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uniform distribution theory

MULTIPLICATIVELY INDEPENDENT INTEGERS AND DENSE MODULO 1 SETS OF SUMS

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ABSTRACT. Let $c \in \mathbb{R}$, $c > 0$, $\beta \in \mathbb{R}$ and $a_1 > a_2 > 1$ and $b_1 > b_2 > 1$ be two distinct pairs of multiplicatively independent integers. If $b_1 > a_1$ and $a_2 > b_2$ or $b_1 < a_1$ and $a_2 < b_2$ then we prove that for every ξ_1, ξ_2, with at least one ξ_i irrational, there exists $q \in \mathbb{N}$ such that the set of sums

$$\{a_1^m a_2^n q\xi_1 + b_1^m b_2^n q\xi_2 + c^{m+n}\beta : m, n \in \mathbb{N}\},$$

is dense modulo 1 for all reals β.

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1. Introduction and main results

In 1967 Furstenberg proved the following

Theorem 1.1 (Furstenberg, [2]). *If $p, q > 1$ are multiplicatively independent integers (i.e., they are not both integer powers of the same integer) then for every irrational ξ the set*

$$\{p^m q^n \xi : m, n \in \mathbb{N}\} \tag{1.2}$$

is dense modulo 1.

An interesting generalization of Furstenberg's theorem can be found in [3].

Theorem 1.3 ([3, Theorem 1.2]). *Suppose that the pairs $p_i, q_i \in \mathbb{N}$, with $1 < p_i < q_i$ for $i = 1, \ldots, k$, $k \in \mathbb{N}$, $(p_i, q_i) \neq (p_j, q_j)$ for $i \neq j$, and $p_1 \leq p_2 \leq$*

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$\ldots \leq p_k$, are multiplicatively independent. Then for distinct $\xi_1, \ldots, \xi_k \in [0,1]$ with at least one $\xi_i \notin \mathbb{Q}$ the set

$$\{\sum_{i=1}^{k} p_i^m q_i^n \xi_i : m, n \in \mathbb{N}\}$$

is dense modulo 1.

For some generalizations of Theorem 1.3 to the case of algebraic numbers see [6, 7].

Another variation on the theme of Furstenberg's theorem is the following result of Berend.

Theorem 1.4 ([1, Proposition III.1])**.** *Let p, q, and c be non-zero integers with p and q multiplicatively independent, ξ an irrational and β arbitrary. Then the set*

$$\{p^m q^n \xi + c^{m+n}\beta : m, n \in \mathbb{N}\} \tag{1.5}$$

is dense modulo 1.

Remark 1.6. Actually, c in Theorem 1.4 may be an arbitrary real number, not necessarily integer. The proof given in [1] works in this case as well.

The aim of this note is to prove the following result, which naturally fits in with the context of Theorem 1.3 and Theorem 1.4.

Theorem 1.7. *Let $a_1 > a_2 > 1$ and $b_1 > b_2 > 1$ be two pairs of multiplicatively independent integers, and let c be a positive real number. Suppose that*

$$a_1 < b_1 \text{ and } a_2 > b_2. \tag{1.8}$$

Then, for any real numbers ξ_1, ξ_2 with at least one ξ_i irrational, there exists $q \in \mathbb{N}$ such that for any real number β, the set

$$\{a_1^m a_2^n q\xi_1 + b_1^m b_2^n q\xi_2 + c^{m+n}\beta : m, n \in \mathbb{N}\} \tag{1.9}$$

is dense modulo 1.

Remark. It is clear that we can consider sets of the form (1.9) with not necessarily all of a_i, b_i and c positive. In fact, using squares of the original parameters we have a subset of (1.9).

Remark. We believe that under the assumptions of Theorem 1.7 the set (1.9) is dense modulo 1 with $q = 1$. In order to prove such a statement a much better understanding of the closed subsets of $\mathbb{T}^2$, invariant under the action of the semigroup S defined in the proof of Proposition 2.1, is required. Specifically, if we knew that under the assumptions of Theorem 1.7 the closure of the orbit

$S\xi$, $\xi = (\xi_1, \xi_2)^t$ contains $(0,0)$ then we would have that for every ξ, with at least one ξ_i irrational, $q = 1$. This seems to be a difficult problem as very little is known about reducible actions of linear semigroups on tori. In particular, description of the closed invariant sets and orbit closures is not known even in the "simplest" case of the semigroup generated by the following two matrices $\left(\begin{smallmatrix} 2 & 0 \\ 0 & 2 \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} 3 & 0 \\ 0 & 3 \end{smallmatrix}\right)$ and only some partial results are available in literature ([3, 5]).

