Uniform Distribution Theory **4** (2009), no.1, 97–116

uniform distribution theory

ON LOCALIZATION IN KRONECKER'S DIOPHANTINE THEOREM

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ABSTRACT. Using a probabilistic approach, we extend for general $\mathbb{Q}$-linearly independent sequences a result of Turán concerning the sequence $(\log p_\ell)$, p_ℓ being the ℓ-th prime. For instance let $\lambda_1, \lambda_2, \dots$ be linearly independent over $\mathbb{Q}$. We prove that there exists a constant C_0 such that for any positive integers N and ω, if $T > \big(\frac{4\omega}{C_0}\sqrt{\log \frac{N\omega}{C_0}}\big)^N/\Xi$, where

$$\Xi = \min_{\substack{u_k \text{ integers} \\ |u_k| \le 6\omega \log(N\omega/C_0) \\ |u_1\lambda_1+\ldots+u_N\lambda_N| \neq 0}} \Big| \sum_{1\le k\le N} \lambda_k u_k \Big|$$

then to any reals $d, \beta_1, \dots, \beta_N$, corresponds a real $t \in [d, d+T]$ such that $\sup_{j=1}^N \|t\lambda_j - \beta_j\| \le 1/\omega$.

Communicated by Oto Strauch

1. Introduction and main result

The well-known theorem of Kronecker on Diophantine approximation asserts that if $\lambda_1, \lambda_2, \dots, \lambda_N$ are linearly independent over $\mathbb{Q}$, then for any given real numbers $\alpha_1, \alpha_2, \dots, \alpha_N$ and any $\varepsilon > 0$, there exists a real number t such that

$$\sup_{j=1}^{N} \|t\lambda_j - \alpha_j\| \le \varepsilon, \tag{1}$$

where $\|x\|$ denotes the distance of x to $\mathbb{Z}$, i.e. $\|x\| = \min_{\nu\in\mathbb{Z}} |x-\nu|$.

A quantitative form of Kronecker's theorem was given by Bacon [2], who proved that if $\lambda_1, \lambda_2, \dots, \lambda_N$ are reals numbers satisfying for some $M \ge 1$

$$\begin{cases} u_1\lambda_1 + \ldots + u_N\lambda_N = 0 \\ |u_1| + \ldots + |u_N| \le M,\ u_k \text{ integers} \end{cases} \Longrightarrow u_1 = u_2 = \ldots = u_N = 0, \tag{2}$$

2000 Mathematics Subject Classification: 11J13, 11K60.
Keywords: Diophantine approximation, discrete random variable, characteristic function.

then for any real numbers $\alpha_1, \alpha_2, \dots, \alpha_N$, there exists a real number t such that

$$\sup_{j=1}^{N} \lVert t\lambda_j - \alpha_j \rVert \le \frac{c(N)}{M}, \qquad c(N) = \frac{1}{2}(N-1)^{3/2}\left(\frac{125}{48}\right)^{(N^3-N)/12}. \tag{3}$$

Recently Chen [3] considerably improved this result, showing that there exists a real number t such that

$$\sum_{n=1}^{N} \lVert t\lambda_n - \alpha_n \rVert^2 \le \frac{\pi^2}{16}\frac{N}{(M+1)^2}. \tag{4}$$

He also considered the case when $\lambda_1, \lambda_2, \dots \lambda_N$ and $\alpha_1, \alpha_2, \dots, \alpha_N$ are real numbers such that for some $M \ge 1$

$$\begin{cases} u_1\lambda_1 + \ldots + u_N\lambda_N \text{ is an integer} \\ |u_1| + \ldots + |u_N| \le M,\ u_k \text{ integers} \end{cases} \Longrightarrow u_1\alpha_1 + \ldots + u_N\alpha_N \text{ is an integer.} \tag{5}$$

No indication is however given on the range of t, and in [3] it was claimed that no estimate for t exists in general. We refer to [11] (see also [10]) for more information about this important facet of Kronecker's theorem. The object of this work is to provide a simple estimate for t.

Theorem 1. There exists a constant C_0 such that for any positive integers N, ω, if $\lambda_1, \lambda_2, \dots, \lambda_N$ are reals satisfying

$$\begin{cases} u_1\lambda_1 + \ldots + u_N\lambda_N = 0 \\ \max_{1\le \ell\le N} |u_\ell| \le 6\omega \log \dfrac{N\omega}{C_0},\ u_k \text{ integers} \end{cases} \Longrightarrow u_1 = u_2 = \ldots = u_N = 0, \tag{6}$$

if $T > \frac{3}{\pi\Xi}\left(\frac{2\sqrt{3}\,\omega}{C_0}\sqrt{\log\ \frac{N\omega}{C_0}}\right)^N$ where

$$\Xi = \Xi(N,\omega) := \min_{\substack{u_k \text{ integers} \\ |u_k|\le 6\omega\log(N\omega/C_0) \\ |u_1\lambda_1+\ldots+u_N\lambda_N|\ne 0}} \Big| \sum_{1\le k\le N} \lambda_k u_k \Big|, \tag{7}$$

then to any reals $d, \beta_1, \dots, \beta_N$ corresponds a real $t \in [d, d+T]$ such that

$$\sup_{j=1}^{N} \lVert t\lambda_j - \beta_j \rVert \le \frac{1}{\omega}. \tag{8}$$

When $\lambda_1, \lambda_2, \dots, \lambda_N$ are linearly independent over $\mathbb{Q}$, condition (6) is trivially satisfied, and so the theorem applies. In the case $\lambda_\ell = \log p_\ell$, p_ℓ being the ℓ-th prime, $\ell = 1, \dots, N$, Turán ([11], Lemma p.313) proved that the conclusion above is satisfied with $T = e^{17\omega N \log^2 N}$ if N is large enough, and $4 \le \omega \le N$. It is possible to estimate Ξ from below.

As $p_j \sim j \log j$, we have $\prod_{\ell=1}^{N} p_\ell \leq e^{(1+o(1))N \log N}$. Further, for $a > b \geq 1$ integers, $\log(a/b) = \log(1 + (a-b)/b) \geq 1/(2b)$. Thus for N large

$$\begin{aligned} \left|\ell_1 \log p_1 + \ldots + \ell_N \log p_N\right| &= \left|\log \prod_{n=1}^{N} p_n^{\ell_n}\right| \geq \frac{1}{2\prod_{n=1}^{N} p_n^{|\ell_n|}} \\ &\geq e^{-\max(|\ell_n|)(1+o(1))N \log N}. \end{aligned} \tag{9}$$

Let $\gamma = (2\log 2)^{-1}$. It follows from Remark 3 that

$$\Xi \geq e^{-(\gamma+o(1))N\omega(\log N\omega)\log N}. \tag{10}$$

From this and Theorem 1, we deduce the slightly better estimate:

Given any $\varepsilon > 0$, for all N large enough, say $N \geq N(\varepsilon)$, and any ω positive integer, one can take

$$T > e^{(\gamma+\varepsilon)\omega N(\log N\omega)\log N}. \tag{11}$$

By considering particular sequences of large prime numbers, it can be however shown, that estimate (10) of Ξ is far from being optimal in general. Let indeed q be some positive integer such that $\max(|\ell_n|) \leq q$. Now let Q be some large integer. Recall there exists a real $1/2 < \alpha < 1$ such that between n and $n + n^\alpha$, n is any integer, there exists a prime P. Applying this for $n = p_j^Q$, provides a prime P_j between p_j^Q and $p_j^Q(1 + p_j^{-(1-\alpha)Q})$. By arguing as for getting (9), we obtain

$$\left|\sum_{j=1}^{N} \ell_j \log P_j\right| \geq \frac{1}{2\prod_{n=1}^{N} P_n^{|\ell_n|}} \geq \frac{1}{2\prod_{n=1}^{N} p_n^{Q|\ell_n|}} \geq e^{-(1+o(1))QqN\log N}. \tag{12}$$

