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UNIFORM DISTRIBUTION PRESERVING MAPPINGS AND VARIATIONAL PROBLEMS

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ABSTRACT. This paper studies two aspects of uniform-distribution preserving functions. We give a general criterion under which a function of this type can be expected to preserve discrepancy and apply it to study linear mappings and their effect on the isotropic discrepancy. Also, we investigate certain types of variational problems for these functions and establish some connections to uniformly distributed sequences which will then allow us to solve similar variational problems for uniformly distributed sequences by means of these functions and their properties.

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1. Introduction

Uniform-distribution preserving functions have been independently discovered by Bosch [2] and Porubský, Šalát & Strauch [17]. The theory was later extended to compact metric spaces by Tichy & Winkler [19]. Several papers dealing with similar results are given in [14].

A function $\phi : [0,1] \to [0,1]$ is said to be **uniform distribution preserving** (henceforth: u.d.p.) if for every equidistributed sequence (x_n) its image $(f(x_n))$ is equidistributed as well. Perhaps the most fundamental property of such functions (and even equivalent to the given definition, see also [17]) is that for each Riemann integrable function f and each u.d.p. function ϕ the composition $f \circ \phi$ is also Riemann integrable and

$$\int_0^1 f(\phi(x))dx = \int_0^1 f(x)dx.$$

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This equation implies that u.d.p. functions can be thought of as suitable measure-preserving rearrangements of the unit interval. We recall that the *outer Jordan measure* of a bounded set A is defined as the greatest lower bound of the areas of the coverings (consisting of finite unions of rectangles) of the set. The *inner Jordan measure* is then defined by choosing any rectangle $B \supset A$ and taking the difference between the area of B and the outer Jordan measure of $B \setminus A$. If both these values coincide, we call the set Jordan measurable and call the common value the Jordan measure of A. Henceforth, we denote the Jordan measure of a set A by $J(A)$. It is known that a bounded set A is Jordan measurable if and only if its characteristic function χ_A is Riemann integrable.

If $A \subseteq [0,1]$ is now any Jordan measurable set, then the theorem cited above implies that for any u.d.p. function ϕ the composition $\chi_A \circ \phi$ is Riemann integrable and hence

$$J(\phi^{-1}(A)) = \int_0^1 \chi_A(\phi(x))dx = \int_0^1 \chi_A(x)dx = J(A).$$

Figuratively speaking, whenever a set is nicely measurable, its size will be preserved whenever a u.d.p. function is applied - this property will be used mostly for intervals. It is noteworthy that there are a great many such functions; from the general construction principles given in [2] it follows that for any continuously differentiable function $f(x)$ satisfying $0 \leq f(0) < 1$ and $f'(x) > 1$ on $[0, f^{-1}(1)]$, there is a u.d.p. function ϕ such that f and ϕ coincide on $[0, f^{-1}(1)]$. Furthermore, the set of u.d.p. functions is closed under composition. Nonetheless, all piecewise linear u.d.p. functions have been characterized in [17].

This paper deals with two different aspects of u.d.p. functions. The first part gives a general condition under which a u.d.p. function ϕ has the property of not only mapping equidistributed sequences to equidistributed sequences but preserving some concept of discrepancy along the way. As an application we study the isotropic discrepancy under linear mappings and give a new proof of a result in the theory of box splines.

The second part gives a generalization of the aforementioned integral equation and solves the resulting variational problem. We will then show that all these results carry over naturally to a related variational problem over equidistributed sequences and that u.d.p. functions provide an efficient tool for solving and constructing solutions. Finally, we sketch some methods to deal with the general problem and solve an interesting special case.

2. The preservation of discrepancy

Our theorem will be formulated for geometric discrepancy systems.

Definition 1 (from [8])**.** Let X be a compact Hausdorff space and μ a positive regular normalized Borel measure on X. A sequence $(x_n)_{n\geq 1}$, $x_n \in X$ is called uniformly distributed with respect to μ ($\mu - u.d.$) if

$$\lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} f(x_n) = \int_X f \, d\mu$$

holds for all continuous functions $f : X \to \mathbb{R}$.

In order to distinguish between different degrees of equidistribution, one usually studies the discrepancy of a sequence. A Borel set $M \subseteq X$ is called a μ-continuity set if $\mu(\partial M) = 0$, where ∂M denotes the boundary of M.

Definition 2 (from [8])**.** A system $\mathcal{D}$ of $\mu-$continuity sets of X is called discrepancy system if

$$\lim_{N\to\infty} \sup_{M\in\mathcal{D}} \left| \frac{1}{N}\sum_{n=1}^{N} \chi_M(x_n) - \mu(M) \right| = 0$$

holds if and only if $(x_n)_{n\geq 1}$ is $\mu - u.d.$ For a discrepancy system $\mathcal{D}$ the supremum

$$D_N^{\mathcal{D}}(x_n) = \sup_{M\in\mathcal{D}} \left| \frac{1}{N}\sum_{n=1}^{N} \chi_M(x_n) - \mu(M) \right|$$

is called discrepancy of $\{x_1, x_2, \ldots, x_N\}$.

The case $X = [0,1]^s$ with $\mu = \lambda_s$ equal to the Lebesgue measure is the one most commonly studied. Two widely used discrepancy system are given by the extreme discrepancy (where $\mathcal{D}$ is given by all hyperrectangles with all sides parallel to the axes) and by the star discrepancy (where the discrepancy system is given by sets of the form $[0, y_1) \times [0, y_2) \times \cdots \times [0, y_s)$). We will now give a definition capturing the idea of elements of a discrepancy system being treated nicely when being transformed. For this, we make use of the the notation $A(I, N, x_n) = \#\{1 \leq i \leq N : x_i \in I\}$.

Definition 3. Let $(X, \mu, \mathcal{D}_1)$ and $(Y, \nu, \mathcal{D}_2)$ be compact Hausdorff spaces X, Y with positive regular normalized Borel measures μ, ν and discrepancy systems $\mathcal{D}_1$, $\mathcal{D}_2$ and let $\phi : X \to Y$ be measure-preserving w.r.t μ and ν. We call ϕ $c-$preserving if there exists a universal constant $c \in \mathbb{N}$ such that for each

$M^* \in \mathcal{D}_2$ there are $\varepsilon_1, \dots, \varepsilon_c \in \{-1, 0, 1\}$ and sets $M_1, \dots, M_c \in \mathcal{D}_1$ such that for all $N \in \mathbb{N}$ and any sequence $(\boldsymbol{z}_n)_{n \geq 0}$ in X

$$A(M^*, N, \phi(\boldsymbol{z}_n)) = \sum_{i=1}^{c} \varepsilon_i A(M_i, N, \boldsymbol{z}_n).$$

Theorem 1. *Under the assumptions of the previous definition, if $\phi : X \to Y$ is $c-$preserving, then for any set $x = \{x_1, \dots, x_N\} \subset X$,*

$$D_N^{\mathcal{D}_2}(\phi(x_n)) \leq c \cdot D_N^{\mathcal{D}_1}(x_n).$$

Proof. Fix $\boldsymbol{x} = \{x_1, \dots, x_N\} \subset X$. For any $M_* \in \mathcal{D}_2$ we choose $M_1, \dots, M_c \in \mathcal{D}_1$ and $\varepsilon_1, \dots, \varepsilon_c \in \{-1, 0, 1\}$ such that

$$A(M^*, N, \phi(\boldsymbol{x})) = \sum_{i=1}^{c} \varepsilon_i A(M_i, N, \boldsymbol{x})$$

Since f is measure-preserving, we have

$$\nu(M^*) = \sum_{i=1}^{c} \varepsilon_i \mu(M_i)$$

Combining this, we get

$$\left| \frac{1}{N} \sum_{n=1}^{N} \chi_{M^*}(x_n) - \nu(M^*) \right| = \left| \frac{1}{N} \sum_{n=1}^{c} \varepsilon_i A(M_i, N, x) - \sum_{i=1}^{c} \varepsilon_i \mu(M_i) \right|$$
$$\leq \sum_{i=1}^{c} \left| \varepsilon_i \left(\frac{A(M_i, N, x)}{N} - \mu(X_i) \right) \right| \leq D_N^{\mathcal{D}_1}(x_n) \sum_{i=1}^{c} |\varepsilon_i| \leq c \cdot D_N^{\mathcal{D}_1}(x_n).$$

□

Remark. It should be strongly emphasized that this theorem is not entirely new but rather the abstract form of a general principle, special cases of which have been used for a long time. The first three of the following four applications constitute a reformulation of known results in this framework.

I. Star Discrepancy. The simplest case of this theorem is probably to choose the function to be the identity and switch the discrepancy system. An example of this is given by the usual estimate between star discrepancy and extreme discrepancy (as it can be found in [8, Lemma 1.7]).

Corollary 1. *Let $(x_n)_{n \geq 1}$ be a sequence in $[0, 1)$, then*

$$D_N^*(x_n) \leq D_N(x_n) \leq 2D_N^*(x_n).$$

Proof. Let $(X, \mu, \mathcal{D}_1) = ([0,1), \lambda_1, \mathcal{D}^*)$, where $\mathcal{D}^*$ denotes the discrepancy system given by $[0, x)$ for $x \in (0,1)$, and $(Y, \nu, \mathcal{D}_2) = ([0,1), \lambda_1, \mathcal{D})$ where the discrepancy system $\mathcal{D}$ consists of intervals of the form $[a, b)$ with $0 \leq a < b \leq 1$ and $f = id$. $\mathcal{D}_1 \subset \mathcal{D}_2$ implies the first inequality. Furthermore,

$$A([a,b), N, x) = 1 \cdot A([0,b), N, x) + (-1) \cdot A([0,a), N, x),$$

and hence $c = 2$. $\square$

The general result $D_N^*(x_n) \leq D_N(x_n) \leq 2^s D_N^*(x_n)$ for a sequence in $[0,1]^s$ can be derived analogously.

