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WEIGHTED LIMIT THEOREMS FOR GENERAL DIRICHLET SERIES. II

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ABSTRACT. Under some hypotheses on the weight function and a function given by general Dirichlet series, weighted limit theorems in the sense of weak convergence of probability measures in the space of meromorphic functions are obtained. If the system of exponents of Dirichlet series is linearly independent over the field of rational numbers, then the explicit form of the limit measure is given.

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1. Introduction

Let $\{a_m : m \in \mathbb{N}\}$ be a sequence of complex numbers, and $\{\lambda_m : m \in \mathbb{N}\}$ be an increasing sequence of real numbers such that $\lim\limits_{m\to\infty} \lambda_m = +\infty$. Then the series

$$\sum_{m=1}^{\infty} a_m \mathrm{e}^{-\lambda_m s}, \quad s = \sigma + it, \tag{1}$$

is called a general Dirichlet series. The region of convergence as well as of absolute convergence of series (1) is a half-plane.

From H. Bohr and B. Jessen fundamental works [2], [3], the asymptotic behaviour of Dirichlet series is characterized by limit theorems in the sense of weak convergence of probability measures. Theorems of such a type for general Dirichlet series were obtained in [5]–[7] and [9]–[13].

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This paper is a continuation of [6], where weighted limit theorems on the complex plane for general Dirichlet series were obtained. Before stating them, we recall the used hypotheses on Dirichlet series and the weight function.

Suppose that the series (1) converges absolutely for $\sigma > \sigma_a$ and has a sum $f(s)$. Additionally, we assume that the function $f(s)$ can be meromorphically continued to the region $\sigma > \sigma_1$, $\sigma_1 < \sigma_a$, all poles in this region are included in a compact set, and that, for $\sigma > \sigma_1$, σ is not the real part of a pole of $f(s)$, the estimates

$$f(\sigma + it) = \mathrm{O}(|t|^a), \quad a = a(\sigma) > 0, \quad |t| \geq t_0 > 0, \tag{2}$$

and

$$\int_{-T}^{T} |f(\sigma + it)|^2 \mathrm{d}t = \mathrm{O}(T), \quad T \to \infty \tag{3}$$

are satisfied.

Let $w(t)$ be a positive function of bounded variation on $[T_0, \infty)$, $T_0 > 0$. Moreover, let

$$U = U(T, w) = \int_{T_0}^{T} w(t)\mathrm{d}t.$$

We suppose that $\lim\limits_{T\to\infty} U(T, w) = +\infty$, and that, for $\sigma > \sigma_1$, σ is not the real part of a pole of $f(s)$, and all $v \in \mathbb{R}$, the estimate

$$\int_{T_0+v}^{T+v} w(t - v)|f(\sigma + it)|^2 \mathrm{d}t = \mathrm{O}(U(1 + |v|)) \tag{4}$$

is satisfied. Note that if $w(t) = t^{-1}$, then the estimate (3) implies (4).

Denote by $\mathcal{B}(S)$ the class of Borel sets of a metric space S, and define the probability measure

$$P_{T,\sigma,w}(A) = \frac{1}{U} \int_{T_0}^{T} w(t) I_{\{t: f(\sigma+it)\in A\}} \mathrm{d}t, \quad A \in \mathcal{B}(\mathbb{C}),$$

where I_A denotes the indicator function of the set A, and $\mathbb{C}$ is the complex plane.

The first theorem of [6] is the following statement.

Theorem 1. [6]. *Suppose that $\sigma > \sigma_1$ and that the function $f(s)$ satisfies the conditions (2) and (4). Then on $(\mathbb{C}, \mathcal{B}(\mathbb{C}))$, there exists a probability measure P_σ such that the measure $P_{T,\sigma,w}$ converges weakly to P_σ as $T \to \infty$.*

The identification of the limit measure P_σ requires some definitions and additional hypotheses on the functions $f(s)$ and $w(t)$. Let $\gamma = \{s \in \mathbb{C} : |s| = 1\}$ denote the unit circle on the complex plane, and

$$\Omega = \prod_{m=1}^{\infty} \gamma_m,$$

where $\gamma_m = \gamma$ for all $m \in \mathbb{N}$. By the Tikhonov theorem, with the product topology and pointwise multiplication, the infinite dimensional torus Ω is a compact topological Abelian group. Therefore, on $(\Omega, \mathcal{B}(\Omega))$ the probability Haar measure m_H can be defined, and this leads to a probability space $(\Omega, \mathcal{B}(\Omega), m_H)$. Denote by $\omega(m)$ the projection of $\omega \in \Omega$ to the coordinate space γ_m, $m \in \mathbb{N}$. Suppose that

$$\lambda_m \geq c(\log m)^\delta \tag{5}$$

with some positive constants c and δ, and on the probability space $(\Omega, \mathcal{B}(\Omega), m_H)$ define the complex-valued random element $f(\sigma, \omega)$ by

$$f(\sigma, \omega) = \sum_{m=1}^{\infty} a_m \omega(m) \mathrm{e}^{-\lambda_m \sigma}, \quad \sigma > \sigma_1.$$

Note that inequality (5) is used to prove in [13] that $f(\sigma, \omega)$ is a random element.