But $Q \log p_j < \log P_j \leq Q \log p_j + \log(1 + p_j^{-(1-\alpha)Q}) \leq Q \log p_j + p_j^{-(1-\alpha)Q}$. Thus, for each $j \leq N$

$$|\log P_j - Q \log p_j| \leq p_j^{-(1-\alpha)Q}.$$

Herefrom

$$\left|\sum_{j=1}^{N} \ell_j(\log P_j - Q\log p_j)\right| \leq q\sum_{j=1}^{N} p_j^{-(1-\alpha)Q} \leq qNp_1^{-(1-\alpha)Q}.$$

Hence

$$\left|\sum_{j=1}^{N} \ell_j \log P_j\right| \geq \frac{Q}{2} e^{-(1+o(1))qN\log N} - qNp_1^{-(1-\alpha)Q}.$$

When $Q \gg qN\log N$, this implies

$$\Big|\sum_{j=1}^{N} \ell_j \log P_j\Big| \ge \frac{Q}{3} e^{-(1+o(1))qN\log N}, \tag{13}$$

which is considerably better than (11).

The proof of Theorem 1 is inspired from Turán's proof of the aforementionned particular case. But we also introduced an important probability structure allowing us to tackle the general case. Let us make some further remarks. By Theorem 1, we can take $T = \big(\frac{4\omega}{C_0}\sqrt{\log\frac{N\omega}{C_0}}\big)^N/\Xi$. Then $\frac{\omega}{C_0}\sqrt{\log\frac{N\omega}{C_0}} = (T\Xi)^{1/N}/4$. Let $w = \omega/C_0$, $\Theta = (T\Xi)^{1/N}/4$. As $w\sqrt{\log Nw} = \Theta$, we get

$$\frac{\Theta}{\sqrt{\log N\Theta}} = \frac{w\sqrt{\log Nw}}{\sqrt{\log N(w\sqrt{\log w})}} \le w = \frac{\omega}{C_0}.$$

Since by (8) , to any reals $d, \beta_1, \dots, \beta_N$, corresponds a real $t \in [d, d+T]$ such that $\sup_{j=1}^N \big|t\lambda_j - \beta_j\big| \le 1/\omega$, we are free to choose $d = T/2$. Then

$$\sup_{j=1}^{N} \big|t\lambda_j - \beta_j\big| \le \frac{1}{\omega} \le \frac{\sqrt{\log N\Theta}}{C_0\Theta} = \frac{\sqrt{\log N(T\Xi)^{1/N}/4}}{C_0(T\Xi)^{1/N}/4} \le 4\frac{\sqrt{\log tN(\Xi)^{1/N}}}{C_0(t\Xi)^{1/N}},$$

or

$$\frac{(t\Xi)^{1/N}}{\sqrt{\log tN(\Xi)^{1/N}}} \sup_{j=1}^{N} \big|t\lambda_j - \beta_j\big| \le \frac{4}{C_0}.$$

And this holds for infinitely many t. We deduce

$$\liminf_{t\to\infty} \frac{t^{1/N}}{\sqrt{\log t}} \cdot \sup_{j=1}^{N} \big|t\lambda_j - \beta_j\big| < \infty. \tag{14}$$

Finally, applications of Theorem 1 to supremums of Dirichlet polynomials and more general polynomials are given at the end of Section 3.

2. Some probabilistic preliminaries

Let $e(x) = e^{2\pi i x}$. Let m be a positive integer. Let $(\Omega, \mathcal{A}, \mathbf{P})$ be a probability space, and let X be a discrete random variable with law defined by:

$$\mathbf{P}\{X = n\} = \begin{cases} \frac{m-|n|}{m^2} & \text{if } 0 \le |n| < m, \\ 0 & \text{if } |n| \ge m. \end{cases} \tag{15}$$

Then $\mathbf{E}\,X = 0$, $\sigma^2 := \mathbf{E}\,X^2 = (m^2-1)/6$, and the characteristic function $\varphi_X(t) = \mathbf{E}\; e(tX)$ satisfies

$$\begin{aligned}\varphi_X(t) &= \sum_{0\le|n|<m} \mathbf{P}\{X=n\}e(tn) = \frac{1}{m^2}\sum_{0\le|n|<m}(m-|n|)e(tn)\\ &= m^{-2}\cdot|A_m(e(t))|^2,\end{aligned}$$

where $A_m(z) = 1+z+\ldots+z^{m-1}$. Indeed we have

$$\begin{aligned}|A_m(z)|^2 &= \sum_{j=0}^{m-1}\sum_{\ell=0}^{m-1} z^{j-\ell} = \sum_{n=-m+1}^{m-1} z^n\#\{0\le j,\ell<m : j-\ell=n\}\\ &= \sum_{0\le|n|<m}\big(m-|n|\big)z^n.\end{aligned}$$

Remark 2. We have $\varphi_X(t) = (2\pi/m)F_m(2\pi t)$, where F_m is the Fejér kernel

$$F_m(u) = \frac{1}{2m\pi}\Big(\frac{\sin mu/2}{\sin u/2}\Big)^2 = \frac{1}{m}\sum_{k=0}^{m-1}D_k(u), \qquad D_m(u) = \frac{1}{2\pi}\sum_{|k|\le m}e^{-iku},$$

D_m being the Dirichlet kernel.

Now let $X_1,\ldots,X_k$ be independent copies of X. Put $S_k = X_1+\ldots+X_k$, and consider its characteristic function $\varphi_{S_k}(t) = \mathbf{E}\; e(tS_k)$. Basic properties of independent random variables imply

$$\varphi_{S_k}(t) = \varphi_X^k(t) = \sum_{0\le|\nu|\le(m-1)k}\mathbf{P}\{S_k=\nu\}e(t\nu) = m^{-2k}\cdot|A_m(e(t))|^{2k}. \tag{16}$$

By the local limit theorem [9] p.187

$$\sup_\nu\left|\sigma\sqrt{k}\mathbf{P}\{S_k=\nu\} - \frac{1}{\sqrt{2\pi}}e^{-\frac{\nu^2}{2\sigma^2k}}\right| = o(1) \qquad k\to\infty.$$

Thereby

$$\mathbf{P}\{S_k=\nu\} = \frac{1}{\sqrt{\pi k(m^2-1)/3}}e^{-\frac{3\nu^2}{(m^2-1)k}} + o(1)\frac{1}{\sqrt{k(m^2-1)/6}},$$

and in particular for each m, as k tends to infinity

$$\begin{aligned}\mathbf{P}\{S_k=0\} &= m^{-2k}\int_0^1|A_m(e(t))|^{2k}dt = \int_0^1\left|\frac{\sin\pi mt}{m\sin\pi t}\right|^{2k}dt\\ &= \left(\frac{3}{\pi}\right)^{1/2}\frac{1}{m\sqrt{k}}(1+o(1)).\end{aligned} \tag{17}$$

When m and k vary simultaneously, some useful estimates are also at disposal ([11]). For k large, and any positive integer m

$$\int_0^1 \left(\frac{\sin \pi m t}{\sin \pi t}\right)^{2k} dt \geq C \frac{m^{2k-1}}{\sqrt{k}},$$

where C is an absolute constant.