II. Low-discrepancy sequences in the simplex. The problem of finding low-discrepancy sequences in the $s-$dimensional simplex

$$\Delta_s = \{(x_1, x_2, \ldots, x_s) \in [0,1)^s : x_1 \leq x_2 \leq \cdots \leq x_s\}$$

was studied by Cools and Pillards in [6] and [7]. Using our terminology, they study a variety of u.d.p. functions $\phi : [0,1]^s \to \Delta_s$ with emphasis on the speed of their numerical evaluation. As an example, they introduce the function $\textsc{Sort}$: $[0,1]^s \to \Delta_s$ which maps each $(x_1, \ldots, x_s) \in [0,1)^s$ to its monotonically increasing permutation. They also define a discrepancy system on Δ_s by modifying the discrepancy system used in defining the star discrepancy $\mathcal{D}_\Delta = \{[0, \boldsymbol{x}] \cap \Delta_s : \boldsymbol{x} \in [0,1]^s\}$ and show for any set of N points $P \subset \Delta_s$ that

$$D_{N \cdot s!}(\textsc{Sort}^{-1}(P)) \leq D_N^{\mathcal{D}_\Delta}(P).$$

III. Scrambling. Applying a u.d.p. transformation ϕ to a uniformly distributed (x_n) such that the resulting sequence $(\phi(x_n))$ has a smaller discrepancy can be regarded as scrambling of a sequence. An example (taken from a survey on recent developements in the field of $(t,s)-$sequences by Niederreiter [13]) for the scrambling of a sequence $\boldsymbol{x}_n$ in $[0,1)^s$ is given by choosing a random $\boldsymbol{r} \in [0,1)^s$ and considering the sequence

$$\boldsymbol{y}_n = \{\boldsymbol{x}_n + \boldsymbol{r}\}.$$

There is a long list of papers on this subject, see for example Owen [15].

IV. Isotropic discrepancy under linear mappings. We now show that a certain class of linear mappings preserves equidistribution and isotropic discrepancy. Recall that the isotropic discrepancy is given by choosing all convex sets as a discrepancy system.

Definition 4 (from [8]). Let $(x_n)_{n=1}^N$ be a finite sequence of points in the $k-$dimensional space $\mathbb{R}^k$. The isotropic discrepancy $J_N = J_N(x_1, \ldots, x_N)$ is defined by

$$J_N = \sup_{C \subseteq \mathbb{T}^k} \left| \frac{A(C, N, x_n)}{N} - \lambda_k(C) \right|,$$

where the supremum is taken over all convex subsets $C \subseteq \mathbb{T}^k$.

For clarification, it should be noted that the counting function $A(\cdot, \cdot, \cdot)$ is here applied to the point set after it has been embedded in the torus. Since the convex sets contain the (hyper)rectangles as a subset, the isotropic discrepancy will generally be larger than the extreme discrepancy and more difficult to determine. A general inequality relating these two quantities is due to Niederreiter [10, p. 95 ff.]

$$D_N(x_n) \le J_N(x_n) \le (4k\sqrt{k} + 1) D_N(x_n)^{\frac{1}{k}}.$$

Theorem 2. *Let $(\boldsymbol{x}_n)_{n \ge 0} = (x_{1,n}, x_{2,n}, \dots, x_{s,n})$ be equidistributed in $[0,1)^s$. Let $A = (a_{i,j}) \in \mathbb{Z}^{t \times s}$ be a fixed matrix. The sequence $(\boldsymbol{y}_n)_{n \ge 0}$ in $[0,1)^t$, defined via*

$$\boldsymbol{y}_n = \begin{pmatrix} y_{1,n} \\ y_{2,n} \\ \dots \\ y_{t,n} \end{pmatrix} = \begin{pmatrix} a_{1,1} & a_{1,2} & \dots & a_{1,s} \\ a_{2,1} & \dots & \dots & a_{2,s} \\ \dots & \dots & \dots & \dots \\ a_{t,1} & a_{t,2} & \dots & a_{t,s} \end{pmatrix} \begin{pmatrix} x_{1,n} \\ x_{2,n} \\ \dots \\ x_{s,n} \end{pmatrix},$$

is equidistributed modulo one if and only if the rows are linearly independent. If this is the case, then

$$J_N(\boldsymbol{y}_n) \le \left(\prod_{j=1}^{t} \sum_{k=1}^{s} |a_{j,k}| \right) J_N(\boldsymbol{x}_n).$$

P r o o f. We first use Weyl's criterion to show that a mapping of this type preserves uniform distribution. Let $\boldsymbol{h} \in \mathbb{Z}^t \setminus \{\boldsymbol{0}\}$ be arbitrary.

$$\lim_{N \to \infty} \frac{1}{N} \sum_{n=0}^{N-1} e^{2\pi i \langle \boldsymbol{y}_n, \boldsymbol{h} \rangle} = \lim_{N \to \infty} \frac{1}{N} \sum_{n=0}^{N-1} e^{2\pi i \langle A\boldsymbol{x}_n, \boldsymbol{h} \rangle} = \lim_{N \to \infty} \frac{1}{N} \sum_{n=0}^{N-1} e^{2\pi i \left\langle \boldsymbol{x}_n, A^T \boldsymbol{h} \right\rangle}.$$

This will only tend to 0 if $A^T \boldsymbol{h} \ne \boldsymbol{0}$ holds for all $\boldsymbol{h} \ne \boldsymbol{0}$ which is equivalent to saying that the rows of A are linearly independent. Conversely, if this is the case, the limit will always be 0 due to Weyl's criterion and the equidistribution of $(\boldsymbol{x}_n)$. This proves the equidistribution of $(\boldsymbol{y}_n)_{n \ge 0}$. As a consequence, mappings of this type preserve measure (technically, we do not require this mapping to be u.d.p. since this follows at once from the inequality we are about to prove - proving this, however, is here probably the shortest way to prove that the measure is preserved).

The proof of the discrepancy inequality is based on the fact that if $A : \mathbb{R}^s \to \mathbb{R}^t$ is a linear function, then the preimage of any bounded, convex set $C \subset \mathbb{R}^t$,

$$A^{-1}C = \{\boldsymbol{x} \in \mathbb{R}^s : A\boldsymbol{x} \in C\},$$

is convex as well, since for any $\boldsymbol{x}, \boldsymbol{y} \in A^{-1}C$, we have, for $0 \leq \alpha \leq 1$, due to the convexity of C,

$$A(\alpha\boldsymbol{x} + (1-\alpha)\boldsymbol{y}) = \alpha A\boldsymbol{x} + (1-\alpha)A\boldsymbol{y} \in C.$$

It is necessary to keep in mind that $(\boldsymbol{y}_n)_{n\geq 0}$ as a set of points is in most cases not a subset of $[0,1)^t$ and only equidistributed when seen as embedded in the torus $\mathbb{R}^t/\mathbb{Z}^t$ (i.e., taking the fractional part $\{\cdot\}$ of each component will yield an equidistributed sequence in $[0,1)^t$). Speaking in pictures, we have

$$(\boldsymbol{x}_n)_{n\geq 0} \xrightarrow{A} (\boldsymbol{y}_n)_{n\geq 0} \xrightarrow{\{\cdot\}} (\{\boldsymbol{y}_n\})_{n\geq 0}.$$

We will now show that the preimage of any convex set $C \subset [0,1)^t$ (i.e., $(A^{-1} \circ \{\cdot\}^{-1} C) \cap [0,1)^s$) can be written as the union of finitely many disjoint convex sets. Let $C \subset [0,1)^t$ be an arbitrary convex body. Inverting the operation of taking the fractional part yields an infinite number of copies of C in $\mathbb{R}^t$, only finitely many of which are in the range of $A \circ [0,1)^s$, since A is linear and hence bounded. We note that any two of those copies of C are disjoint and will now bound the number of copies in the range of $A \circ [0,1)^s$ from above. Every component of $(\boldsymbol{x}_n)_{n\geq 0}$ is contained in the unit interval, therefore we can contain the image of $[0,1)^s$ under A in a hyperrectangle of the form

$$\{A\boldsymbol{x} | \boldsymbol{x} \in [0,1)^s\} \subseteq \prod_{j=1}^{t} \left[\sum_{k=1}^{s} \min(0, a_{j,k}), \sum_{k=1}^{s} \max(0, a_{j,k})\right].$$

Hence the set $\{A\boldsymbol{x} | \boldsymbol{x} \in [0,1)^s\} \cap (C + \mathbb{Z}^t)$ consists of at most

$$\prod_{j=1}^{t} \left(\sum_{k=1}^{s} \max(0, a_{j,k}) - \sum_{k=1}^{s} \min(0, a_{j,k})\right) = \prod_{j=1}^{t} \sum_{k=1}^{s} |a_{j,k}|$$

disjoint convex sets, the preimage of them being, as we have already seen, convex themselves (and therefore also their intersection with $[0,1)^s$). Finally, we consider the set

$$\left\{\boldsymbol{x} \in [0,1)^s : A\boldsymbol{x} \in C + \mathbb{Z}^t\right\}$$

and need to make sure that it consists of disjoint convex sets (because the union of two convex sets is generally not convex and non-convex sets are not contained in our discrepancy system); if there are two convex preimages which share at least one common point, then they are also pathwise-connected and pathwise-connected sets remain so under linear functions. Since $\{A\boldsymbol{x}|\boldsymbol{x} \in [0,1)^s\} \cap (C+\mathbb{Z}^t)$ contains only isolated convex sets, the proof is finished. $\square$

Remark. In the proof it was seen that the constant arises from bounding

$$\#\left\{\boldsymbol{h} \in \mathbb{Z}^t | \ \exists\ \boldsymbol{x} \in [0,1)^s : A\boldsymbol{x} \in [0,1)^t + \boldsymbol{h}\right\} \leq \prod_{j=1}^{t} \sum_{k=1}^{s} |a_{j,k}|.$$

We will quickly improve this estimate (and hence the arising constant) and hint further below how one could extend this line of thought even further.

