$

We say $(a_k)_{k=1}^{\infty}$ is L^p good universal if $\lim_{N\to\infty} A_N f(x)$ exists almost everywhere. Answering a question ascribed variously to A. Bellow, H. Furstenberg and M. Herman it was shown by J. Bourgain that the sequence $a_k = k^2 (k = 1, 2, \dots)$ is L^2 good universal [Bo2]. Later this was extended to show $a_k = k^r (k = 1, 2 \dots)$ for natural numbers $r \geq 1$ was L^p good universal for $p > 1$. The author showed [Na1] that if ϕ is a polynomial mapping the natural numbers to themselves, then $a_k = \phi(p_k)(k = 1, 2 \dots)$ is L^p good universal for $p > 1$. The case $\phi(x) = x$ of this result appears in [Wi] which itself improves [Bo1], where the same result appears for $p > \frac{\sqrt{3}+1}{2}$. What happens when $p = 1$, to the best of the author's knowledge, is still open. The property of these sequences that makes the identification of the limit interesting as in the case of our theorem, is that the sequences $(k^r)_{k=1}^{\infty}$ and $(\phi(p_k))_{k=1}^{\infty}$ are not generally uniformly distributed modulo m for all natural numbers $m \geq 2$. Another class of sequences for which L^p $(p > 1)$ good universality can be proved are condition H sequences introduced in [Na3] and constructed as follows. Denote by $\lfloor y \rfloor$ the largest integer not greater than y and let $\{y\}$ denote its fractional part $y - \lfloor y \rfloor$. Set $a_k = \lfloor g(k) \rfloor$ $(k = 1, 2, \dots)$, where g is a differentiable function from $[1, \infty)$ to itself whose derivative increases with its argument. Let A_M denote the cardinality of the set $\{k : a_k \leq M\}$ and suppose for some function $a : [1, \infty) \to [1, \infty)$ increasing to infinity as it argument does, we set

$$b(M) = \sup_{\{\alpha\} \in \left[\frac{1}{a(M)}, \frac{1}{2}\right)} \left| \sum_{k : a_k \leq M} e(\alpha a_k) \right|,$$

where for a real number x we have used $e(x)$ to denote $e^{2\pi i x}$. Suppose also for some decreasing function $c : [1, \infty) \to [1, \infty)$ and some positive constant C that

$$\frac{b(M) + A_{\lfloor a(M) \rfloor} + \frac{M}{a(M)}}{A_M} \leq C c(M).$$

Then if we have

$$\sum_{s=1}^{\infty} c(\theta^s) < \infty \qquad \text{for every} \quad \theta > 1,$$

we say that $(a_k)_{k \geq 1}$ satisfies *condition* H. Examples include sequences of integers $a_k = \lfloor g(k) \rfloor$ $(k = 1, 2, \dots)$ for $g(k) = k^{\omega}$, for non-integer $\omega > 1$, $g(k) = e^{(\log k)^{\gamma}}$, for γ in $(1, \frac{3}{2})$ and $g(k) = P(k)$, where $P(k) = \alpha_r k^r + \cdots + \alpha_1 k + \alpha_0$ and the real numbers $\alpha_1, \dots, \alpha_r$ are not all different rational multiples of the same real number. An important property of condition H sequences is that they are Hartman uniformly distributed, i.e., $(a_k)_{k=1}^{\infty}$ is uniformly distributed modulo m for each natural number $m \geq 2$ and $(\{a_k \theta\})_{k=1}^{\infty}$ is uniformly distributed modulo 1 for each irrational number θ. Note that this means $a_k = \phi(p_k)$ $(k = 1, 2 \ldots)$ is not Hartman uniformly distributed. Hartman uniformly distributed sequences are very useful with regard to applications as the limit of the pointwise limit of the ergodic averages is identical to those in Birkhoff's ergodic theorem. Other good universal sequences of integers that are Hartman uniformly distributed include various random constructions exemplified by those in [BW] and [Bo1] and a construction, based on Hardy fields in [BKQW] which seems to the author to satisfy condition H and hence be a special case in [Na3, of Theorem 4], proved by much the same means. J. Rosenblatt [R] has shown that even L^∞ good universality fails if there exists $q > 1$ such that $a_{k+1} \geq qa_k$ $(k = 1, 2 \ldots)$. For further background see the survey [T].