Indeed, with the variable change $t = u/m\pi$ and since $\cos x \leq (\frac{\sin x}{x})$ if $0 < x \leq \pi/2$, we may write for any positive integers m and k

$$\begin{aligned}
\int_0^1 \left|\frac{\sin \pi m t}{\sin \pi t}\right|^{2k} dt &= \frac{1}{m\pi}\int_0^{m\pi} \left|\frac{\sin u}{\sin u/m}\right|^{2k} du \geq \frac{1}{m\pi}\int_0^{m\pi} \left|\frac{\sin u}{(u/m)}\right|^{2k} du \\
&\geq \frac{m^{2k-1}}{\pi}\int_0^{m\pi} \left|\frac{\sin u}{u}\right|^{2k} du \geq \frac{m^{2k-1}}{\pi}\int_0^{\frac{1}{\sqrt{k}}} \left|\frac{\sin u}{u}\right|^{2k} du \\
&\geq \frac{m^{2k-1}}{\pi}\int_0^{\frac{1}{\sqrt{k}}} \cos^{2k} u du = \frac{m^{2k-1}}{\pi}\int_0^{\frac{1}{\sqrt{k}}} (1-\sin^2 u)^k du \\
&\geq \frac{m^{2k-1}}{\pi}(1-\frac{1}{k})^k \int_0^{\frac{1}{\sqrt{k}}} du \geq \frac{m^{2k-1}}{\pi\sqrt{k}} e^{-1}.
\end{aligned}$$

And so there exists constant $C_0 > 0$ such that for any positive integers m, k

$$\mathbf{P}\{S_k = 0\} \geq \frac{C_0}{m\sqrt{k}}. \tag{18}$$

One can take $C_0 = 1/e\pi$, and we notice (see (23)) that $C_0 < 1/4$. Conversely, for any $m > m_0$, and any positive k

$$\int_0^1 \left|\frac{\sin \pi m t}{\sin \pi t}\right|^{2k} dt \leq 2^{2k+1} m^{4k^2/(2k+1)} = C \cdot m^{2k-1+1/(2k+1)}.$$

Finally, as $\sin \pi t \geq (2/\pi)\pi t = 2t$, for $0 \leq t \leq 1/2$, we have

$$\varphi_{S_k}(t) = \Big(\frac{\sin \pi m t}{m \sin \pi t}\Big)^{2k} \leq \Big(\frac{1}{2m|t|}\Big)^{2k} \wedge 1. \tag{19}$$

3. Proof of Theorem 1

Let $\beta_1, \ldots, \beta_N$ be given reals. Let $Y_1, \ldots, Y_N$ be independent copies of S_k. Consider the random vector $\mathbf{Y} = (Y_1, \ldots, Y_N)$ and let $\underline{\beta} = (\beta_1, \ldots, \beta_N)$, $\underline{t} = (t\lambda_1 - \beta_1, \ldots, t\lambda_N - \beta_N)$. Put

$$\mathbf{\Upsilon}(t, \underline{\beta}) := \mathbf{E}\ e(\langle \underline{t}, \mathbf{Y}\rangle) = \mathbf{E}\ e\bigg(t\sum_{\ell=1}^N \lambda_\ell Y_\ell - \sum_{\ell=1}^N \beta_\ell Y_\ell\bigg) = \prod_{\ell=1}^N \varphi_{S_k}(t\lambda_\ell - \beta_\ell),$$

and for $j = 1, \dots, N$

$$\boldsymbol{\Upsilon}_j(t,\underline{\beta}) := \mathbf{E}\, e\Big(t \sum_{\substack{1\le \ell\le N\\ \ell\neq j}} \lambda_\ell Y_\ell - \sum_{\substack{1\le \ell\le N\\ \ell\neq j}} \beta_\ell Y_\ell\Big) = \prod_{\substack{1\le \ell\le N\\ \ell\neq j}} \varphi_{S_k}(t\lambda_\ell - \beta_\ell).$$

Let $\omega \ge 1$. Let d be another given real and let $T > 0$. Suppose that to any $t \in [d, d+T]$, corresponds an index $j = j_t \in \{1, \dots, N\}$, such that

$$\big|t\lambda_j - \beta_j\big| > 1/\omega. \tag{20}$$

We will show that this can happen only if T is not too large. By (19)

$$\varphi_{S_k}(t\lambda_j - \beta_j) \le \Big(\frac{1}{2m\big|t\lambda_j - \beta_j\big|}\Big)^{2k} \le \Big(\frac{\omega}{2m}\Big)^{2k}.$$

and so

$$\boldsymbol{\Upsilon}(t,\underline{\beta}) \le \Big(\frac{\omega}{2m}\Big)^{2k} \sum_{j=1}^{N} \chi\{j_t = j\} \boldsymbol{\Upsilon}_j(t,\underline{\beta}).$$

Integrating this inequality over $[d, d+T]$ yields

$$\begin{aligned}\int_d^{d+T} \boldsymbol{\Upsilon}(t,\underline{\beta})dt &\le \Big(\frac{\omega}{2m}\Big)^{2k} \sum_{j=1}^{N} \int_d^{d+T} \chi\{j_t = j\} \boldsymbol{\Upsilon}_j(t,\underline{\beta})dt \\ &\le \Big(\frac{\omega}{2m}\Big)^{2k} \sum_{j=1}^{N} \int_d^{d+T} \boldsymbol{\Upsilon}_j(t,\underline{\beta})dt. \end{aligned} \tag{21}$$

But

$$\begin{aligned}\boldsymbol{\Upsilon}_j(t,\underline{\beta}) &= \prod_{\substack{1\le \ell\le N\\ j\neq \ell}} \varphi_{S_k}(t\lambda_\ell - \beta_\ell) \\ &= \prod_{\substack{1\le \ell\le N\\ \ell\neq j}} \Big(\sum_{0\le|\nu|\le(m-1)k} \mathbf{P}\{S_k = \nu\} e\big((t\lambda_\ell - \beta_\ell)\nu\big)\Big) \\ &= \mathbf{P}\{S_k = 0\}^{N-1}\end{aligned}$$

$$+ \sum_{0<\sup_{\ell\neq j}|\nu_\ell|\le(m-1)k} \Big(\prod_{\substack{1\le \ell\le N\\ j\neq \ell}} \mathbf{P}\{S_k = \nu_\ell\}\Big) e\Big(-\sum_{\substack{1\le \ell\le N\\ j\neq \ell}} \beta_\ell \nu_\ell\Big).e\Big(t\sum_{\substack{1\le \ell\le N\\ j\neq \ell}} \lambda_\ell \nu_\ell\Big). \tag{22}$$

Let C_0 be the constant from (18). As $C_0 < 1/4$, it follows that $N\omega > 4C_0$. Choose

$$m = 2\omega, \qquad k = \inf\Big\{j \ge 1 : \frac{N\omega}{C_0} \le \frac{4^{2j-1}}{\sqrt{j}}\Big\}. \tag{23}$$

Then k is well defined, $k \geq 2$, and

$$\frac{N\omega}{C_0} \leq \frac{4^{2k-1}}{\sqrt{k}} \qquad \text{and} \qquad \frac{N\omega}{C_0} > \frac{4^{2k-3}}{\sqrt{k-1}}.$$

Further

$$k \leq 3 \log\big(\frac{N\omega}{C_0}\big). \tag{24}$$

Indeed, put for a while $X = N\omega/C_0$ and observe that $k \leq 2^k$ and $(7k/2) - 6 \geq k/2$ when $k \geq 2$. Then

$$X > \frac{4^{2k-3}}{\sqrt{k-1}} > \frac{4^{2k-3}}{\sqrt{k}} \geq 2^{4k-6-k/2} = 2^{(7k/2)-6} \geq 2^{k/2}.$$