Corollary 2. *Let $A \in \mathbb{Z}^{t \times s}$ be as above, $(\boldsymbol{x}_n)$ be any sequence in $[0,1]^s$ and $\boldsymbol{y}_n = \{A\boldsymbol{x}_n\}$. Then*

$$\begin{aligned} J_N(\boldsymbol{y}_n) &\leq \#\left\{\boldsymbol{h} \in \mathbb{Z}^t : \boldsymbol{h} \in \mathbb{B}_{\|A\|_2+\sqrt{t}}(\boldsymbol{0})\right\} J_N(\boldsymbol{x}_n) \\ &\leq \left(c_t \|A\|_2^t + \mathcal{O}\left(\|A\|_2^{t-1}\right)\right) J_N(\boldsymbol{x}_n), \end{aligned}$$

where c_t denotes the volume of the $t-$dimensional sphere with radius 1, $\mathbb{B}_r(\boldsymbol{x})$ the ball of radius r with center $\boldsymbol{x}$ and $\|A\|_2$ the spectral norm of A.

P r o o f. From the definition of the spectral norm, we can infer that

$$\{A\boldsymbol{x} : \boldsymbol{x} \in [0,1)^s\} \subseteq \mathbb{B}_{\|A\|_2}(\boldsymbol{0}).$$

The diagonal of a unit cube $[0,1]^t$ is $\sqrt{t}$ and hence

$$\left\{\boldsymbol{h} \in \mathbb{Z}^t : \left(\boldsymbol{h} + [0,1)^t\right) \cap \mathbb{B}_{\|A\|_2}(\boldsymbol{0}) \neq \emptyset\right\} \subseteq \left\{\boldsymbol{h} \in \mathbb{Z}^t : \boldsymbol{h} \in \mathbb{B}_{\|A\|_2+\sqrt{t}}(\boldsymbol{0})\right\}.$$

This quantity is, of course, well-understood (it is merely the lattice point problem of Gauss) and will be roughly equal to the volume of the hypersphere (the error term is of very minor importance to us and we made use of the most elementary estimate; there are considerably better results in the literature). $\square$

Since Gauss, who tried to determine the number of lattice points within a circle, this problem has been generalized to different shapes. Even better estimates would result from using that $A \circ [0,1)^s$ is convex and

$$\lambda_t(\{A\boldsymbol{x} : \boldsymbol{x} \in [0,1)^s\}) = \sqrt{\det(A^T A)}$$

in combination with results regarding more general Gauss lattice point problems on convex domains. This, however, would require additional information about the set $A \circ [0,1)^s$ and is beyond the scope of this paper.

Application to box splines. This theorem will now be applied in the theory of box splines. We use a simplified definition of box splines, which will cover all the cases relevant to this theorem. For the full definition, we refer the reader to the monograph [5].

Definition 5. Let $t < s$ and let $\Xi \in \mathbb{R}^{t\times s}$ be a matrix of full rank. The associated box spline is given by $b_\Xi : \mathbb{R}^t \to \mathbb{R}$, defined via

$$b_\Xi(\boldsymbol{x}) = \frac{1}{\sqrt{\det \Xi\Xi^T}} \lambda_{s-t}\left(\Xi^{-1}\{\boldsymbol{x}\} \cap [0,1]^s\right).$$

This definition is rather convoluted, its geometric image, however, very appealing: One is given the $s-$dimensional unit cube $[0,1]^s$ and a hyperplane of dimension t (given by Ξ). One now moves this hyperplane through the cube and measures the intersection, which can somehow be thought of as being similar to the length of a shadow casted by a box (hence the name). It is clear from the geometric image that these functions are continuous. The constant arises from normalization, i.e., is chosen because

$$\frac{1}{\sqrt{\det \Xi\Xi^T}} \int_{\mathbb{R}^t} \lambda_{s-t}\left(\Xi^{-1}\{\boldsymbol{x}\} \cap [0,1]^s\right) d\boldsymbol{x} = 1.$$

A simple example (taken from [5]) is given by $t = 1$, $s = 2$ and the $t \times s$-matrix $\Xi = \begin{pmatrix}1 & 1\end{pmatrix}$. Geometrically, the box spline $b_\Xi(x)$ is now given by moving lines of the form $x + y = const$ through the unit square $[0,1]^2$ and measuring the length of the intersection, which is then the value of $b(x)$ at precisely the point where said line intersects the real axis. The function can now be exactly computed and after normalization evaluates to

$$b_{\begin{pmatrix}1 & 1\end{pmatrix}}(x) = \begin{cases} x, & \text{if } \ 0 \le x \le 1 \\ 2 - x, & \text{if } \ 1 \le x \le 2 \\ 0 & \text{otherwise.} \end{cases}$$

In this example the sum of all the translates of the function form a partition of unity, since

$$\sum_{i\in\mathbb{Z}} b_\Xi(x+i) = b_\Xi(\{x\}) + b_\Xi(1 + \{x\}) = \{x\} + 2 - (1 + \{x\}) = 1.$$

This is no coincidence and was first proven to be always the case (provided the matrix has only integer entries) by de Boor and Höllig in [4]. This theorem is of some importance since it implies that one can represent constants via box splines. The original proof can also be found in the monograph by Boor, Höllig and Riemenschneider [5]. We give a new and a simpler proof.

Theorem 3. *Let $t < s$, $\Xi \in \mathbb{Z}^{t\times s}$ with rank $\Xi = t$ and let $b_\Xi(\boldsymbol{x})$ be the associated box spline. Then, for each $\boldsymbol{x} \in \mathbb{R}^t$,*

$$\sum_{\boldsymbol{i}\in\mathbb{Z}^t} b(\boldsymbol{x}+\boldsymbol{i}) = 1.$$

P r o o f. Box splines are continuous functions and their support is bounded (which comes from the boundedness of the linear operator). As a consequence, the sum ranges only over finitely many terms and is continuous. If for some $\boldsymbol{x}$

$$\sum_{\boldsymbol{i}\in\mathbb{Z}^t} b(\boldsymbol{x}+\boldsymbol{i}) > 1,$$

then there is an entire neighbourhood where this holds. Let C be a convex set of nonzero measure (any circle will do) fully contained in this neighbourhood. Due to continuity,

$$\int_C \sum_{\boldsymbol{i}\in\mathbb{Z}^t} b(\boldsymbol{x}+\boldsymbol{i})d\boldsymbol{x} > \lambda_t(C).$$

Therefore, for any uniformly distributed $(\boldsymbol{x}_n)_{n\geq 0}$ in $[0,1)^s$, the density of the sequence $(\{\Xi\boldsymbol{x}_n\})_{n\geq 0}$ in C does not tend to $\lambda_t(C)$ - a contradiction to Theorem 2. The other case is analogous. $\square$

We finally state Theorem 1 in the one-dimensional case, where it has a particularly simple form.

Corollary 3. *Let $\phi : [0,1) \to [0,1)$ be a piecewise continuous u.d.p. function consisting of k pieces (i.e., k intervals forming a partition of $[0,1)$ such that any restriction $\phi\big|_I$ is continuous). Then, for any point set $\{x_0, x_1, \dots, x_{N-1}\}$,*

$$D_N(\phi(x_0), \phi(x_1), \dots, \phi(x_{N-1})) \leq k \cdot D_N(x_0, x_1, \dots, x_{N-1}).$$

P r o o f. Follows immediately from Theorem 1. $\square$

One easily sees that this constant is best possible. Consider the set $\mathcal{P} = \left\{\frac{i}{k} : 0 \leq i \leq k-1\right\}$ having discrepancy $D_k(\mathcal{P}) = k^{-1}$. Applying the function $\phi(x) = \{kx\}$ (being continuous on k intervals) to this point set, leads to all points being equal to 1. This corollary allows the creation of strange-looking low-discrepancy sequences.

Example. Let $(x_n)_{n\geq 1}$ be the van-der-Corput sequence in base 2. Let the sequence $(y_n)_{n\geq 1}$ be defined via

$$y_n = \begin{cases} \frac{4}{3}x_n(x_n+1) & \text{if } n \text{ is even} \\ \sqrt{\frac{49}{64} - \frac{3}{4}x_n} - x_n + \frac{7}{8} & \text{if } n \text{ is odd.} \end{cases}$$

Then, for any $N \in \mathbb{N}$, we have

$$ND_N^*(x_n) \leq \frac{2\log N}{3\log 2} + 2.$$

P r o o f. This is just a combination of a u.d.p. function constructed by Bosch [2, Example 2.10], certain discrepancy estimates for the dyadic van-der-Corput sequence (see, for example, [1]) and Corollary 3. □

3. Variational problems

In this section, we consider a variational problem from different points of view. In its first form, it reads as follows. Let $\mathbb{U}$ denote the set of u.d.p. functions. For given Riemann integrable functions f, g, what is the range of the functional

$$I : \mathbb{U} \times \mathbb{U} \to \mathbb{R} \qquad I(\phi, \psi) = \int_0^1 f(\phi(x))g(\psi(x))dx?$$

Recall, that for $g \equiv 1$ this is answered by the integral equation mentioned in the introduction and in this case the functional is constant, since

$$I(\phi, \psi) = \int_0^1 f(\phi(x))dx = \int_0^1 f(x)dx.$$

We will at first consider this problem, then study some seemingly unrelated interpolation problems and finally combine them to solve a similar problem for uniformly distributed sequences.