Proof of Theorem. The existence of the limit (1) is assured in [Na1, by Theorem 1]. This means that all we have to do is to identify this limit as $\int_G f(g)dg$. We first prove that

$$\lim_{N\to\infty} \left\| \frac{1}{N} \sum_{n=1}^{N} f\left(T^{\phi(p_n)}x\right) - \int_G f(g)dg \right\|_2 = 0, \tag{3}$$

where as standard

$$\|f\|_p = \left(\int_X |f|^p(g)dg \right)^{\frac{1}{p}} \qquad \text{for } p \in [1, \infty).$$

Let $\langle \,.\,, \,.\, \rangle$ denote the standard inner product on $L^2(X, \beta, \mu)$. Also let $Uf(x) = f(Tx)$ and for natural number n let U^{-n} denote the adjoint of U^n. A basic fact from ergodic theory is that there exists a measure ω_f defined on $\mathbf{T}$, that is non-atomic if T is weak mixing [Na2] such that

$$\langle\, U^n f, f \,\rangle = \int_{\mathbf{T}} z^n d\omega_f(z) \qquad (n \in \mathbf{Z}).$$

In the special case where T is an affine map of a compact connected metric abelian group G, its being weak mixing is equivalent to the condition that $\gamma \equiv 1$ is the only character of G such that $\gamma \circ A^n = \gamma$ for some natural number n. See [Wa, Theorem 1.11 p. 31 and Theorem 1.29 p. 50], for details. In proving (3) we assume, as we can do without loss of generality, that $\int_G f(g)dg = 0$.

This implies that

$$\begin{aligned}\left\|\frac{1}{N+1}\sum_{k=0}^{N} f\left(T^{\phi(p_k)}x\right)\right\|_2^2 &= \frac{1}{(N+1)^2}\sum_{0\le k_1,\,k_2\le N}\left\langle U^{(\phi(p_{k_1})-\phi(p_{k_2}))}f, f\right\rangle \\ &= \frac{1}{(N+1)^2}\sum_{0\le k_1,\,k_2\le N}\int_{\mathbf{T}} z^{(\phi(p_{k_1})-\phi(p_{k_2}))}d\omega_f(z) \\ &= \int_{\mathbf{T}}\left|\frac{1}{N+1}\sum_{k=0}^{N} z^{\phi(p_k)}\right|^2 d\omega_f(z). \qquad (4)\end{aligned}$$

This means that to prove (3) it suffices to show that the integrand on the right hand side of (4) tends to zero for $z = e(t)$ with irrational t. This follows immediately from the fact, taken from [R], that for any real polynomial P the sequence $(P(p_n))_{n=1}^{\infty}$ is uniformly distributed modulo 1 if $P(x) - P(0)$ has an irrational coefficient. We have established (3).

Now if $g \in L^{\infty}(G)$ then, using the bounded convergence theorem, there exists $g^* \in L^p(G)$ with

$$\lim_{N\to\infty}\left\|\frac{1}{N}\sum_{n=1}^{N} g\left(T^{\phi(p_n)}x\right) - g^*(h)\right\|_p = 0.$$

For general g in $L^p(G)$, the same observation follows, using the fact that $L^{\infty}(G)$ is dense in $L^p(G)$, that T is measure preserving and the triangle inequality. Also using (3) and the triangle inequality one verifies $f^*(x) = \int_G f(h)dh$ for all f in $L^2(G)$ and for f in $L^p(G)$ with $p \in (1,2)$ because $\|f\|_p$ is bounded by $\|f\|_2$. We now need to show the norm limit is the same as the almost everywhere limit. Choose natural numbers $(N_t)_{t=1}^{\infty}$ such that we have

$$\left\|A_{N_t}f - \int_G f(h)dh\right\|_p \le \frac{1}{t}.$$

Then

$$\sum_{t=1}^{\infty}\int_G\left|A_{N_t}f(h) - \int_G f(h)dh\right|^p dh < \infty,$$

which using Fatou's lemma gives

$$\int_G\left(\sum_{t=1}^{\infty}\left|A_{N_t}f(h) - \int_G f(h)dh\right|^p\right)dh < \infty.$$