2. Proof of Theorem 1.7

The proof of Theorem 1.7 will follow from the following result.

Proposition 2.1. *Let $a_1 > a_2 > 1$ and $b_1 > b_2 > 1$ be two pairs of multiplicatively independent integers such that*

$$a_1 < b_1 \text{ and } a_2 > b_2.$$

Then for any real numbers ξ_1, ξ_2 with at least one ξ_i irrational, and for every $\varepsilon > 0$ there exist $q \in \mathbb{N}$ and $M_0 \in \mathbb{N}$ such that for every $M \geq M_0$ the set:

$$qA_M := \{a_1^m a_2^{M-m} q\xi_1 + b_1^m b_2^{M-m} q\xi_2 : 0 \leq m \leq M\}$$

is ε-dense modulo 1.

Remark. Equivalently, Proposition 2.1 says that the set $A_M := \frac{1}{q}(qA_M)$ is ε-dense modulo $1/q$, or, in other words, that A_M is ε-dense in $\frac{1}{q}\mathbb{T}$.

Before we give the proof of Proposition 2.1 we present the proof of Theorem 1.7.

Proof of Theorem 1.7. We may assume that both ξ_1 and ξ_2 are non-zero; if one of them is zero then Theorem 1.7 follows from Theorem 1.4 (and Remark 1.6). The result follows immediately from Proposition 2.1. In fact, for a fixed $M = m + n$ the expression in (1.9) is a translate by $c^M\beta$ of the ε-dense modulo 1 set qA_M. □

In order to prove Proposition 2.1 we generalize the original proof from [1] using certain results from [3] on the diagonal semigroups acting on the 2-dimensional torus, and some ideas from [6] adapted to our setting.

Proof of Proposition 2.1. Consider a semigroup $S = \langle s_1, s_2 \rangle \subset \mathrm{End}(\mathbb{T}^2)$ of toral endomorphisms generated by the following two matrices:

$$s_1 = \begin{pmatrix} a_1 & 0 \\ 0 & b_1 \end{pmatrix}, \quad s_2 = \begin{pmatrix} a_2 & 0 \\ 0 & b_2 \end{pmatrix}.$$

Let $\xi = (\xi_1, \xi_2) + \mathbb{Z}^2 \in \mathbb{T}^2$ and denote by F the closure of the the orbit of the point ξ under the action of the semigroup S :

$$F = \overline{S\xi}.$$

Clearly, F is closed and S-invariant subset of $\mathbb{T}^2$. We show that F is infinite. By the assumption one of ξ_i's is irrational. Suppose that ξ_1 (ξ_2, resp.) is irrational. Then, by Theorem 1.1, for every $x \in \mathbb{T}$ ($y \in \mathbb{T}$, resp.) there are subsequences n_k and $m_k \subset \mathbb{N}$ such that $a_1^{n_k} a_2^{m_k} \xi_1 \to x$ ($b_1^{n_k} b_2^{m_k} \xi_2 \to y$, resp.) as $k \to \infty$. Since $\mathbb{T}$ is compact it follows that there exist $y \in \mathbb{T}$ ($x \in \mathbb{T}$, resp.) such that $(x, y) \in F$. Hence F is infinite.

By [3, Corollary 3.2], it follows that F contains a non-isolated rational point $r = p/q$, $q \in \mathbb{N}$, $p \in \mathbb{Z}^2$. Define

$$F' = qF.$$

Then $(0, 0) \in F'$, and we have the following.

Lemma 2.2. *The set F' contains at least one of the following sets*

$$\begin{aligned} T_1 =& \mathbb{T} \times \{0\}, \\ T_2 =& \{0\} \times \mathbb{T}. \end{aligned} \tag{2.3}$$

Remark. Lemma 2.2 follows from [3, Lemma 3.4] since (1.8) implies that the condition (3) of [3, Lemma 3.4] can not hold. However, we include the detailed proof here because (1) we want the paper to be self-contained as much as possible (2) the formulation in [3] is not exactly what we need here, and finally (3) the proof of [3, Lemma 3.4] given in [3] is in some parts very sketchy and, moreover, seems to be more complicated than the proof we present.

Remark. It will be clear from the proof of Lemma 2.2 that in fact a more general result holds true. Namely, it is enough to assume that F' is an arbitrary closed, infinite, s_1- and s_2-invariant subset of $\mathbb{T}^2$ containing 0 (not necessarily the closure of the orbit of some point which contains 0).