3.1. Functions

Our description of the solution hinges on what is sometimes called the monotonic rearrangement of a function f (see also [9]). The picture behind this definition is to take a possibly oscillating function f, cut the underlying interval $[0,1]$ into many small parts and rearrange them such that the new function f^* is monotonically increasing. For this we make use of (very elementary) results in rearrangement theory, see for example [11].

Definition 6. For any Lebesgue integrable functions $f : [0,1] \to \mathbb{R}$, we define the (increasing) rearrangement of f by $f^*(0) = \inf_{0 \leq t \leq 1} f(t)$ and

$$f^*(x) := \inf \left\{ t \in \mathbb{R} : \left[\int_0^1 \mathbf{1}_{f(u) \leq t} du \right] \geq x \right\}$$

for all $x > 0$ (note that this integral is to be understood in the Lebesgue sense).

The theory of rearrangements has been extended to more general spaces and measures; we will only apply it in the elementary case of Riemann integrable functions and hence the following easy lemma, which has analogues in more general settings, is sufficient.

Lemma 1. *Let $f : [0,1] \to \mathbb{R}$ be Riemann integrable and let f^* be defined as above. Then*

(1) *f^* is Riemann integrable, monotonically increasing and*

$$\int_0^1 f(x)dx = \int_0^1 f^*(x)dx.$$

(2) *If f is monotonically increasing, then $f^* = f$. If f is monotonically decreasing, then $f^* = f(1-x)$.*

(3) *For any $a, b \in \mathbb{R}$ we have*

$$(af(x)+b)^* = \begin{cases} af^*(x)+b, & \text{if } a \geq 0 \\ af^*(1-x)+b, & \text{otherwise.} \end{cases}$$

Proof. Any Riemann integrable function is bounded and Jordan measurable; hence f^* exists. It is easily seen that f^* is monotonically increasing and bounded, hence also Riemann integrable. The fact that the integrals coincide, is merely a matter of definition. The second statement is clear. For the third statement, first note that for $b \in \mathbb{R}$ the relation $(f+b)^* = f^* + b$ is obvious. The multiplication with a positive constant has apparently the stated effect while multiplication with a negative constant will turn the function around. □

Theorem 4. *Let f, g be Riemann integrable functions on $[0,1]$. Let ϕ, ψ be arbitrary u.d.p. transformations. Then*

$$\int_0^1 f^*(x)g^*(1-x)dx \leq \int_0^1 f(\phi(x))g(\psi(x))dx \leq \int_0^1 f^*(x)g^*(x)dx$$

and these bounds are best possible. Also, every number within the bounds is attained by some $\phi, \psi \in \mathbb{U}$.

Proof. The Hardy-Littlewood inequality [9, Theorem 378] states that

$$\int_0^1 f(x)g(x)\ dx \leq \int_0^1 f^*(x)g^*(x)\ dx.$$

As we know (see [17, Theorem 4]), u.d.p. functions do not affect the measure, i.e., $J(\phi^{-1}(I)) = J(I)$ for every interval I, and therefore (it is of no importance whether these integrals are taken in the Riemann or Lebesgue sense)

$$\begin{aligned}(f \circ \phi)^*(x) &= \inf\left\{t \in \mathbb{R} : \left[\int_0^1 \mathbf{1}_{f(\phi(u)) \le t} du\right] \ge x\right\} \\ &= \inf\left\{t \in \mathbb{R} : \left[\int_0^1 \mathbf{1}_{f(u) \le t} du\right] \ge x\right\} = f^*(x).\end{aligned}$$

The Hardy-Littewood inequality now implies the upper bound and, in combination with Lemma 1 and since f can be replaced by $-f$, also the lower bound. For showing the bounds to be sharp, it is sufficient to show that for each Riemann integrable function f and each $\varepsilon > 0$ there exists a u.d.p function ϕ such that

$$\int_0^1 |f^*(x) - f(\phi(x))|\, dx \le \varepsilon,$$

for if this the case, the boundedness of Riemann integrable functions and the inequality

$$\begin{aligned}&|f^*(x)g^*(x) - f(\phi(x))g(\psi(x))| \\ &\qquad = |f^*(x)g^*(x) - f(\phi(x))g^*(x) + f(\phi(x))g^*(x) - f(\phi(x))g(\psi(x))| \\ &\qquad \le \sup_{0\le x\le 1} |g(x)||f^*(x) - f(\phi(x))| + \sup_{0\le x\le 1} |f(x)||g^*(x) - g(\psi(x))|\end{aligned}$$

imply the existence of suitable ϕ, ψ such that the right-hand side of the inequality

$$\begin{aligned}&\left|\int_0^1 f^*(x)g^*(x)dx - \int_0^1 f(\phi(x))g(\psi(x))dx\right| \\ &\qquad\qquad \le \int_0^1 |f^*(x)g^*(x) - f(\phi(x))g(\psi(x))|\, dx\end{aligned}$$

becomes arbitrarily small - and hence the left-hand side as well. It is clear that once this approximation property is ensured, it also holds for $f^*(1-x)$ as one then only needs to consider $\phi(1-x)$.

Let $0 \le a < b < c < d \le 1$ with $b - a = d - c$. It is easy to see that

$$\phi(x) = \begin{cases} c + (x - a) & \text{if } x \in [a, b] \\ a + (x - c) & \text{if } x \in [c, d] \\ x & \text{otherwise} \end{cases}$$

is in $\mathbb{U}$ and exchanges the intervals $[a, b]$ and $[c, d]$. Let now $\varepsilon > 0$ be arbitrary and let $N \in \mathbb{N}$ be so large that the upper and lower Riemann sum with respect to the equidistant partition $0 < \frac{1}{N} < \cdots < \frac{N-1}{N} < 1$ differ by no more than ε^2. Let the permutation $\pi : \{0, \dots, N-1\} \to \{0, \dots, N-1\}$ be such that $\pi(i) < \pi(j)$ implies

$$\left(\inf_{\frac{i}{N} \leq t \leq \frac{i+1}{N}} f(t) < \inf_{\frac{j}{N} \leq t \leq \frac{j+1}{N}} f(t) \right) \vee$$
$$\vee \left(\inf_{\frac{i}{N} \leq t \leq \frac{i+1}{N}} f(t) = \inf_{\frac{j}{N} \leq t \leq \frac{j+1}{N}} f(t) \wedge \sup_{\frac{i}{N} \leq t \leq \frac{i+1}{N}} f(t) \leq \sup_{\frac{j}{N} \leq t \leq \frac{j+1}{N}} f(t) \right).$$

Let us define $\phi \in \mathbb{U}$ as the function which orders the intervals $[\frac{i}{N}, \frac{i+1}{N}]$ with respect to π. Since every permutation can be written as a composition of transpositions, transpositions can be realized in $\mathbb{U}$ and $\mathbb{U}$ is closed under composition, such a function exists. We may now partition these small intervals into two sets $A \cup B$, where

$$A = \left\{ \left[\frac{i}{n}, \frac{i+1}{n} \right] : 0 \leq i \leq n-1 \wedge \left| \sup_{t \in \left[\frac{i}{n}, \frac{i+1}{n}\right]} f(t) - \inf_{t \in \left[\frac{i}{n}, \frac{i+1}{n}\right]} f(t) \right| \leq \varepsilon \right\}$$

and

$$B = [0,1] \setminus \bigcup_{x \in A} x.$$

Since the integration error with respect to this partition is less than ε^2 we have $J(B) \leq \varepsilon$ and hence

$$\int_0^1 |f^*(x) - f(\phi(x))| \, dx \leq \int_{\bigcup_{x \in A} x} |f^*(x) - f(\phi(x))| \, dx + \varepsilon.$$

$f\big|_{\bigcup_{x \in A} x}$ behaves very regularly and is nicely rearranged. By making the partition sufficiently fine, the result follows. We finally show that each number α strictly between the bounds is reached. For this we note that for any $\phi, \psi \in \mathbb{U}$ and any $0 < t < 1$ the function

$$\eta(x) = \begin{cases} \phi\left(\frac{x}{t}\right) & \text{if } 0 \leq x \leq t, \\ \psi\left(\frac{x-t}{1-t}\right) & \text{if } t < x \leq 1, \end{cases}$$

is in $\mathbb{U}$. From the previous consideration, we know that there are $\phi_1, \phi_2, \psi_1, \psi_2 \in \mathbb{U}$ such that

$$\int_0^1 f(\phi_1(x)) g(\psi_1(x)) dx < \alpha < \int_0^1 f(\phi_2(x)) g(\psi_2(x)) dx$$

and hence it is possible to write α as the linear combination of these integrals, i.e.,

$$\alpha = \lambda \left(\int_0^1 f(\phi_1(x))g(\psi_1(x))dx \right) + (1-\lambda) \left(\int_0^1 f(\phi_2(x))g(\psi_2(x))dx \right)$$

with $\quad 0 < \lambda < 1$. If we then define

$$\phi_3(x) = \begin{cases} \phi_1\left(\frac{x}{\lambda}\right) & \text{if } 0 \le x \le \lambda, \\ \phi_2\left(\frac{x-\lambda}{1-\lambda}\right) & \text{if } \lambda < x \le 1, \end{cases} \text{ and } \psi_3(x) = \begin{cases} \psi_1\left(\frac{x}{\lambda}\right) & \text{if } 0 \le x \le \lambda, \\ \psi_2\left(\frac{x-\lambda}{1-\lambda}\right) & \text{if } \lambda < x \le 1, \end{cases}$$

we have

$$\int_0^1 f(\phi_3(x))g(\psi_3(x))dx = \alpha.$$

□

As a corollary, we briefly collect some properties of the set $\mathbb{U}$ when seen as a subset of L_2 and use the previous theorem to derive a statement about the angle (as induced by the inner product of L_2) between two u.d.p. functions. We also prove that for each function, which is not u.d.p., there is an entire (L_2-)neighbourhood not containing any u.d.p. functions.