Hence $k \leq (2/\log 2) \log X < 3 \log X$. Thus $mk \leq 6\omega \log \frac{N\omega}{C_0}$. By the assumption made, the argument $\sum_{1\leq \ell\leq N\,, j\neq \ell} \nu_\ell\lambda_\ell$ appearing in the last sum of (22) is non-vanishing. Therefore, if $\sup\{|\nu_\ell| : \ell \neq j\} > 0$

$$\int_d^{d+T} e\Big(t \sum_{\substack{1\leq \ell\leq N\\ j\neq \ell}} \lambda_\ell\nu_\ell\Big) dt = \frac{e(d\sum_{\substack{1\leq \ell\leq N\\ j\neq \ell}} \lambda_\ell\nu_\ell)\Big(e(T\sum_{\substack{1\leq \ell\leq N\\ j\neq \ell}} \lambda_\ell\nu_\ell) - 1\Big)}{2i\pi \sum_{\substack{1\leq \ell\leq N\\ j\neq \ell}} \lambda_\ell\nu_\ell}.$$

And so

$$\Big|\int_d^{d+T} e\Big(t \sum_{\substack{1\leq \ell\leq N\\ j\neq \ell}} \lambda_\ell\nu_\ell\Big) dt\Big| = \Big|\frac{\sin \pi T \sum_{\substack{1\leq \ell\leq N\\ j\neq \ell}} \lambda_\ell\nu_\ell}{\pi \sum_{\substack{1\leq \ell\leq N\\ j\neq \ell}} \lambda_\ell\nu_\ell}\Big|.$$

Consequently

$$\int_d^{d+T} \mathbf{\Upsilon}_j(t, \underline{\beta}) dt = T\mathbf{P}\{S_k = 0\}^{N-1} + H_j,$$

with

$$H_j = \sum_{0<\sup_{\ell\neq j}|\nu_\ell|\leq (m-1)k} \Big(\prod_{\substack{1\leq \ell\leq N\\ j\neq \ell}} \mathbf{P}\{S_k = \nu_\ell\}\Big) \int_d^{d+T} e\Big(\sum_{\substack{1\leq \ell\leq N\\ j\neq \ell}} (t\lambda_\ell - \beta_\ell)\nu_\ell\Big) dt,$$

and

$$|H_j| \leq \sum_{0<\sup_{\ell\neq j}|\nu_\ell|\leq (m-1)k} \Big(\prod_{\ell\neq j} \mathbf{P}\{Y_\ell = \nu_\ell\}\Big) \Big|\frac{\sin \pi T \sum_{\substack{1\leq \ell\leq N\\ j\neq \ell}} \lambda_\ell\nu_\ell}{\pi \sum_{\substack{1\leq \ell\leq N\\ j\neq \ell}} \lambda_\ell\nu_\ell}\Big|.$$

It follows that

$$\int_d^{d+T} \mathbf{\Upsilon}(t, \underline{\beta}) dt \leq \Big(\frac{\omega}{2m}\Big)^{2k} \Big(NT\mathbf{P}\{S_k = 0\}^{N-1} + \sum_{j=1}^{N} |H_j|\Big). \tag{25}$$

Similarly

$$\int_d^{d+T} \Upsilon(t,\underline{\beta})dt = T\mathbf{P}\{S_k=0\}^N + H,$$

and

$$|H| \le \sum_{0<\sup_\ell|\nu_\ell|\le(m-1)k} \Big(\prod_\ell \mathbf{P}\{Y_\ell=\nu_\ell\}\Big)\bigg|\frac{\sin \pi T\sum_{1\le\ell\le N}\lambda_\ell\nu_\ell}{\pi\sum_{1\le\ell\le N}\lambda_\ell\nu_\ell}\bigg|.$$

So that

$$\begin{aligned} &T\mathbf{P}\{S_k=0\}^N - |H| \\ &\le \Big(\frac{\omega}{2m}\Big)^{2k}\Big(\frac{N}{\mathbf{P}\{S_k=0\}}\Big)T\mathbf{P}\{S_k=0\}^N + \Big(\frac{\omega}{2m}\Big)^{2k}\sum_{j=1}^N |H_j|. \end{aligned} \tag{26}$$

By the choice made in (23) of m and k, we have

$$\Big(\frac{\omega}{2m}\Big)^{2k} N = \frac{N}{4^{2k}} = \frac{N\omega}{2m4^{2k-1}} \le \frac{C_0}{2m\sqrt{k}} \le \frac12\mathbf{P}\{S_k=0\}. \tag{27}$$

We get from (3.7) and (3.8)

$$T\mathbf{P}\{S_k=0\}^N \le 2\bigg(|H| + \Big(\frac{\omega}{2m}\Big)^{2k}\sum_{j=1}^N|H_j|\bigg) \le 2\bigg(|H| + \frac{\mathbf{P}\{S_k=0\}}{2N}\sum_{j=1}^N|H_j|\bigg).$$

Hence

$$T\mathbf{P}\{S_k=0\}^N \le 3\max\big(|H|, \max_{j=1}^N |H_j|\big). \tag{28}$$

We shall now bound $|H_j|$ and $|H|$. We begin with $|H|$ and put $\mathcal{Z}_N = \sum_{1\le\ell\le N}\lambda_\ell Y_\ell$. We have

$$|H| \le \mathbf{E}\,\Big|\frac{\sin\pi T\mathcal{Z}_N}{\pi\mathcal{Z}_N}\Big|\cdot\chi\{\mathcal{Z}_N\neq 0\}. \tag{29}$$

Indeed

$$\begin{aligned} |H| \ &\le \ \sum_{0<\sup_\ell|\nu_\ell|\le(m-1)k} \Big(\prod_\ell \mathbf{P}\{Y_\ell=\nu_\ell\}\Big)\bigg|\frac{\sin \pi T\sum_{1\le\ell\le N}\lambda_\ell\nu_\ell}{\pi\sum_{1\le\ell\le N}\lambda_\ell\nu_\ell}\bigg| \\ &= \ \mathbf{E}\,\Big|\frac{\sin\pi T\mathcal{Z}_N}{\pi\mathcal{Z}_N}\Big|\cdot\chi\{\mathcal{Z}_N\neq 0\}. \end{aligned}$$

And we have the trivial bound

$$\mathbf{E}\,\Big|\frac{\sin\pi T\mathcal{Z}_N}{\pi\mathcal{Z}_N}\Big|\cdot\chi\{\mathcal{Z}_N\neq 0\} \quad\le\quad \mathbf{E}\,\frac{1}{\pi|\mathcal{Z}_N|}\cdot\chi\{\mathcal{Z}_N\neq 0\}$$

$$\le \quad \frac{1}{\pi \min_{0<\sup_\ell|\nu_\ell|\le(m-1)k}\big\{\big|\sum_{1\le\ell\le N}\lambda_\ell\nu_\ell\big|\big\}}.$$