From this we know that we have,

$$\sum_{t=1}^{\infty} \left| A_{N_t} f(h) - \int_G f(h)dh \right|^p < \infty,$$

almost everywhere. Hence we have

$$\lim_{t\to\infty} \frac{1}{N_t} \sum_{t=1}^{N^t} f\left(T^{\phi(p_n)}x\right) = \int_G f(g)dg$$

almost everywhere with respect to Haar measure on G. So $(A_N f)_{N=1}^{\infty}$ converges almost everywhere with respect to Haar measure on G to $\int_G f(h)dh$ as required. □

Remark. As the referee has also observed, the only fact about $Tx = a + A(x)$ we have used is that it is weak mixing under our condition on characters and our result is true for any dynamical system (X, β, μ, T) that is weak mixing and in fact under some weaker conditions. In particular, affine maps on non-commutative groups are weak mixing if they satisfy a condition on the unitary representations of the group [C]. As the referee also points out, for the limit to be the $\int_X f(t)d\mu$ μ almost everywhere all that is required is that the dynamical system (X, β, μ, T) be ergodic and have no measurable eigenfunctions with a root of unity for an eigenvalue. I thank the referee for very detailed suggestions improving the presentation of this paper.

REFERENCES

[AN] ASMAR, N.H. – NAIR, R.: *Certain averages on the a-adic numbers*, Proc. Amer. Math. Soc. **114** (1992), no. 1, 21-28.

[BKQW] BOSHERNITZAN, M. – KOLESNIK, G. – QUAS, A. – WIERDL, M.: *Ergodic averaging sequences*, J. Anal. Math. **95** (2005), 63–103.

[BW] BOUHKARI, F. – WEBER, M.: *Almost sure convergence of weighted series of contractions*, Illinois J. Math. **46** (2002), no. 1, 1–21.

[Bo1] BOURGAIN, J.: *An approach to pointwise ergodic theorems*, Geometric aspects of functional analysis (1986/87), in: Lecture Notes in Math. **1317**, Springer, Berlin, 1988. pp. 204–223,

[Bo2] BOURGAIN, J.: *On the maximal ergodic theorem for certain subsets of the integers*, Israel J. Math. **61**(1988), no. 1, 39–72.

[Bo3] BOURGAIN, J.: *Pointwise ergodic theorems for arithmetic sets. With an appendix by the author, Harry Furstenberg, Yitzhak Katznelson and Donald S. Ornstein*, Inst. Hautes Études Sci. Publ. Math. **69** (1989), 5–45.

[C] CHU, H.: *Some results on affine transformations of compact groups*, Invent. Math. **28**, (1975) 161–183.

[Na1] NAIR, R.: *On polynomials in primes and J. Bourgain's circle method approach to ergodic theorems. II*, Studia. Math. **105** (1993), no. 3, 207–233.

[Na2] NAIR, R.: *On the metrical theory of continued fractions*, Proc. Amer. Math. Soc. **120** (1994), no. 4, 1041–1046.

[Na3] NAIR, R.: *On uniformly distributed sequences of integers and Poincaré recurrence. II*, Indag. Math. (N.S.) **9** (1998), no. 3, 405–415.

[R] RHIN, G.: *Sur la répartition modulo 1 des suites $f(p)$*, Acta. Arith. **23** (1973), 217–248.

[S] SCHNEIDER, D.: *Convergence presque sûre de moyennes ergodiques perturbées*, C. R. Acad. Sci. Paris Sér. I Math. **319** (1994), no. 11, 1201–1206.

[T] THOUVENOT, J.P.: *La convergence presque sûre des moyennes ergodiques suivant certaines sous-suites d'entiers (d'après Jean Bourgain)*, Séminaire Bourbaki **1989–90**, Astrisque no. 189–190 (1990), Exp. no. 719, 133–153.

[Wa] WALTERS, P.: *An Introduction to Ergodic Theory*, Graduate Texts in Mathematics **79**, Springer-Verlag, New York-Berlin, 1982.

[Wi] WIERDL, M.: *Pointwise ergodic theorem along the prime numbers*, Israel J. Math. **64**, (1988) no. 3, 315–336 (1989).

Received April 7, 2008
Accepted October 14, 2009

Radhakrishnan Nair
Mathematical Sciences
University of Liverpool,
Liverpool L69 7ZL,
UNITED KINGDOM
E-mail: nair@liverpool.ac.uk