Corollary 4. *$\mathbb{U}$ is a subset of the L_2-sphere with radius $\frac{1}{\sqrt{3}}$, every element of $\mathbb{U}$ is an accumulation point of $\mathbb{U}$, for any $\phi, \psi \in \mathbb{U}$*

$$\{t\phi + (1-t)\psi : 0 < t < 1\} \cap \mathbb{U} = \emptyset$$

and for each $\phi \in \mathbb{U}$ the entire set $\mathbb{U}$ is located in a small sector around ϕ

$$\mathbb{U} \subset \left\{ f \in L_2([0,1]) : \|f\| = \frac{1}{\sqrt{3}} \wedge \angle(\phi, f) \le \frac{\pi}{3} \right\}.$$

Furthermore, if $\mathcal{R}([0,1]) \subset L_2([0,1])$ denotes the space of Riemann integrable functions on $[0,1]$, then $\mathcal{R}([0,1]) \backslash \mathbb{U}$ is open. Finally, no element of $\mathbb{U}$ is isolated.

Proof. The first part is clear. Since for every $\phi \in \mathbb{U}$ and $\varepsilon > 0$ also $\psi(x) = \{\phi(x) + \varepsilon\} \in \mathbb{U}$ and

$$\|\phi(x) - \psi(x)\|_{L_2}^2 \le \int_{\phi(x) \le 1-\varepsilon} \varepsilon^2 \, dx + \int_{\phi(x) \ge 1-\varepsilon} (1-\varepsilon)^2 \, dx \le \varepsilon,$$

every point is an accumulation point. This also proves that $\mathbb{U}$ contains no isolated elements. The fact that the convex combination contains no u.d.p. functions follows from the fact that the Banach space L_p is strictly convex for

$1 < p < \infty$. The result for the angle follows directly from our result above for $f(x) = g(x) = x$, which is

$$\frac{1}{6} \leq \langle \phi(x), \psi(x) \rangle_{L_2} = \int_0^1 \phi(x)\psi(x)\; dx \leq \frac{1}{3}.$$

Let now $f \in \mathcal{R}([0,1]) \setminus \mathbb{U}$. We show that there is an entire neighbourhood containing no elements of $\mathbb{U}$. There exists an interval $I = [a,b] \subseteq [0,1]$ such that $J(f^{-1}[a,b]) \neq b - a$. We may w.l.o.g. assume that $J(f^{-1}[a,b]) < b - a$ since $J(f^{-1}[a,b]) > b - a$ implies that the desired equation holds for either the interval $[0,a]$ or $[b,1]$. Let now $\phi \in \mathbb{U}$ be arbitrary. Then

$$\|f - \phi\|_{L_2}^2 = \int_0^1 (f(x) - \phi(x))^2\; dx \geq \int_{\phi^{-1}(I) \setminus f^{-1}(I)} (f(x) - \phi(x))^2\; dx.$$

The values $f(x)$ do not lie within the interval I and we can bound the integral from below by assuming that they lie on its boundary.

$$\int_{\phi^{-1}(I) \setminus f^{-1}(I)} (f(x) - \phi(x))^2\; dx \geq \int_{\phi^{-1}(I) \setminus f^{-1}(I)} \min(\phi(x) - a, b - \phi(x))^2\; dx$$

Since $\phi(x)$ is measure-preserving, it cannot be near the boundaries of the interval all the time and

$$\begin{aligned} &\int_{\phi^{-1}(I) \setminus f^{-1}(I)} \min(\phi(x) - a, b - \phi(x))^2\; dx \\ &\geq 2 \int_0^{\frac{1}{2} J(\phi^{-1}(I) \setminus f^{-1}(I))} x^2\; dx = \frac{1}{12} J\left(\phi^{-1}(I) \setminus f^{-1}(I)\right)^2. \end{aligned}$$

We end by combining $J(\phi^{-1}(I)) = J(I)$ and $J(A \setminus B) \geq J(A) - J(B)$ to get

$$\frac{1}{12} J\left(\phi^{-1}(I) \setminus f^{-1}(I)\right)^2 \geq \frac{1}{12}\left((b-a) - J(f^{-1}([a,b]))\right)^2 > 0.$$

□

3.2. From functions to sequences

Given a uniformly distributed sequence (x_n) and a u.d.p. function ϕ, we can build a new uniformly distributed sequence $(\phi(x_n))$. Conversely, for general $(x_n), (y_n)$, is it possible to find a ϕ such that

$$(x_n) \quad \xrightarrow{\phi} \quad (y_n)?$$

Generally not and any in $[0,1)^2$ equidistributed sequence $(\boldsymbol{x}_n) = ((x_n), (y_n))$ will serve as an counterexample. Assume $\phi(x_n) = y_n$, then for any $0 \leq a < b \leq 1$

$$\inf_{a<t<b} \phi(t) = 0 \qquad \text{and} \qquad \sup_{a<t<b} \phi(t) = 1$$

due to the equidistribution of $(\boldsymbol{x}_n)$. Such a function is then surely not Riemann integrable. If we ask the same question for finite point sets, say, the first N terms of the two equidistributed sequences, the huge number of u.d.p. functions will provide uncountably many solutions to this interpolation problem. Most of these functions will behave erratically and connect the points seemingly by accident; however, there also exist piecewise linear solutions with interesting properties.

Definition 7. Let $0 = x_0 < x_1 < \cdots < x_{N-1} \leq 1$ be a partition of the unit interval and $y_0, \dots, y_{N-1}$ be N points (not necessarily distinct) in $[0,1]$. We define their interpolant $h : [0,1] \to [0,1]$ by

$$h(x)\big|_{x_n \leq x < x_{n+1}} = y_n \qquad \text{and} \qquad h(x)\big|_{x_{N-1} \leq x \leq 1} = y_{N-1}.$$

The interpolant is not u.d.p. (it is piecewise constant), but there is a function $\phi \in \mathbb{U}$ approximating the interpolant h in the L_1 norm, where the degree of approximation can be explicitely bounded in terms of the discrepancies of the partition and the set of points. In the case of a uniform partition (this we will use later), the approximation is even uniform.

Theorem 5. *Let $0 = x_0 < x_1 < \cdots < x_{N-1} \leq 1$ and $y_0, \dots, y_{N-1}$ as above and let h be their interpolant. Then there exists a u.d.p. function ϕ such that*

$$\|h - \phi\|_{L^1} = \int_0^1 |h(x) - \phi(x)|\, dx \leq 3D_N(x_n) + 2D_N(y_n).$$

Moreover, if the partition is equidistant, we have for all $0 \leq x \leq 1$

$$|h(x) - \phi(x)| \leq D_N^*(y_n) + \frac{1}{N}.$$

P r o o f. We can w.l.o.g. assume that $y_0 \leq \cdots \leq y_{N-1}$ (otherwise we cut and rearrange the interval as it has been done above). We choose the u.d.p. function $\phi(x) = x$. The function $|h(x) - \phi(x)|$ then grows like the identity on (x_n, x_{n+1}) and has jumps of height $y_{n+1} - y_n$ at the point x_{n+1}. Hence, the total variation of $|h(x) - \phi(x)|$ can be bounded by 2. Applying Koksma's inequality yields

$$\int_0^1 |h(x) - \phi(x)|\, dx \leq \frac{1}{N} \sum_{n=0}^{N-1} |h(x_n) - \phi(x_n)| + 2D_N(x_n).$$

We bound the sum via the triangle inequality by

$$\frac{1}{N} \sum_{n=0}^{N-1} |h(x_n) - \phi(x_n)| \leq \frac{1}{N} \sum_{n=0}^{N-1} \Big|h(x_n) - \frac{n}{N}\Big| + \frac{1}{N} \sum_{n=0}^{N-1} \Big|\frac{n}{N} - \phi(x_n)\Big|.$$

We recall that $h(x_n) = y_n$ and $\phi(x_n) = x_n$ and hence

$$\begin{aligned}\frac{1}{N}\sum_{n=0}^{N-1}\left|h(x_n) - \frac{n}{N}\right| &+ \frac{1}{N}\sum_{n=0}^{N-1}\left|\frac{n}{N} - \phi(x_n)\right| \\ &= \frac{1}{N}\sum_{n=0}^{N-1}\left|y_n - \frac{n}{N}\right| + \frac{1}{N}\sum_{n=0}^{N-1}\left|\frac{n}{N} - x_n\right|.\end{aligned}$$

For the second sum, since all the x_i's are distinct, we have

$$\left|\frac{n}{N} - x_n\right| = \left|\frac{A([0,x_n),N,x_n)}{N} - \lambda([0,x_n))\right| \le D_N^*(x_n).$$