Proof of Lemma 2.2. We follow [3] and [6]. Clearly, F' is a non-empty, s_1- and s_2-invariant closed subset of $\mathbb{T}^2$, and contains $(0, 0)$. By [3, Lemma 3.1] the intersection of F with the "axes" T_1 and T_2 is either empty, contains finitely many rational points, or equals T_i, $i = 1, 2$. Assume that for $i = 1, 2$,

$$F \cap T_i \neq T_i \text{ and } 1 < \mathrm{Card}(F \cap T_i) < +\infty. \tag{2.4}$$

We will show that this assumption leads to a contradiction.

Since $(0, 0) \in F'$, there exists a sequence $\{(x_n, y_n) + \mathbb{Z}^2\} \subset F'$ tending to $(0, 0)$ in $\mathbb{T}^2$. By (2.4) we may assume that $x_n + \mathbb{Z} \neq 0$ and $y_n + \mathbb{Z} \neq 0$. Without loss of generality, choosing an appropriate representative from $x_n + \mathbb{Z}$ ($y_n + \mathbb{Z}$, resp.), we can assume that $0 \neq x_n \to 0$, and $0 \neq y_n \to 0$. Choosing an appropriate subsequence, we can assume that

$$\lim_{k\to\infty} \frac{|y_n|}{|x_n|} = c \in [0, +\infty]. \tag{2.5}$$

Consider the case when $c \neq 0$ or the limit in (2.5) is infinite. By the assumption (1.8) there are $k, l \in \mathbb{N}$ such that

$$b_1^k b_2^l > a_1^k a_2^l.$$

Hence, for every $j \in \mathbb{N}$,

$$\frac{|(b_1^k b_2^l)^j y_n|}{|(a_1^k a_2^l)^j x_n|} \geq \rho^j \frac{|y_n|}{|x_n|} \text{ with } \rho > 1. \tag{2.6}$$

Lemma 2.7. *For every integer $r \geq 1$ we can choose a subsequence $v_{n_i} = (x_{n_i}, y_{n_i})$, and a subsequence $\{j_{n_i}\} \subset \mathbb{N}$ tending to infinity, such that*

$$\lim_{i\to\infty} (b_1^k b_2^l)^{j_{n_i}} y_{n_i} = y \neq 0 \text{ and } (1/b_1^k b_2^l)^{r+1} \leq |y| \leq (1/b_1^k b_2^l)^r. \tag{2.8}$$

P r o o f. In fact, without loss of generality we can assume that $|y_n| \leq (1/b_1^k b_2^l)^r$. Thus for every n, there exists the smallest natural number j_n such that

$$(1/b_1^k b_2^l)^{r+1} \leq |(b_1^k b_2^l)^{j_n} y_n| \leq (1/b_1^k b_2^l)^r.$$

Hence, by compactness, we can chose a subsequence $\{n_i\} \subset \mathbb{N}$ such that (2.8) holds. ☐

Let $r \geq 1$ be fixed. Since the limit c in (2.5) is non-zero, it follows from (2.6) and Lemma 2.7 that $(a_1^k a_2^l)^{j_{n_i}} x_{n_i} \to 0$. Moreover, we have the sequence

$$\{(s_1^k s_2^l)^{j_{n_i}} v_{n_i}\}_i \subset F'$$

such that $(s_1^k s_2^l)^{j_{n_i}} v_{n_i} \to (0, y_r) \in \mathbb{T}^2$, with $(1/b_1^k b_2^l)^{r+1} \leq |y_r| \leq (1/b_1^k b_2^l)^r$. Taking $r \to \infty$ we get a sequence of different points $(0,0) \neq (0, y_r) \to (0,0)$, and laying on T_2. This contradicts (2.4).

Now, let $c = 0$ in (2.5). By the assumption (1.8) there are $k', l' \in \mathbb{N}$ such that

$$b_1^{k'} b_2^{l'} < a_1^{k'} a_2^{l'},$$

and consequently, for every $j \in \mathbb{N}$,

$$\frac{|(a_1^{k'} a_2^{l'})^j x_n|}{|(b_1^{k'} b_2^{l'})^j y_n|} \geq \rho^j \frac{|x_n|}{|y_n|} \text{ with } \rho > 1.$$

Now we proceed analogously to the previous case changing the role of x_n with y_n and get, for every $r \geq 1$ a sequence $\{(s_1^{k'} s_2^{l'})^{j_{n_i}} v_{n_i}\}_i \subset F'$ such that, as $i \to \infty$, $(s_1^k s_2^l)^{j_{n_i}} v_{n_i} \to (x_r, 0) \in \mathbb{T}^2$, with $(0,0) \neq (x_r, 0) \to (0,0)$ and

$$(1/b_1^{k'} b_2^{l'})^{r+1} \leq |x_r| \leq (1/b_1^{k'} b_2^{l'})^{r+1}.$$

Hence we can produce infinitely many points in $F' \cap T_1$. This again contradicts (2.4). $\square$