By (23), (24), $mk = 2\omega k \le 6\omega\log(N\omega/C_0)$. Notice by using assumption (6) that

$$\min\big\{\big|\sum_{1\le\ell\le N}\lambda_\ell\nu_\ell\big| : 0<\sup_\ell|\nu_\ell|\le(m-1)k\big\} = \min_{\substack{\nu_k \text{ integers}\\ |\nu_\ell|\le(m-1)k\\ |\nu_1\lambda_1+\ldots+\nu_N\lambda_N|\neq0}}\Big|\sum_{1\le\ell\le N}\lambda_\ell\nu_\ell\Big|$$

$$\ge \min_{\substack{\nu_k \text{ integers}\\ |\nu_\ell|\le 6\omega\log(N\omega/C_0)\\ |\nu_1\lambda_1+\ldots+\nu_N\lambda_N|\neq0}}\Big|\sum_{1\le\ell\le N}\lambda_\ell\nu_\ell\Big| = \Xi.$$

Thus

$$|H| \le \frac{1}{\pi\Xi}. \tag{30}$$

Similarly, letting $\mathcal{Z}_{N,j} = \sum_{\substack{1\le\ell\le N\\ \ell\neq j}}\lambda_\ell Y_\ell$, we have

$$\begin{aligned}|H_j| &\le \sum_{0<\sup_{\ell\neq j}|\nu_\ell|\le(m-1)k}\Big(\prod_\ell \mathbf{P}\{Y_\ell=\nu_\ell\}\Big)\left|\frac{\sin\pi T\sum_{\substack{1\le\ell\le N\\ \ell\neq j}}\lambda_\ell\nu_\ell}{\pi\sum_{\substack{1\le\ell\le N\\ \ell\neq j}}\lambda_\ell\nu_\ell}\right|\\ &= \mathbf{E}\left|\frac{\sin\pi T\mathcal{Z}_{N,j}}{\pi\mathcal{Z}_{N,j}}\right|\cdot\chi\{\mathcal{Z}_{N,j}\neq0\}.\end{aligned}$$

And so,

$$\begin{aligned}|H_j| &\le \mathbf{E}\left|\frac{\sin\pi T\mathcal{Z}_{N,j}}{\pi\mathcal{Z}_{N,j}}\right|\cdot\chi\{\mathcal{Z}_{N,j}\neq0\}\\ &\le \frac{1}{\pi\min_{0<\sup_{\ell\neq j}|\nu_\ell|\le(m-1)k}\big\{\big|\sum_{1\le\ell\le N}\lambda_\ell\nu_\ell\big|\big\}} \le \frac{1}{\pi\Xi}.\end{aligned} \tag{31}$$

By inserting these estimates into (28), we get

$$T\mathbf{P}\{S_k=0\}^N \le \frac{3}{\pi\Xi}. \tag{32}$$

By (18), $\mathbf{P}\{S_k=0\} \ge C_0/(m\sqrt{k}) = C_0/(2\omega\sqrt{k})$, and by reporting this into (32) and using (23), (24), we arrive to

$$T \le \frac{3}{\pi\Xi}\Big(\frac{2\omega\sqrt{k}}{C_0}\Big)^N \le \frac{3}{\pi\Xi}\Bigg(\frac{2\omega\sqrt{3\log\big(\frac{N\omega}{C_0}\big)}}{C_0}\Bigg)^N. \tag{33}$$

Consequently, if

$$T > \frac{3}{\pi\Xi}\Big(\frac{2\sqrt{3}\,\omega}{C_0}\sqrt{\log\ \frac{N\omega}{C_0}}\Big)^N,$$

then to any reals $d, \beta_1, \ldots, \beta_N$ corresponds a real $t \in [d, d+T]$ such that

$$\sup_{j=1}^{N} \|t\lambda_j - \beta_j\| \le \frac{1}{\omega}. \tag{34}$$

The proof is now complete. □

Remark 3. By (23), $k \sim (4 \log 2)^{-1} \log(\frac{N\omega}{C_0})$, as $N\omega \to \infty$. Thus

$$k \le \left(\frac{1}{4 \log 2} + o(1)\right) \log\left(\frac{N\omega}{C_0}\right), \qquad N\omega \to \infty,$$

which is better than (24). Hence also $mk \le \big((2 \log 2)^{-1} + o(1)\big)\omega \log(N\omega/C_0)$. Let $\gamma = (2 \log 2)^{-1}$. Consequently, one can take

$$\Xi = \min_{\substack{u_k \text{ integers} \\ |u_k| \le (\gamma + o(1))\omega \log (N\omega/C_0) \\ |u_1\lambda_1 + \ldots + u_N\lambda_N| \ne 0}} \Big| \sum_{1 \le k \le N} \lambda_k u_k \Big|,$$

in this case.

Remark 4. Let I be any bounded interval and let $|I|$ denote its length. Let $\beta_1, \ldots, \beta_N$ be given reals, and put

$$\delta_I = \inf_{t \in I} \sup_{1 \le j \le N} \|t\lambda_j - \beta_j\|.$$

Suppose $\delta_I > 0$, and let ω be the unique integer such that $\omega \ge 2$ and $1/\omega \le \delta_I \le 1/(\omega - 1)$. The following is a simple extrapolation of our proof, and more specifically of estimates (28) and (27), (24).

There is an absolute constant K such that for any interval I any integer $N \ge 1$

$$\begin{aligned} &\delta_I^N \left(\log \frac{N}{\delta_I}\right)^{-1/2} \\ \le\ & K \max\left(\mathbf{E} \left|\frac{\sin \pi |I| \mathcal{Z}_N}{\pi |I| \mathcal{Z}_N}\right| \chi\{\mathcal{Z}_N \ne 0\}, \max_{j=1}^{N} \mathbf{E} \left|\frac{\sin \pi |I| \mathcal{Z}_{N,j}}{\pi |I| \mathcal{Z}_{N,j}}\right| \chi\{\mathcal{Z}_{N,j} \ne 0\}\right). \end{aligned}$$

Theorem 1 has interesting consequences for Dirichlet polynomials and more general polynomials. We shall investigate them. Let $\alpha_1, \ldots, \alpha_L$ be given reals and consider the Dirichlet polynomials $D_L(t) = \sum_{n=1}^{L} \alpha_n n^{it}$. Let $\pi(x) = \#\{p \,\text{prime} \le x\}$ be the prime number function. Choose $N = \pi(L)$. Using the prime factor decomposition, $n = p_1^{a_1} \ldots p_N^{a_N}$, $a_j(n) \ge 0$, $1 \le j \le N$, $1 \le n \le L$, we get

$$D_L(t) = \sum_{n=1}^{L} \alpha_n e^{it \sum_{j=1}^{N} a_j(n) \log p_j}. \tag{35}$$

Let $\vartheta_1,\ldots,\vartheta_N\in[0,1[$. Let $\Omega(n)=\sum_{j=1}^N a_j(n)$ denote the prime divisor function. As

$$\begin{aligned}
&\left|e^{2i\pi t\sum_{j=1}^N a_j(n)\log p_j}-e^{2i\pi\sum_{j=1}^N a_j(n)\vartheta_j}\right|\\
=\ &\left|e^{2i\pi\sum_{j=1}^N a_j(n)[(t\log p_j-\nu_j-\vartheta_j)+\nu_j+\vartheta_j]}-e^{2i\pi\sum_{j=1}^N a_j(n)\vartheta_j}\right|\\
=\ &\left|e^{2i\pi\sum_{j=1}^N a_j(n)(t\log p_j-\nu_j-\vartheta_j)}-1\right|\\
\le\ &2\pi\sum_{j=1}^N a_j(n)\big|t\log p_j-\nu_j-\vartheta_j\big|,
\end{aligned}$$

by taking the infimum over all ν_j, we get

$$\begin{aligned}
\left|e^{2i\pi t\sum_{j=1}^N a_j(n)\log p_j}-e^{2i\pi\sum_{j=1}^N a_j(n)\vartheta_j}\right| &\le 2\pi\sum_{j=1}^N a_j(n)\|t\log p_j-\vartheta_j\|\\
&\le 2\pi\big(\sup_{j=1}^N\|t\log p_j-\vartheta_j\|\big)\Omega(n).
\end{aligned}$$