This cannot be done in this form for the first sum because the sequence (y_i) might have clusters (i.e., several points at the same position). However, the sequence cannot be arbitrarily dense at one point and for any $z \in (0,1)$

$$\begin{aligned}\#\left\{i \in \{0,\dots,N-1\} : y_i = z\right\} &= N\lim_{\varepsilon\to 0}\left|\frac{A([z-\varepsilon,z+\varepsilon),N,y_n)}{N} - 2\varepsilon\right| \\ &\le ND_N(y_n).\end{aligned}$$

The same inequality holds, of course, for $z \in \{0,1\}$ as well as can be seen by making the limits one-sided. Clearly, since the index in $y_0 \le \cdots \le y_{N-1}$ starts at 0,

$$n = A([0,y_n],N,x_n) - \#\left\{i \in \{n+1,\dots,N\} : y_i = y_n\right\}.$$

Using this and once again the triangle inequality, we arrive at

$$\begin{aligned}\left|y_n - \frac{n}{N}\right| &= \left|\lambda([0,y_n]) - \frac{A([0,y_n],N,x_n)}{N} + \frac{\#\left\{i \in \{n+1,\dots,N\} : y_i = y_n\right\}}{N}\right| \\ &\le \left|\frac{A([0,y_n],N,x_n)}{N} - \lambda([0,y_n])\right| + D_N(y_n) \le D_N^*(y_n) + D_N(y_n).\end{aligned}$$

This proves the statement. As for the special case, we note that for any $x \in [0,1]$,

$$\begin{aligned}|h(x) - \phi(x)| &= \left|\frac{1}{N}\#\left\{0 \le i \le N-1 : y_i \le \frac{\lfloor Nx\rfloor}{N}\right\} - \frac{\lfloor Nx\rfloor}{N} + \frac{\lfloor Nx\rfloor}{N} - x\right| \\ &\le D_N^*(y_n) + \frac{1}{N}.\end{aligned}$$

□

We now reverse the direction of this theorem and show that in some sense a given u.d.p. function ϕ never differs much from a permutation (a permutation in the sense that one chooses an equidistant partition of the unit interval for both the x_n and y_n, whereas the point sets might be scrambled). This statement is almost obvious but will be very useful later.

Theorem 6. *Let $\phi \in \mathbb{U}$. For each $\varepsilon > 0$ there exists a $N \in \mathbb{N}$ and permutation π of the set $\{0, 1, \dots, N-1\}$ so that the interpolant $h(x)$ with respect to the equidistant partition $x_i = \frac{i}{N}$ and the point set $y_j = \frac{\pi(j)}{N}$ satisfies*

$$\int_0^1 |\phi(x) - h(x)|\, dx \le \varepsilon.$$

Proof. We take any $[a, b] \subset [0, 1]$ and consider the Jordan measurable preimage $\phi^{-1}([a, b])$. This preimage has Jordan measure $b-a$ and by choosing a sufficiently fine partition of the unit interval the result follows. $\square$

3.3. Sequences

The variational problem we are going to study for equidistributed sequences is as follows. Given Riemann integrable functions f and g, what are

$$\inf_{(x_n),(y_n)} \liminf_{N\to\infty} \frac{1}{N} \sum_{k=1}^{N} f(x_k) g(y_k) \qquad \text{and} \qquad \sup_{(x_n),(y_n)} \limsup_{N\to\infty} \frac{1}{N} \sum_{k=1}^{N} f(x_k) g(y_k)$$

where (x_n) and (y_n) are uniformly distributed sequences? Naturally, the range assumed by these values is at least as large as for the corresponding problem for u.d.p. functions, i.e.,

$$\begin{aligned} &\inf_{(x_n),(y_n)} \liminf_{N\to\infty} \frac{1}{N} \sum_{k=1}^{N} f(x_k) g(y_k) \le \int_0^1 f^*(x) g^*(1-x) dx \\ &= \inf_{\phi,\psi} \int_0^1 f(\phi(x)) g(\psi(x)) dx \le \sup_{\phi,\psi} \int_0^1 f(\phi(x)) g(\psi(x)) dx \\ &= \int_0^1 f^*(x) g^*(x) dx \le \sup_{(x_n),(y_n)} \limsup_{N\to\infty} \frac{1}{N} \sum_{k=1}^{N} f(x_k) g(y_k) \end{aligned}$$

because one can always take any uniformly distributed sequence (z_n) and set $x_n := \phi(z_n)$ and $y_n := \psi(z_n)$. We will now show that the bounds for both problems coincide.

Theorem 7. *Let f, g be Riemann integrable functions on $[0, 1]$. Let $(x_n), (y_n)$ be arbitrary equidistributed sequences. Then*

$$\begin{aligned} &\int_0^1 f^*(x) g^*(1-x) dx \le \liminf_{N\to\infty} \frac{1}{N} \sum_{k=1}^{N} f(x_k) g(y_k) \\ &\le \limsup_{N\to\infty} \frac{1}{N} \sum_{k=1}^{N} f(x_k) g(y_k) \le \int_0^1 f^*(x) g^*(x) dx \end{aligned}$$

and these bounds are best possible. Furthermore, any value between the bounds is attained for some pair of uniformly distributed sequences.

P r o o f. Assume that (the other case is analogous) there exist $(x_n), (y_n)$ with

$$\liminf_{N\to\infty} \frac{1}{N}\sum_{k=1}^{N} f(x_k)g(y_k) < \int_0^1 f^*(x)g^*(1-x)dx.$$

We fix a sufficiently small $\varepsilon > 0$ and find a sequence of natural integers $(N_i)_{i\geq 1}$ such that

$$\frac{1}{N_i}\sum_{k=1}^{N_i} f(x_k)g(y_k) < \int_0^1 f^*(x)g^*(1-x)dx - \varepsilon$$

for each N_i. Let now $N \in \mathbb{N}$ be arbitrary, let $h_{x,N}$ be the interpolant of the first N terms of (x_n) with respect to the equidistant partition $0, \frac{1}{N}, \dots, \frac{N-1}{N}$ of $[0,1]$, let $h_{y,N}$ be likewise defined for (y_n) and let $\phi_{x,N}$ and $\phi_{y,N}$ denote the u.d.p. transformations which approximate the respective interpolants in the sense of Theorem 5. We first note that from the definition of the interpolant and our using an equidistant partition

$$\int_0^1 f(h_{x,N}(t))g(h_{y,N}(t))dt = \frac{1}{N}\sum_{k=1}^{N} f(x_k)g(y_k).$$

From the usual $|ab - AB| = |ab - Ab + Ab - AB| \leq |b|\,|a-A| + |A|\,|b-B|$ estimate, we get

$$\begin{aligned}
&\left|\frac{1}{N}\sum_{k=1}^{N} f(x_k)g(y_k) - \int_0^1 f(\phi_{x,N}(t))g(\phi_{y,N}(t))dt\right| \\
&= \left|\int_0^1 f(h_{x,N}(t))g(h_{y,N}(t))dt - \int_0^1 f(\phi_{x,N}(t))g(\phi_{y,N}(t))\right| \\
&\leq \sup_{0\leq t\leq 1}|g(t)|\int_0^1 |f(h_{x,N}(t)) - f(\phi_{x,N}(t))|\,dt \\
&+ \sup_{0\leq t\leq 1}|f(t)|\int_0^1 |g(h_{y,N}(t)) - g(\phi_{y,N}(t))|\,dt.
\end{aligned}$$

As we have already seen, the expression $|h_{x,N}(t) - \phi_{x,N}(t)|$ can be estimated in terms of the star discrepancy of (x_n) and will, provided (x_n) is uniformly distributed, tend to 0 as N tends to infinity (and likewise for the second integral). Since the Riemann integrable functions $f(t), g(t)$ are bounded, the entire expression will tend to 0.

As N_i increases, the error from integrating via interpolants tends to 0 and hence we can fix a sufficiently large N_i and use the u.d.p. functions $\phi(t), \psi(t)$ approximating the interpolants $h_{x,N_i}(t), h_{y,N_i}(t)$ to create a counterexample to the statement of Theorem 4. This contradiction concludes the argument. The general construction principle mentioned before the statement of the theorem also immediately allows us to see that the bounds are sharp and that the attained values contain any number within the bounds. $\square$

Example. In [16] it was shown that for any two equidistributed sequences $(x_n), (y_n)$

$$\limsup_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} |x_n - y_n| \le \frac{1}{2}$$

and that this bound is sharp. An explicit construction was used to show that every value in $[0, 0.5]$ is the limit for appropriate sequences $(x_n), (y_n)$. Applying Theorem 7 immediately gives the related result that for any uniformly distributed $(x_n), (y_n)$

$$\begin{aligned} 0 &\le \liminf_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} (x_n - y_n)^2 \le \limsup_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} (x_n - y_n)^2 \\ &= \limsup_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} (x_n^2 - 2x_n y_n + y_n^2) \\ &\le \limsup_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} (x_n^2 + y_n^2) - \liminf_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} 2x_n y_n \\ &= \frac{2}{3} - \liminf_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} 2x_n y_n \le \frac{1}{3} \end{aligned}$$

with equality if $y_n = x_n$ and $y_n = 1 - x_n$, respectively. For any $0 < \lambda < 1$, let

$$\phi(x) = \begin{cases} \frac{x}{\lambda} & \text{if } 0 \le x \le \lambda \\ \frac{x-\lambda}{1-\lambda} & \text{if } \lambda < x \le 1 \end{cases} \qquad \text{and} \qquad \psi(x) = \begin{cases} \frac{x}{\lambda} & \text{if } 0 \le x \le \lambda \\ \frac{1-x}{1-\lambda} & \text{if } \lambda < x \le 1 \end{cases}$$

and choose any uniformly distributed (z_n) and set $x_n = \phi(z_n)$ and $y_n = \psi(z_n)$. Then

$$\lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} (x_n - y_n)^2 = \frac{\lambda + 1}{6}$$

and since $0 < \lambda < 1$, we see that every value between the bounds is attained.