Also, we need a condition for $w(t)$ which generalizes the classical Birkhoff-Khintchine theorem, see, for example, [4]. Denote by $\mathbb{E}X$ the expectation of the random element X. Let $X(t, \omega)$ be an arbitrary ergodic process, $\mathbb{E}|X(t, \omega)| < \infty$, with sample paths integrable almost surely in the Riemann sense over every finite interval. We suppose that

$$\frac{1}{U} \int_{T_0}^{T} w(t) X(t + v, \omega) \mathrm{d}t = \mathbb{E}X(0, \omega) + r_T \left(1 + |v|\right)^\alpha \tag{6}$$

almost surely for all $v \in \mathbb{R}$ with some $\alpha > 0$, where $r_T \to 0$ as $T \to \infty$. If in (6) $w(t) \equiv 1$ and $v = 0$, then (6) becomes the Birkhoff-Khintchine theorem. Let $\mu(T) = \inf_{t \in [T_0, T]} w(t)$. If

$$w(T) \mu^{-1}(T) = \mathrm{O}(1),$$

then the weight function $w(t)$ satisfies (6) with $\alpha = 1$ (see [14]).

Theorem 2. [6]. *Let $\sigma > \sigma_1$. Suppose that the system $\{\lambda_m : m \in \mathbb{N}\}$ is linearly independent over the field of rational numbers, and inequality* (5) *holds. Moreover, suppose that the weight function $w(t)$ satisfies* (6), *and, for the function $f(s)$, estimates* (2) *and* (4) *hold. Then the probability measure $P_{T,\sigma,w}$ converges weakly to the distribution $P_{f,\sigma}$ of the random element $f(\sigma, \omega)$ as $T \to \infty$.*

We recall that $P_{f,\sigma}$ is defined by

$$P_{f,\sigma}(A) = m_H\left(\omega \in \Omega : f(\sigma,\omega) \in A\right), \quad A \in \mathcal{B}(\mathbb{C}).$$

The aim of this paper is to prove weighted limit theorems in the space of meromorphic functions for the function $f(s)$. The first limit theorem in this space for the function $f(s)$ was obtained in [10].

Let $\mathbb{C}_\infty = \mathbb{C} \cup \{\infty\}$ be the Riemann sphere, and let $d(s_1, s_2)$ denote the spherical metric, i. e., for $s_1, s_2, s \in \mathbb{C}$,

$$d(s_1,s_2) = \frac{2|s_2 - s_1|}{\sqrt{1+|s_1|^2}\sqrt{1+|s_2|^2}}, \quad d(s,\infty) = \frac{2}{\sqrt{1+|s|^2}}, \quad d(\infty,\infty) = 0.$$

Denote by D the half-plane $\{s \in \mathbb{C} : \sigma > \sigma_1\}$, and let $M(D)$ stand for the space of meromorphic functions $g : D \to (\mathbb{C}_\infty, d)$ equipped with the topology of uniform convergence on compacta. In this topology, a sequence $\{g_n\} \in M(D)$ converges to $g \in M(G)$ if

$$\sup_{s \in K} d(g_n(s), g(s)) \underset{n\to\infty}{\longrightarrow} 0$$

for every compact subset K of D.

Analytic on D functions form a subspace $H(D)$ of $M(G)$.

Denote by meas$\{A\}$ the Lebesgue measure of a measurable set $A \subset \mathbb{R}$.

Theorem 3. [10]. *Suppose that the function $f(s)$ satisfies the hypotheses* (2) *and* (3). *Then on $(M(D), \mathcal{B}(M(D)))$, there exists a probability measure P such that the measure*

$$\frac{1}{T}\text{meas}\{\tau \in [0,T] : f(s+i\tau) \in A\}, \quad A \in \mathcal{B}(M(D)),$$

converges weakly to P as $T \to \infty$.