Herefrom

$$\left|\sum_{n=1}^L\alpha_n n^{2i\pi t}-\sum_{n=1}^L\alpha_n e^{2i\pi\sum_{j=1}^N a_j(n)\vartheta_j}\right|\le 2\pi\big(\sup_{j=1}^N\|t\log p_j-\vartheta_j\|\big)\sum_{n=1}^L|\alpha_n|\Omega(n).$$

Let ω be some positive integer. By the comments made after Theorem 1 concerning Túran's result, if $T>T(N,\omega):=e^{2\omega N\log(N\omega/C_0)\log N}$, then for any real d, any reals $\vartheta_1,\ldots,\vartheta_N$, there exists $\tau\in[d,d+T]$ such that

$$\sup_{j=1}^N\|\tau\log p_j-\vartheta_j\|\le 1/\omega.$$

Thus

$$\left|\sum_{n=1}^L\alpha_n n^{2i\pi\tau}-\sum_{n=1}^L\alpha_n e^{2i\pi\sum_{j=1}^N a_j(n)\vartheta_j}\right|\le\frac{2\pi}{\omega}\cdot\sum_{n=1}^L|\alpha_n|\Omega(n).$$

Let $\mathbf{T}=\mathbb{R}/\mathbb{Z}$ be the circle and put for $(\vartheta_1,\ldots,\vartheta_N)\in\mathbf{T}^N$,

$$Q(\vartheta_1,\ldots,\vartheta_N)=\sum_{n=1}^L\alpha_n e^{2i\pi\sum_{j=1}^N a_j(n)\vartheta_j}.$$

Consequently, given any $(\vartheta_1,\ldots,\vartheta_N)\in\mathbf{T}^N$, $Q(\vartheta_1,\ldots,\vartheta_N)$ is well approached by $D_L(2\pi\tau)$, for some $\tau\in[d,d+T]$ with an error term precised by the above estimate. Now by (35),

$$D_L(2\pi\tau)=\sum_{n=1}^L\alpha_n e^{2i\pi\tau\sum_{j=1}^N a_j(n)\log p_j}=Q(\|\tau\log p_1\|,\ldots,\|\tau\log p_N\|).$$

Thereby

$$\begin{aligned} 0 &\le \sup_{(\vartheta_1,\ldots,\vartheta_N)\in\mathbf{T}^N}\Big|\sum_{n=1}^L \alpha_n e^{2i\pi\sum_{j=1}^N a_j(n)\vartheta_j}\Big| - \sup_{d\le\tau\le d+T}\big|D_L(2\pi\tau)\big| \\ &\le \frac{2\pi}{\omega}\cdot\sum_{n=1}^L |\alpha_n|\Omega(n). \end{aligned} \tag{36}$$

Letting d and T tend to infinity, next ω tend to infinity yields (Bohr's reduction argument)

$$\sup_{t\in\mathbb{R}}\big|D_L(t)\big| = \sup_{(\vartheta_1,\ldots,\vartheta_N)\in\mathbf{T}^N}\Big|\sum_{n=1}^L \alpha_n e^{2i\pi\sum_{j=1}^N a_j(n)\vartheta_j}\Big|.$$

Thus (36) means that

$$0 \le \sup_{t\in\mathbb{R}}|D_L(t)| - \sup_{2\pi d\le t\le 2\pi(d+T)}|D_L(t)| \le \frac{2\pi}{\omega}\cdot\sum_{n=1}^L|\alpha_n|\Omega(n), \tag{37}$$

Therefore the supremum of the Dirichlet polynomials D_L over large intervals (of length greater than $T(N,\omega)$) is comparable to the supremum over the real line. And the error made is controlled by the degree of accuracy existing for the Kronecker theorem within this interval. Further estimate (37) is *uniform* over d.

It would be interesting to know below which size of the interval this property breaks down. Notice by the Dirichlet Theorem, that for any reals $\varphi_1,\ldots,\varphi_N$ we may choose $t\le\omega^N$ such that

$$\sup_{j=1}^N \|t\varphi_j\| \le \frac{1}{\omega}. \tag{38}$$

(corresponding to the particular case $\beta_1=\ldots=\beta_N=d=0$ in Theorem 1). Further this is nearly optimal, see Erdös and Rényi's article [6] for a discussion and for some related results and the references therein, notably Hajós paper. Therefore this size cannot be smaller than ω^N.

More generally, let A be some positive real and let $\lambda_1,\lambda_2,\ldots,\lambda_N$ be reals satisfying

$$\begin{cases} u_1\lambda_1+\ldots+u_N\lambda_N=0 \\ \max_{1\le j\le N}|u_j|\le 2A, u_j\in\mathbb{Z}\end{cases} \implies u_1=u_2=\ldots=u_N=0. \tag{39}$$

Let $a_j:\mathbb{Z}\to\mathbb{Z}$, $1\le j\le N$ be arbitrary mappings, and put

$$\mathcal{N}_A=\Big\{b(n)=\sum_{j=1}^N a_j(n)\lambda_j : \max_{1\le j\le N}|a_j(n)|\le A, n\in\mathbb{Z}\Big\}, \qquad B(n)=\sum_{j=1}^N a_j(n).$$

Because of assumption (39), to any $b \in \mathcal{N}_A$ corresponds a unique n such that $b = b(n)$. Given N reals $\alpha_1, \ldots, \alpha_N$, consider the polynomials $\mathcal{D}_A(t) = \sum_{n\in\mathcal{N}_A} \alpha_n e^{itb(n)}$. Let $\vartheta_1, \ldots, \vartheta_N \in [0, 1[$. Similarly

$$\begin{aligned}\left|e^{2i\pi t\sum_{j=1}^N a_j(n)\lambda_j} - e^{2i\pi\sum_{j=1}^N a_j(n)\vartheta_j}\right| &\le 2\pi \sum_{j=1}^N a_j(n)\|t\lambda_j - \vartheta_j\| \\ &\le 2\pi\big(\sup_{j=1}^N \|t\lambda_j - \vartheta_j\|\big)B(n).\end{aligned}$$

And

$$\Big|\sum_{n\in\mathcal{N}_A} \alpha_n n^{2i\pi t} - \sum_{n\in\mathcal{N}_A} \alpha_n e^{2i\pi\sum_{j=1}^N a_j(n)\vartheta_j}\Big| \le 2\pi\big(\sup_{j=1}^N \|t\lambda_j - \vartheta_j\|\big). \sum_{n\in\mathcal{N}_A} |\alpha_n| B(n).$$