Remark. For an actual computation, it is sometimes helpful to restrain from calculating the rearrangements directly and instead search for a u.d.p. functions η, ϕ, ψ such that $f \circ \phi = f^* \circ \eta$ and $g \circ \psi = g^* \circ \eta$ and use

$$\int_0^1 (f \circ \phi)(x)(g \circ \psi)(x)dx = \int_0^1 f^*(x)g^*(x)dx.$$

Example. For any equidistributed sequences $(x_n), (y_n)$ we have

$$-\frac{1+e}{1+\pi^2} \leq \liminf_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} \cos(2\pi x_n)e^{y_n} \leq \limsup_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} \cos(2\pi x_n)e^{y_n} \leq \frac{1+e}{1+\pi^2}$$

P r o o f. Instead of calculating the rearrangements of $\cos(2\pi x)$ and e^x, we note that for

$$\phi(x) = \begin{cases} \frac{1}{2} - x & \text{if } 0 \leq x \leq \frac{1}{2} \\ x & \text{if } x > \frac{1}{2} \end{cases}$$

we have $\cos(2\pi\phi(x)) = \cos^*(2\pi x) \circ \{2x\}$ (and hence, in this example, $\eta(x) = \{2x\}$). Naturally, $\exp(\{2x\}) = (e^x)^* \circ \{2x\}$. Combining this with the integral equation, we get that the extreme values are given by

$$\int_0^1 \cos^*(2\pi x)e^{1-x}dx = \int_0^1 \cos(2\pi\phi(x))e^{\{2(1-x)\}}dx = -\frac{1+e}{1+\pi^2} \quad \text{and}$$

$$\int_0^1 \cos^*(2\pi x)e^{x}dx = \int_0^1 \cos(2\pi\phi(x))e^{\{2x\}}dx = \frac{1+e}{1+\pi^2}.$$

Furthermore, examples for which equality is attained can be constructed by taking any equidistributed sequence (z_n) and setting $x_n := \phi(z_n)$, $y_n = \{2x_n\}$ and $x_n := \phi(z_n)$, $y_n = 1 - \{2x_n\}$, respectively. □

3.4. Generalizations

A natural problem is whether all of this can be generalized to give bounds for

$$I : \mathbb{U}^n \to \mathbb{R} \qquad I(\phi_1, \ldots, \phi_n) = \int_0^1 f_1(\phi_1(x))f_2(\phi_2(x)) \ldots f_n(\phi_n(x))dx.$$

This turns out to be surprisingly complicated since it hinges on the existence of analogues to the rearrangement inequality for more than two sequences, apparently a very hard problem (the monographs on inequalities by Bullen [3] and Hardy, Littlewood & Polya [9] both remark on the difficulty of this subject). In this section, we discuss some methods to derive bounds and apply them to the special case where all $f_i(x) = x$, which will serve as a model problem. It corresponds to the very interesting task of finding

$$\inf_{(x_k^{(1)}),\dots,(x_k^{(n)})} \liminf_{N\to\infty} \frac{1}{N}\sum_{k=1}^{N} x_k^{(1)}x_k^{(2)}\dots x_k^{(n)} \quad \text{and}$$
$$\sup_{(x_k^{(1)}),\dots,(x_k^{(n)})} \limsup_{N\to\infty} \frac{1}{N}\sum_{k=1}^{N} x_k^{(1)}x_k^{(2)}\dots x_k^{(n)},$$

where the product ranges over all choices of n uniformly distributed sequences $(x_k^{(1)}), (x_k^{(2)}), \dots, (x_k^{(n)})$. Finding the precise value of the supremum will turn out to be easy; the value of the infimum will be of greater interest (roughly speaking it measures how good uniformly distributed sequences can damp each other down).

Upper bounds. An natural upper bound for the supremum is given by using a generalized form of Hölder's inequality to seperate the products into several terms, each of which then no longer depends on the u.d.p. functions involved.

Generalized Hölder inequality from [3, p. 128]. *Suppose that $r_i > 0$, $1 \le i \le m$, and $a_{ij} > 0$, $1 \le i \le m$, $1 \le j \le n$ and define*

$$\frac{1}{\rho_m} = \sum_{i=1}^{m} \frac{1}{r_i}$$

then

$$\left(\sum_{j=1}^{n}\prod_{i=1}^{m} a_{ij}^{\rho_m}\right)^{\frac{1}{\rho_m}} \le \prod_{i=1}^{m}\left(\sum_{j=1}^{n} a_{ij}^{r_i}\right)^{\frac{1}{r_i}}$$

We will use the integral version of this theorem with $r_i = m$ and $\rho_m = 1$.

Theorem 8. *Let $f_1, \dots, f_n$ be Riemann integrable functions. Then*

$$\sup_{\phi_1,\dots,\phi_n} \int_0^1 f_1(\phi_1(x))f_2(\phi_2(x))\dots f_n(\phi_n(x))dx$$
$$\le \left(\prod_{i=1}^{n}\int_0^1 |f_i(\phi_i(x))|^n\, dx\right)^{\frac{1}{n}} = \left(\prod_{i=1}^{n}\int_0^1 |f_i(x)|^n\, dx\right)^{\frac{1}{n}}.$$

P r o o f. This is just a generalized version of Hölder's inequality in combination with the fact that if $f_i(x)$ is Riemann integrable, then so is $|f_i(x)|$ and the integral equation for u.d.p. functions. □

Especially, if all f_i coincide, the bound is sharp and equality is attained if all u.d.p. functions are the same. Especially,

$$\limsup_{N\to\infty} \frac{1}{N}\sum_{k=1}^{N} x_k^{(1)} x_k^{(2)} \dots x_k^{(n)} \le \frac{1}{n+1}$$

with equality if all equidistributed sequences coincide.

Building examples. A natural way to construct explicit u.d.p. functions for which the expression attaines extreme values is to solve the problem for two functions and then to iterate the argument. Since we already know an upper bound for the model problem, we apply this algorithm to construct small solutions to get an idea how the infimum behaves and how small it is. We can calculate the minimum for two functions exactly (and already did this once, when considering the L_2 inner product); set $f(x) = x$, then

$$\begin{aligned}\int_0^1 \phi_1(x)\phi_2(x)dx &= \int_0^1 f(\phi_1(x))f(\phi_2(x))dx \\ &\ge \int_0^1 f^*(x)f^*(1-x)dx = \int_0^1 x(1-x)dx = \frac{1}{6}.\end{aligned}$$

In the next step, we define $g(x) = x(1-x)$ and solve the minimization problem

$$\int_0^1 g(\phi_1(x))\phi_2(x)dx \quad \text{via} \quad \phi_1(x) = x \quad \text{and} \quad \phi_2(x) = h(x) := \begin{cases} 1-2x & \text{if } x \le \frac{1}{2} \\ 2x-1 & \text{if } x \ge \frac{1}{2}. \end{cases}$$

This construction gives an upper bound for the infimum for three sequences, i.e.,

$$\inf_{\phi_1,\phi_2,\phi_3} \int_0^1 \phi_1(x)\phi_2(x)\phi_3(x)dx \le \int_0^1 x(1-x)h(x)dx = \frac{1}{16}.$$

The next step requires to set $g(x) = x(1-x)h(x)$ and minimize once again over $g(\phi_1(x))\phi_2(x)$. The construction then yields

$$\begin{aligned}&\inf_{\phi_1,\phi_2,\phi_3,\phi_4}\int_0^1 \phi_1(x)\phi_2(x)\phi_3(x)\phi_4(x)dx \\ &\qquad \le \inf_{\phi_1,\phi_2}\int_0^1 \phi_1(x)(1-\phi_1(x))h(\phi_1(x))\phi_2(x)dx \sim \frac{1}{43.44\dots}\end{aligned}$$

These easily constructed examples already show that it is a fair guess to assume the asymptotic behaviour of the infimum to be exponential in nature. Proving that the asymptotic behaviour is at least exponential can be done easily. The following result will be improved later. We include it nonetheless because its proof relies merely on the fact that $\phi(x) \in \mathbb{U}$ is equivalent to $1-\phi(x) \in \mathbb{U}$.

Remark. Let $\phi_1(x), \dots, \phi_n(x) \in \mathbb{U}$ be arbitrary. Then there is a subset $A \subseteq \{1, \dots, n\}$ such that

$$\int_0^1 \prod_{i \in A} \phi_i(x) \prod_{i \notin A} (1 - \phi_i(x))dx \le \frac{1}{2^n}.$$

Proof. Summing over all $A \subseteq \{1, \dots, n\}$ gives

$$\sum_{A \subseteq \{1,\dots,n\}} \int_0^1 \prod_{i \in A} \phi_i(x) \prod_{i \notin A} (1 - \phi_i(x))dx$$
$$= \int_0^1 \sum_{A \subseteq \{1,\dots,n\}} \prod_{i \in A} \phi_i(x) \prod_{i \notin A} (1 - \phi_i(x))dx = \int_0^1 1 \; dx = 1$$

and hence at least one of the 2^n summands is not larger than $\frac{1}{2^n}$. □

Lower bounds. Our focus will now be on obtaining lower bounds for the infimum. Using the fact that u.d.p. functions preserve measure, we can find a lower bound by solving an optimization problem over the real numbers.