The first attempt for identification of the limit measure P in Theorem 3 was made in [13], using the hypothesis that the set $\{\log 2\} \bigcup \{\lambda_m : m \in \mathbb{N}\}$ is linearly independent over the field of rational numbers. In [5], the number $\log 2$ was removed from the hypothesis. Suppose that inequality (5) is satisfied, and for $s \in D$, on the probability space $(\Omega, \mathcal{B}(\Omega), m_H)$, define the $H(D)$-valued random element $f(s,\omega)$ by the formula

$$f(s,\omega) = \sum_{m=1}^{\infty} a_m \omega(m) \mathrm{e}^{-\lambda_m s}.$$

Theorem 4. [5]. *Suppose that the system* $\{\lambda_m : m \in \mathbb{N}\}$ *is linearly independent over the field of rational numbers, and that conditions* (2), (3) *and* (5) *are satisfied. Then the measure*

$$\frac{1}{T}\text{meas}\{\tau \in [0,T] : f(s+i\tau) \in A\}, \quad A \in \mathcal{B}(M(D)),$$

converges weakly to the distribution of the $H(D)$*-valued random element* $f(s,\omega)$ *as* $T \to \infty$.

Now we state new weighted limit theorems in the space of meromorphic functions for $f(s)$. Define

$$P_{T,w}(A) = \frac{1}{U}\int\limits_{T_0}^{T} w(\tau) I_{\{\tau : f(s+i\tau)\in A\}} \mathrm{d}\tau, \quad A \in \mathcal{B}(M(D)).$$

Theorem 5. *Suppose that conditions* (2) *and* (4) *are satisfied. Then on* $(M(D), \mathcal{B}(M(D)))$, *there exists a probability measure* P_w *such that* $P_{T,w}$ *converges weakly to* P_w *as* $T \to \infty$.

The next theorem gives the explicit form for the limit measure P_w in Theorem 5.

Theorem 6. *Suppose that the system* $\{\lambda_m : m \in \mathbb{N}\}$ *is linearly independent over the field of rational numbers, and inequality* (5) *holds. Moreover, suppose that the weight function* $w(t)$ *satisfies* (6), *and, for the function* $f(s)$, *estimates* (2) *and* (4) *hold. Then the probability measure* $P_{T,w}$ *converges weakly to the distribution* P_f *of the random element* $f(s,\omega)$ *as* $T \to \infty$.

Note that the distribution P_f is defined by

$$P_f(A) = m_H\left(\omega \in \Omega : f(s,\omega) \in A\right), \quad A \in \mathcal{B}(H(D)).$$

Theorems 5 and 6 show, in some sense, the regularity of the asymptotic behaviuor of the function $f(s)$. History and bibliography on probabilistic results for zeta-functions and Dirichlet series are given in [6], the first part of our work.

2. Limit theorems for absolutely convergent Dirichlet series

Since all poles of $f(s)$ in the half-plane D are included in a compact set, their number is finite. Denote the poles of $f(s)$ in the region D by $s_1, ..., s_r$. Without loss of generality we can assume that every of these poles has order 1. Define

$$f_1(s) = \prod_{j=1}^{r} \left(1 - \mathrm{e}^{\lambda_1(s_j - s)}\right).$$

Then, obviously, $f_1(s_j) = 0$ for $j = 1, ..., r$. Therefore, the function

$$f_2(s) = f_1(s) f(s)$$

is regular on D. Denote by $|A|$ the number of elements of the set A. Then, for $\sigma > \sigma_a$, the function $f_2(s)$ can be written in the form

$$f_2(s) = \sum_{A \subseteq \{1,...,r\}} \sum_{m=1}^{\infty} a_m \mathrm{e}^{\lambda_1 \sum_{j\in A} s_j} (-1)^{|A|} \mathrm{e}^{-(\lambda_m + |A|\lambda_1)s} =$$

$$= \sum_{j=0}^{r} \sum_{m=1}^{\infty} a_{m,j} \mathrm{e}^{-(\lambda_m + j\lambda_1)s}$$

by the absolute convergence with the coefficients $a_{m,j} = \mathrm{O}(|a_m|)$, $m \in \mathbb{N}$, $j = 1, ..., r$.