Let ω be a positive integer such that $A < \omega \log \frac{N\omega}{C_0}$. By Theorem 1, if

$$T > \frac{3}{\pi\Xi(N,\omega)}\bigg(\frac{2\sqrt{3}\,\omega}{C_0}\sqrt{\log\ \frac{N\omega}{C_0}}\bigg)^N,$$

where

$$\Xi(N,\omega) = \min_{\substack{u_k \text{ integers} \\ |u_k|\le\omega\log(N\omega/C_0) \\ |u_1\lambda_1+\ldots+u_N\lambda_N|\neq 0}} \Big|\sum_{1\le k\le N} \lambda_k u_k\Big|,$$

then to any reals $d, \beta_1, \ldots, \beta_N$ corresponds a real $\tau \in [d, d+T]$ such that $\sup_{j=1}^N \|\tau\lambda_j - \beta_j\| \le 1/\omega$. Consequently, by similar considerations

$$\begin{aligned}0 &\le \sup_{(\vartheta_1,\ldots,\vartheta_N)\in\mathbf{T}^N} \Big|\sum_{n\in\mathcal{N}_A} \alpha_n e^{2i\pi\sum_{j=1}^N a_j(n)\vartheta_j}\Big| - \sup_{d\le\tau\le d+T} \big|\mathcal{D}_A(2\pi\tau)\big| \\ &\le \frac{2\pi}{\omega}\cdot\sum_{n\in\mathcal{N}_A} |\alpha_n| B(n). \qquad (40)\end{aligned}$$

Since, by letting d, T, next ω tend to infinity

$$\sup_{t\in\mathbb{R}} \big|\mathcal{D}_A(t)\big| = \sup_{(\vartheta_1,\ldots,\vartheta_N)\in\mathbf{T}^N} \Big|\sum_{n\in\mathcal{N}_A} \alpha_n e^{2i\pi\sum_{j=1}^N a_j(n)\vartheta_j}\Big|,$$

the same comments concerning the supremums of the polynomials $\mathcal{D}_A$ over large intervals are in order.

4. Concludings remarks

We conclude this work by making several remarks related to the proof above and some key expressions having appeared in it, as well as to some related questions.

1. The central point of the proof is inequality (28):

$$T\mathbf{P}\{S_k = 0\}^N \le 3\max\big(|H|, \max_{j=1}^{N} |H_j|\big).$$

To get it, we had to adjust parameters m and k so that the factor

$$\big(\omega/2m\big)^{2k}(N/\mathbf{P}\{S_k = 0\})$$

of $T\mathbf{P}\{S_k = 0\}^N$ in (26), can be made less than 1/2. This operation seems inherent to the proof, thereby making the choice of m and k made in (23) unavoidable. Next $|H|$ and $|H_J|$ are controlled in exactly the same manner. For H for instance, in (29) we obtained the interesting bound

$$|H| \le \mathbf{E}\,\Big|\frac{\sin \pi T\mathcal{Z}_N}{\pi\mathcal{Z}_N}\Big| \cdot \chi\{\mathcal{Z}_N \ne 0\},$$

and next continued with the rather brutal estimate

$$\begin{aligned}\mathbf{E}\,\Big|\frac{\sin \pi T\mathcal{Z}_N}{\pi\mathcal{Z}_N}\Big| \cdot \chi\{\mathcal{Z}_N \ne 0\} \quad &\le \quad \mathbf{E}\,\frac{1}{\pi|\mathcal{Z}_N|} \cdot \chi\{\mathcal{Z}_N \ne 0\} \\ &\le \quad \frac{1}{\pi \min_{0<\sup_\ell |\nu_\ell| \le (m-1)k} \big|\sum_{1\le \ell\le N} \lambda_\ell \nu_\ell\big|},\end{aligned}$$

leading to (30). At this stage, the question naturally arises whether this bound is really the best possible, in other words how to compute

$$\mathbf{E}\,\Big|\frac{\sin \pi T\mathcal{Z}_N}{\pi\mathcal{Z}_N}\Big|\chi\{\mathcal{Z}_N \ne 0\}. \tag{41}$$

We believe that this is an important question. When in place of $\mathcal{Z}_N$, we have a random variable U with density distribution G, it is possible to evaluate $\mathbf{E}\,\big|\frac{\sin \pi TU}{\pi U}\big|$, by using the formula (see for instance [7] p.430) for any real $0 < r < 2$

$$|x|^r = \frac{1}{2K(r)}\int_{-\infty}^{\infty} \frac{1-\cos xt}{|t|^{r+1}}\,dt = \frac{1}{K(r)}\int_{-\infty}^{\infty} \frac{\sin^2(\frac{xt}{2})}{|t|^{r+1}}\,dt \tag{42}$$

where x is real and

$$K(r) = \frac{\Gamma(2-r)}{r(1-r)}\sin\Big[(1-r)\frac{\pi}{2}\Big]. \tag{43}$$

Choose $1 < r < 2$. By writing that $|t| = |t|^{(\frac{r+1}{2})} \cdot |t|^{1-(\frac{r+1}{2})}$, next using the Cauchy-Schwarz inequality, we get by the aforementionned formula

$$\begin{aligned}
\mathbf{E}\left|\frac{\sin xU}{U}\right| &= \int_{\mathbb{R}}\left|\frac{\sin\frac{2xt}{2})}{t}\right|G(t)dt \le \\
&\le \Big[\int_{\mathbb{R}}\frac{\sin^2\frac{2xt}{2}}{|t|^{r+1}}dt\Big]^{1/2}\cdot\Big[\int_{\mathbb{R}}\frac{G^2(t)}{|t|^{2[1-(\frac{r+1}{2})]}}dt\Big]^{1/2} \\
&= \Big(\frac{|2x|^r}{2K(r)}\Big)^{1/2}\cdot\Big[\int_{\mathbb{R}}|t|^{r-1}G^2(t)dt\Big]^{1/2}, \qquad (44)
\end{aligned}$$

since $2[1-(\frac{r+1}{2})] = 2(\frac{1-r}{2}) = 1-r$. Let V be a random variable with density distribution $A_U^{-1}\cdot G^2(t)$ where $A_U = \int_{\mathbb{R}} G^2(t)dt$. Thereby $\mathbf{E}\left|\frac{\sin xU}{U}\right| \le \big(\frac{|2x|^r}{2K(r)}A_U\cdot \mathbf{E}\,|V|^{r-1}\big)^{1/2}$. Letting $x = \pi T$, we obtain

$$\mathbf{E}\left|\frac{\sin \pi TU}{U}\right| \le C_r\Big[T^r A_U\cdot\mathbf{E}\,|V|^{r-1}\Big]^{1/2}.$$

2. The construction made in Section 2 leads to an interesting observation concerning the general study of small deviations in probability theory. The problem of evaluating

$$\mathbf{P}\{|\mathcal{Z}_N| < \varepsilon\}$$

which is clearly related to the one of estimating $\mathbf{E}\left|\frac{\sin\pi T\mathcal{Z}_N}{\pi\mathcal{Z}_N}\right|\chi\{\mathcal{Z}_N\neq 0\}$, is of an arithmetic nature. And so it seems that in general, one cannot expect to find estimates of the small deviations of sums of i.i.d. random variables (even discrete and bounded) by means on purely probabilist arguments only. The intriguing remainding question is then to know which kind of conditions on the sequence λ_n, $n \le N$, would permit to get sharp estimates of the small deviations.

3. In a very recent work, we obtained an estimate of integral (41). The proof is rather delicate and will be published elsewhere. Although the bounds we found are sharp, there are unfortunately not sharp enough to be incorporated in the proof (section 3), and to provide significant new results. But we showed that the integral in (41) appears in a rather wide context and obtained other applications.