By Lemma 2.2 we can assume that $F' = qF$ either contains

$$\mathbb{T} \times \{0\} \text{ or } \{0\} \times \mathbb{T}.$$

First we consider the case when

$$F' \supset \mathbb{T} \times \{0\}. \tag{2.9}$$

It is known that there exists a point $\omega \in [0,1]$ such that the set

$$\{(a_1/a_2)^n \omega : n \in \mathbb{N}\} \tag{2.10}$$

is dense modulo 1 (recall that $a_1 > a_2$). Actually the set defined in (2.10) is dense modulo 1 (and even uniformly distributed modulo 1) for almost all real numbers ω (see [4, Corollary 4.3]). Let $\varepsilon > 0$ be fixed. By continuity, there exists a neighborhood U of ω and a natural number N such that for every $u \in U$ the set

$$\{(a_1/a_2)^n u : 0 \leq n \leq N\}$$

is ε-dense modulo 1. Define $U_0 = U/a_2^N$ and choose $0 < \delta < \frac{\varepsilon}{b_1^N b_2^N}$. By (2.9) we can take $m_0, n_0 \in \mathbb{N}$ such that

$$q s_1^{m_0} s_2^{n_0} \xi = (\theta_1, \theta_2) + (l_1, l_2), \tag{2.11}$$

where $\theta_1 \in U_0$, $\theta_2 \in [-\delta, \delta]$ and $(l_1, l_2) \in \mathbb{Z}^2$.

Now we consider the following sets

$$\begin{aligned} B_N :=& \{a_1^{m_0+j} a_2^{n_0+N-j} \xi_1 + b_1^{m_0+j} b_2^{n_0+N-j} \xi_2 : 0 \leq j \leq N\} \\ =& \{(a_1/a_2)^j a_2^N a_1^{m_0} a_2^{n_0} \xi_1 + (b_1/b_2)^j b_2^N b_1^{m_0} b_2^{n_0} \xi_2 : 0 \leq j \leq N\} \end{aligned}$$

and

$$\begin{aligned} qB_N =& \{a_1^{m_0+j} a_2^{n_0+N-j} q\xi_1 + b_1^{m_0+j} b_2^{n_0+N-j} q\xi_2 : 0 \leq j \leq N\} \\ =& \{(a_1/a_2)^j a_2^N a_1^{m_0} a_2^{n_0} q\xi_1 + (b_1/b_2)^j b_2^N b_1^{m_0} b_2^{n_0} q\xi_2 : 0 \leq j \leq N\}. \end{aligned}$$

By (2.11) the set qB_N is equal (modulo 1) to

$$\{(a_1/a_2)^j a_2^N \theta_1 + (b_1/b_2)^j b_2^N \theta_2 : 0 \leq j \leq N\}.$$

Since $a_2^N\theta_1 \in U$ and $|(b_1/b_2)^j b_2^N\theta_2| < b_1^N b_2^N\delta < \varepsilon$, we conclude that qB_N is 2ε-dense modulo 1. Taking $M = m_0 + n_0 + N$ we see that $qA_M \supset qB_N$. Hence, qA_M is 2ε-dense modulo 1 and so the result follows under the assumption (2.9).

If $F' \supset \{0\} \times \mathbb{T}$, the proof is the same. It is enough to consider b_1 and b_2 instead of a_1 and a_2. □

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REFERENCES

[1] BEREND, D.: *Actions of sets of integers on irrationals*, Acta Arith. **48** (1987), 275–306.

[2] FURSTENBERG, H.: *Disjointness in ergodic theory, minimal sets, and a problem in Diophantine approximation*, Math. Systems Theory **1** (1967), 1–49.

[3] KRA, B.: *A generalization of Furstenberg's Diophantine theorem*, Proc. Amer. Math. Soc. **127**(7) (1999), 1951–1956.

[4] KUIPERS, L.– NIEDERREITER, H.: *Uniform Distribution of Sequences*, John Wiley & Sons, New York-London-Sydney, 1974 (Reprint edition in Dover Publications, Inc. Mineola, New York, 2006).

[5] MEIRI, D. – PERES, Y.: *Bi-invariant sets and measures have integer Hausdorff dimension*, Ergodic Theory Dynam. Systems **19** (1999), no. 2, 523–534.

[6] URBAN, R.: *On density modulo* 1 *of some expressions containing algebraic integers*, Acta Arith. **127** (2007), no. 3, 217–229.

[7] URBAN, R.: *Algebraic numbers and density modulo* 1, J. Number Theory **128** (2008), no. 3, 645–661.

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