Theorem 9. *Let the functions $f_i(x) \ge 0$ be Riemann integrable and $\alpha_1, \dots, \alpha_n$ denote real numbers. Then*

$$\inf_{\phi_1,\dots,\phi_n} \int_0^1 f_1(\phi_1(x)) f_2(\phi_2(x)) \dots f_n(\phi_n(x))dx$$
$$\ge \sup_{\substack{a_k \ge 0 \\ \sum_{k=1}^n a_k \le 1}} \left(1 - \sum_{k=1}^n a_k\right) \prod_{k=1}^n f_k^*(\alpha_k)$$

Proof. Fix arbitrary real numbers $a_1, \dots, a_n$ satisfying the constraints and arbitrary $\phi_1, \dots, \phi_n$. We consider the set

$$\mathcal{A} = \bigcup_{k=1}^n \{x \in [0,1] : f_k(\phi_k(x)) \le f_k^*(\alpha_k)\}.$$

Clearly, from the fact that u.d.p. functions preserve the Jordan measure,

$$J\left(\bigcup_{k=1}^n \{x \in [0,1] : f_k(\phi_k(x)) \le f_k^*(\alpha_k)\}\right)$$
$$\le \sum_{k=1}^n J\left(\{x \in [0,1] : f_k(\phi_k(x)) \le f_k^*(\alpha_k)\}\right) = \sum_{k=1}^n a_k.$$

Therefore the set $[0,1] \setminus \mathcal{A}$ has at least Jordan measure $1 - \sum_{k=1}^{n} a_k$ and so

$$\int_{[0,1]\setminus\mathcal{A}} \prod_{k=1}^{n} f_k(\phi_k(x))dx \geq \left(\inf_{x\in[0,1]\setminus\mathcal{A}} \prod_{k=1}^{n} f_k(\phi_k(x)) \right) J([0,1] \setminus \mathcal{A})$$
$$\geq \prod_{k=1}^{n} f_k^*(\alpha_k) \left(1 - \sum_{k=1}^{n} a_k \right).$$

□

If we apply this to our model problem, we obtain, by setting all $\alpha_i = \frac{1}{n+1}$,

$$\inf_{\phi_1,\dots,\phi_n} \int_0^1 \phi_1(x)\phi_2(x)\dots\phi_n(x)dx \geq \left(\frac{1}{n+1} \right)^{n+1}.$$

Another approach is suggested by the fact that functions in $\mathbb{U}$ can be arbitrarily well approximated by permutations. This requires additional knowledge about the asymptotic behaviour of certain occuring expressions (for the special case, we need to know how $n!$ grows asymptotically).

Theorem 10. *Let $n \in \mathbb{N}$. Then*

$$\inf_{\phi_1,\dots,\phi_n} \int_0^1 \phi_1(x)\phi_2(x)\dots\phi_n(x)dx \geq \frac{1}{e^n}.$$

Proof. Recall that u.d.p. functions can be arbitrarily good approximated by permutations (in the sense of Theorem 6) and therefore

$$\inf_{\phi_1,\dots,\phi_n} \int_0^1 \phi_1(x)\phi_2(x)\dots\phi_n(x)dx \geq \inf_{m\in\mathbb{N}} \inf_{\pi_1,\dots,\pi_n\in S_m} \frac{1}{m} \sum_{i=1}^{m} \prod_{k=1}^{n} \frac{\pi_k(i)}{m}$$

where S_m denotes the set of all permutations of $\{1,\dots,m\}$. From the inequality between arithmetic and geometric mean and Stirling's formula, we know that

$$\inf_{m\in\mathbb{N}} \frac{1}{m} \sum_{i=1}^{m} \prod_{k=1}^{n} \frac{\pi_k(i)}{m} \geq \inf_{m\in\mathbb{N}} \left(\prod_{i=1}^{m} \prod_{k=1}^{n} \frac{\pi_k(i)}{m} \right)^{\frac{1}{m}}$$
$$= \inf_{m\in\mathbb{N}} \left(\prod_{k=1}^{n} \prod_{i=1}^{m} \frac{\pi_k(i)}{m} \right)^{\frac{1}{m}} = \inf_{m\in\mathbb{N}} \left(\frac{m!^{\frac{1}{m}}}{m} \right)^{n} = \frac{1}{e^n}.$$

□

Finally, we show that this result captures the true growth and is sharp up to a constant by constructing an explicit example. The example, which consists in combining shifted versions of what is sometimes called the tent map in the study of dynamical system, allows us to make use of a classical result of Euler.

Theorem 11. *Let $n \geq 3$. Then*

$$\inf_{\phi_1,\ldots,\phi_n} \int_0^1 \phi_1(x)\phi_2(x)\ldots\phi_n(x)dx \leq e^{\frac{1}{6n}} \frac{n}{n-2}\frac{4}{\pi}\frac{1}{e^n}.$$

P r o o f. Let the function $f : [0,2] \to \mathbb{R}$ be defined via

$$f(x) = \begin{cases} 2x & \text{if } 0 \leq x \leq \frac{1}{2}, \\ 2-2x & \text{if } \frac{1}{2} \leq x \leq 1, \\ f(x-1) & \text{otherwise .} \end{cases}$$

Then for each $n \in \mathbb{N}$ and $0 \leq k \leq n-1$ we have $f(x+\frac{k}{n})\big|_{[0,1]} \in \mathbb{U}$. We will now consider the integral over the product of these functions. We first assume that $n \geq 3$ is an even integer. Then, from the definition of the function $f(x)$,

$$\int_0^1 \prod_{k=0}^{n-1} f\left(x+\frac{k}{n}\right)dx = 2^n n \int_0^{\frac{1}{n}} \prod_{k=0}^{\frac{n}{2}-1}\left(x+\frac{k}{n}\right)\prod_{k=\frac{n}{2}}^{n-1}\left(\frac{k}{n}-x\right)dx.$$

We rearrange the product, pull some number before the integral and substitute $z = nx$.

$$\begin{aligned} &2^n n \int_0^{\frac{1}{n}} \prod_{k=0}^{\frac{n}{2}-1}\left(x+\frac{k}{n}\right)\prod_{k=\frac{n}{2}}^{n-1}\left(\frac{k}{n}-x\right)dx \\ &= 2^n n \int_0^{\frac{1}{n}} x\left(\frac{1}{2}-x\right)\prod_{k=1}^{\frac{n}{2}-1}\left(\frac{k^2}{n^2}-x^2\right)dx \\ &= 2^n n\frac{(\frac{n}{2}-1)!^2}{n^{n-2}}\int_0^{\frac{1}{n}} x\left(\frac{1}{2}-x\right)\prod_{k=1}^{\frac{n}{2}-1}\left(1-\frac{(nx)^2}{k^2}\right)dx \\ &= 2^n\frac{(\frac{n}{2}-1)!^2}{n^n}\int_0^1 z\left(\frac{n}{2}-z\right)\prod_{k=1}^{\frac{n}{2}-1}\left(1-\frac{z^2}{k^2}\right)dz \end{aligned}$$

The product converges uniformly on $[0,1]$ and from Euler's representation of $\sin \pi x$ as

$$\sin(\pi x) = \pi x \prod_{k=1}^{\infty}\left(1-\frac{x^2}{k^2}\right)$$

we may rewrite the integrand as

$$\int_0^1 z\left(\frac{n}{2}-z\right)\prod_{k=1}^{\frac{n}{2}-1}\left(1-\frac{z^2}{k^2}\right)dz = \int_0^1 \left(\frac{n}{2}-z\right)\frac{\sin(\pi z)}{\pi}\prod_{k=\frac{n}{2}}^{\infty}\frac{1}{\left(1-\frac{z^2}{k^2}\right)}dz.$$

The infinite product is maximal when $z = 1$, in this case, due to

$$\frac{1}{\left(1-\frac{1}{k}\right)\left(1+\frac{1}{k}\right)} = \frac{k^2}{(k-1)(k+1)} \quad \text{we know that} \quad \prod_{k=\frac{n}{2}}^{\infty} \frac{1}{\left(1-\frac{1}{k^2}\right)} = \frac{n}{n-2}.$$

By using this bound and evaluating the integral, we get

$$\begin{aligned}\int_0^1 \left(\frac{n}{2}-z\right)\frac{\sin(\pi z)}{\pi}\prod_{k=\frac{n}{2}}^{\infty}\frac{1}{\left(1-\frac{z^2}{k^2}\right)}dz &\le \frac{n}{n-2}\int_0^1\left(\frac{n}{2}-z\right)\frac{\sin(\pi z)}{\pi}dz \\ &= \frac{n}{n-2}\frac{n-1}{\pi^2}.\end{aligned}$$

Using the following inequality due to [18],

$$n! \le \sqrt{2\pi n}\left(\frac{n}{e}\right)^n e^{\frac{1}{12n}},$$

then results in

$$\begin{aligned}&\int_0^1 \prod_{k=0}^{n-1} f(x+\frac{k}{n})dx \le 2^n \frac{(\frac{n}{2}-1)!^2}{n^n}\frac{n}{n-2}\frac{n-1}{\pi^2} \\ &\le \frac{4}{\pi^2 n}\frac{2^n}{n^n}\frac{n}{n-2}\left(\frac{n}{2}!\right)^2 \le \frac{n}{n-2}e^{\frac{1}{6n}}\frac{4}{\pi}\frac{1}{e^n}.\end{aligned}$$

For odd n, we can decompose the integral into

$$\int_0^1 \prod_{k=0}^{n-1} f(x+\frac{k}{n})dx = 2^{n+1}n\int_0^{\frac{1}{2n}}\prod_{k=0}^{\frac{n-1}{2}}\left(x+\frac{k}{n}\right)\prod_{k=\frac{n+1}{2}}^{n-1}\left(\frac{k}{n}-x\right)dx$$

and then proceed along the same lines. □

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