Final note. While writing down the paper, Chen [4] (December 2007) informed us that his theorem 1 in [3] can also provide another estimate for t, but different than ours and concerning $\sum_{n=1}^N \| t\lambda_n - \alpha_n\|^2$. More precisely let $\lambda_1, \lambda_2, \ldots, \lambda_N$ be linearly independent over $\mathbb{Q}$. Given $\varepsilon > 0$,

$$M_0 = \Big[\big(\frac{N\pi^2}{8\varepsilon}\big)^{1/2}\Big], \qquad \Lambda = \min_{\substack{u_j \text{ integers}\\ |u_j|\le M_0\\ |u_1\lambda_1+\ldots+u_N\lambda_N|\neq 0}} \big|u_1\lambda_1+\ldots+u_N\lambda_N\big|.$$

Put

$$T_0(\varepsilon, (\lambda_j)) = \frac{NM_0^N}{2\pi\Lambda}.$$

Let $\alpha_1, \alpha_2, \ldots, \alpha_N$ be real numbers. Then in any interval J of length greater than $T_0(\varepsilon, (\lambda_j))$, there exists a t such that $\sum_{n=1}^N \|t\lambda_n - \alpha_n\|^2 \leq \varepsilon$. Although the two quantities $\sum_{n=1}^N \|t\lambda_n - \alpha_n\|^2$ and $\sup_{n=1}^N \|t\lambda_n - \alpha_n\|$ are not really comparable, it is however interesting to compare the bounds for T obtained in each case, call them T_C and T_W respectively. Besides, Chen's approach and our are radically different.

i) Suppose we want to bound $\sup_{n=1}^N \|t\lambda_n - \alpha_n\|$. Let $\varepsilon = 1/\omega^2$. Compare first Ξ and Λ. If $(\omega/N)^{1/2} \log(N\omega) = \mathcal{O}(1)$, then $\Xi \gg \Lambda$. Next $\log(T_C\Lambda) \sim N \log(N\omega)$ and $\log(T_W\Xi) \sim (N \log \omega + \log\log(N\omega))$. Thus $T_W \ll T_C$. Now if ω is large, namely if $(\omega/N)^{1/2} \log(N\omega) \neq \mathcal{O}(1)$, then $\Xi \ll \Lambda$, the two preceding estimates of T_C and T_W remain valid, but we do not see how to compare them.

ii) Suppose now we want to bound $\sum_{n=1}^N \|t\lambda_n - \alpha_n\|$. Let $\varepsilon = \mu^{-1}$, μ integer and $\omega = \sqrt{N\mu}$. Then $\log(T_W\Xi) \sim N \log(N\mu) \sim \log(T_C\Lambda)$. The same comments on Ξ and Λ are in order.

Acknowledgement. I thank Mikhaïl Lifshits, who carefully read the preliminary versions of the present paper and helped to improve it. I also thank Yong-Gao Chen for helpful comments and Dany Berend for a useful remark.

Appendix. [1] The problem of distribution of $(t\lambda_1, \ldots, t\lambda_N)$, $t \in [0, \infty)$, can also be solved modulo 1 by the following method:

Let H_0 be a constant characterizing independence of coordinates of the vector $(\lambda_1, \ldots, \lambda_N)$ such that if $\overline{h} = (h_1, \ldots, h_N) \in \mathbb{Z}^N$, $0 < ||\overline{h}||_\infty \leq H_0$, then $\sum_{l=i}^N h_i\lambda_i \neq 0$ and put as in (7)

$$\Xi = \min_{0 < ||\overline{h}||_\infty \leq H_0} |\sum_{l=i}^N h_i\lambda_i|, \text{ where } ||\overline{h}||_\infty = \max_{1 \leq i \leq N} |h_i|.$$

Define continuous discrepancy D_T as

$$D_T = \sup_{d \in [0,\infty)} \sup_{I \subset [0,1)^N} \left| \frac{1}{T} \int_d^{d+T} \chi_I(\{(t\lambda_1, \ldots, t\lambda_N)\}) dt - |I| \right|, \tag{45}$$

where I is an interval, $|I|$ is the volume of I, χ_I is the characteristic function of I and $\{x\}$ is the fractional part of x.

[1] This was sent to the author by Oto Strauch.

By definition (45), if $D_T < |I|$, then for every d there exists $t \in [d, d+T]$ such that $(t\lambda_1, \dots, t\lambda_N) \bmod 1 \in I$ because in an opposite case the right-hand side of (45) contains $|0 - |I||$ and then $|I| \le D_T$.

An upper bound of D_T is given by a continuous variant of Erdős-Turán-Koksma inequality

$$D_T \le \sup_{d\in[0,\infty)} \left(\frac{3}{2}\right)^N \left(\frac{2}{H+1} + \sum_{0<||\overline{h}||_\infty \le H} \frac{1}{r(\overline{h})} \left| \frac{1}{T} \int_d^{d+T} e(t \sum_{i=1}^N h_i\lambda_i) dt \right| \right), \tag{46}$$

where $r(\overline{h}) = \prod_{i=1}^N \max(1, |h_i|)$ and $H > 0$ is an arbitrary integer. Note that the discrepancy D_T in [5, p. 278, Def. 2.74] and also the continuous Erdős-Turán-Koksma inequality in [5, p. 279, Th.2.77] is formed for fixed $d = 0$, but for arbitrary d this inequality is clear, too.

If we put $H = H_0$, then as in p. 104

$$\left| \int_d^{d+T} e\left(t \sum_{i=1}^N h_i\lambda_i\right) dt \right| \le \frac{1}{\pi\Xi}. \tag{47}$$

Furthermore we have

$$\sum_{0<||\overline{h}||_\infty \le H_0} \frac{1}{r(\overline{h})} = \tag{48}$$

$$= \left(\frac{1}{|-H_0|} + \frac{1}{|-(H_0-1)|} + \dots + 1 + \frac{1}{1} + \frac{1}{2} + \dots + \frac{1}{H_0} \right)^N - 1$$

$$\le (3 + 2\log H_0)^N. \tag{49}$$

Insert (47) and (49) in (46), then the inequality $D_T < |I|$ follows from

$$\left(\frac{3}{2}\right)^N \left(\frac{2}{H_0+1} + \frac{1}{T\Xi\pi} (3 + 2\log H_0)^N \right) < |I|. \tag{50}$$

Solving (50) with respect to T we have

$$T > \frac{1}{\Xi\pi} \cdot \frac{(3 + 2\log H_0)^N}{|I| \left(\frac{2}{3}\right)^N - \frac{2}{H_0+1}} \tag{51}$$

but in this case we must assume

$$|I| > \left(\frac{3}{2}\right)^N \cdot \frac{2}{H_0+1}. \tag{52}$$

Thus, if the given interval $I \subset [0,1)^N$ satisfies (52) and for T holds (51), then for every d there exists $t \in [d, d+T]$ such that $(\{t\lambda_1\}, \dots, \{t\lambda_N\}) \in I$.

To compare (51) with Theorem 1 we put

$$H_0 + 1 = \frac{3}{|I| \left(\frac{2}{3}\right)^N}. \tag{53}$$

Then (51) has the form

$$T > \frac{1}{\Xi\pi}(H_0 + 1)(3 + 2\log H_0)^N. \tag{54}$$

Insert $|I| = \left(\frac{1}{\omega}\right)^N$ in (54) we find

$$T > \frac{1}{\Xi\pi} 3\left(\frac{3}{2}\right)^N \omega^N \left(3 + 2\log\left(3\left(\frac{3}{2}\right)^N \omega^N - 1\right)\right)^N \tag{55}$$

with $H_0 + 1 = 3\left(\frac{3}{2}\right)^N \omega^N$. We see that the result in Theorem 1

$$T > \frac{3}{\Xi\pi}\left(\frac{2\sqrt{3}\omega}{C_0}\sqrt{\log\frac{N\omega}{C_0}}\right)^N$$

with $H_0 = 6\omega \log \frac{N\omega}{C_0}$, is better than (55).

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Received October 14, 2008
Accepted May 22, 2009

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