It is clear that the definition of $f_2(s)$, and (2) and (4), for $\sigma > \sigma_1$, imply the estimate

$$f_2(\sigma + it) = \mathrm{O}(|t|^a), \quad a = a(\sigma) > 0, \quad |t| \geq t_0 > 0, \tag{7}$$

and, for all $v \in \mathbb{R}$, the estimate

$$\int_{-T+v}^{T+v} w(t - v)|f_2(\sigma + it)|^2 \mathrm{d}t = \mathrm{O}(U(1 + |v|)). \tag{8}$$

Let $\sigma_2 > \sigma_a - \sigma_1$ be a fixed number, $v(m, n) = \exp\{-\mathrm{e}^{(\lambda_m - \lambda_n)\sigma_2}\}$ and

$$g_n(s) = \sum_{j=0}^{r} \sum_{m=1}^{\infty} a_{m,j} v(m, n) \mathrm{e}^{-(\lambda_m + j\lambda_1)s}.$$

In view of (7) and (8), in the same way as in [6], it follows that the series for $g_n(s)$ converges absolutely for $\sigma > \sigma_1$. In this section, first we consider the weak convergence of the probability measure

$$P_{T,n,w}(A) = \frac{1}{U} \int_{T_0}^{T} w(\tau) I_{\{\tau : g_n(s+i\tau) \in A\}} \mathrm{d}\tau, \quad A \in \mathcal{B}(H(D)).$$

For this, we recall a limit theorem on the torus Ω.

Let

$$Q_{T,w}(A) = \frac{1}{U}\int\limits_{T_0}^{T} w(\tau) I_{\{\tau:(\mathrm{e}^{-i\lambda_m\tau}:m\in\mathbb{N})\in A\}}\mathrm{d}\tau, \quad A\in\mathcal{B}(\Omega).$$

Lemma 1. [6]. *On $(\Omega,\mathcal{B}(\Omega))$, there exists a probability measure Q_w such that the measure $Q_{T,w}$ converges weakly to Q_w as $T\to\infty$. If the system $\{\lambda_m : m\in\mathbb{N}\}$ is linearly independent over the field of rational numbers, then the measure $Q_{T,w}$ converges weakly to the Haar measure m_H as $T\to\infty$.*

Theorem 7. *On $(H(D),\mathcal{B}(H(D)))$, there exists a probability measure $P_{n,w}$ such that the measure $P_{T,n,w}$ converges weakly to $P_{n,w}$ as $T\to\infty$.*

P r o o f. Define the function $h_n:\Omega\to H(D)$ by the formula

$$h_n(\omega) = \sum_{j=0}^{r}\sum_{m=1}^{\infty} a_{m,j} v(m,n)\omega(m)\omega^j(1)\mathrm{e}^{-(\lambda_m+j\lambda_1)s}.$$

Then the function h_n is continuous, and

$$h_n\left((\mathrm{e}^{-i\lambda_m\tau}:m\in\mathbb{N})\right) = g_n(s+i\tau).$$

Consequently, $P_{T,n,w} = Q_{T,w}h_n^{-1}$, and Lemma 1 and Theorem 5.1 of [1] imply the weak convergence of $P_{T,n,w}$ to $Q_w h_n^{-1}$ as $T\to\infty$. $\square$

For $\omega\in\Omega$, let

$$g_n(s,\omega) = \sum_{j=0}^{r}\sum_{m=1}^{\infty} a_{m,j} v(m,n)\omega(m)\omega^j(1)\mathrm{e}^{-(\lambda_m+j\lambda_1)s}.$$

Obviously, the latter series also converges absolutely for $\sigma>\sigma_1$. Let $\hat{\omega}$ be a fixed element of Ω, and

$$\hat{P}_{T,n,w}(A) = \frac{1}{U}\int\limits_{T_0}^{T} w(\tau) I_{\{\tau:g_n(s+i\tau,\hat{\omega})\in A\}}\mathrm{d}\tau, \quad A\in\mathcal{B}(H(D)).$$

Theorem 8. *Suppose that the system $\{\lambda_m : m\in\mathbb{N}\}$ is linearly independent over the field of rational numbers. Then on $(H(D),\mathcal{B}(H(D)))$, there exists a probability measure P_n such that the measures $P_{T,n,w}$ and $\hat{P}_{T,n,w}$ converge weakly to P_n as $T\to\infty$.*

P r o o f. Since the system $\{\lambda_m : m\in\mathbb{N}\}$ is linearly independent over the field of rational numbers, by the second part of Lemma 1 and the proof of Theorem 7 we have that $P_{T,n,w}$ converges weakly to $m_H h_n^{-1}$ as $T\to\infty$.

Define $\hat{h}_n : \Omega \to H(D)$ by the formula

$$\hat{h}_n(\omega) = \sum_{j=0}^{r} \sum_{m=1}^{\infty} a_{m,j} v(m,n) \hat{\omega}(m) \hat{\omega}^j(1) \omega(m) \omega^j(1) \mathrm{e}^{-(\lambda_m + j\lambda_1)s}.$$

Then, in the same way as in the case of $P_{T,n,w}$, we obtain that $\hat{P}_{T,n,w}$ converges weakly to $m_H \hat{h}_n^{-1}$ as $T \to \infty$. Now we take $h : \Omega \to \Omega$ defined by $h(\omega) = \omega\hat{\omega}$. Then, clearly, $\hat{h}_n(\omega) = h_n(h(\omega))$. Since the Haar measure is invariant, the equality

$$m_H \hat{h}_n^{-1} = m_H (h_n(h))^{-1} = (m_H h^{-1}) h_n^{-1} = m h_n^{-1}$$

holds, and the theorem is proved. □

3. Approximation by the mean

For $s \in D$ and $\omega \in \Omega$, define

$$f_2(s, \omega) = \sum_{j=0}^{r} \sum_{m=1}^{\infty} a_{m,j} \omega(m) \omega^j(1) \mathrm{e}^{-(\lambda_m + j\lambda_1)s}.$$

In this section, we approximate $f_2(s)$ by $g_n(s)$ as well as $f_2(s, \omega)$ by $g_n(s, \omega)$ in the mean. This is necessary to deduce Theorems 5 and 6 from Theorems 7 and 8, respectively.

Theorem 9. *Let K be a compact subset of the half-plane D. Then*

$$\lim_{n \to \infty} \limsup_{T \to \infty} \frac{1}{U} \int_{T_0}^{T} w(\tau) \sup_{s \in K} |f_2(s + i\tau) - g_n(s + i\tau)| \mathrm{d}\tau = 0.$$

Proof. Let σ_2 be the same as in definition of $g_n(s)$, and

$$l_n(s) = \frac{s}{\sigma_2} \Gamma\left(\frac{s}{\sigma_2}\right) \mathrm{e}^{\lambda_n s}, \quad n \in \mathbb{N}.$$

Then the function $g_n(s)$ can be written in the form in [5]

$$g_n(s) = \frac{1}{2\pi i} \int_{\sigma_2 - i\infty}^{\sigma_2 + i\infty} f_2(s + z) l_n(z) \frac{\mathrm{d}z}{z}.$$

From this, for $\sigma_3 > \sigma_1$ and $\sigma_3 < \sigma$, using the residue theorem we derive

$$g_n(s) = \frac{1}{2\pi i} \int\limits_{\sigma_3-\sigma-i\infty}^{\sigma_3-\sigma+i\infty} f_2(s+z) l_n(z) \frac{\mathrm{d}z}{z} + f_2(s). \tag{9}$$

Let L be a simple closed contour lying in D and enclosing the set K. Denote by δ the distance of L from the set K. Then by the Cauchy integral formula

$$\sup_{s\in K} |f_2(s+i\tau) - g_n(s+i\tau)| \le \frac{1}{2\pi\delta} \int\limits_L |f_2(z+i\tau) - g_n(z+i\tau)||\mathrm{d}z|.$$

Hence, we obtain

$$\frac{1}{U}\int\limits_{T_0}^{T} w(\tau) \sup_{s\in K} |f_2(s+i\tau) - g_n(s+i\tau)|\mathrm{d}\tau \ll$$

$$\frac{1}{U\delta}\int\limits_L |\mathrm{d}z| \int\limits_{T_0+\mathrm{Im}z}^{T+\mathrm{Im}z} w(\tau - \mathrm{Im}z)|f_2(\mathrm{Re}z + i\tau) - g_n(\mathrm{Re}z + i\tau)|\mathrm{d}\tau \ll$$

$$\frac{|L|}{U\delta} \sup_{\sigma+iu\in L} \int\limits_{T_0+u}^{T+u} w(t-u)|f_2(\sigma+it) - g_n(\sigma+it)|\mathrm{d}t, \tag{10}$$

where $|L|$ denotes the length of the contour L. Suppose that $\min\{\sigma : s \in K\} = \sigma_1 + \varepsilon$, $\varepsilon > 0$, and $\max\{\sigma : s \in K\} = A$. We put $\sigma_3 = \sigma_1 + \frac{\varepsilon}{2}$. The contour L can be chosen so that, for $s \in L$, the inequalities $\sigma \ge \sigma_1 + \frac{3\varepsilon}{4}$ and $\delta \ge \frac{\varepsilon}{4}$ hold. In view of (9),

$$f_2(\sigma+it) - g_n(\sigma+it) \ll \int\limits_{-\infty}^{\infty} |f_2(\sigma_3 + it + i\tau)||l_n(\sigma_3 - \sigma + i\tau)|\mathrm{d}\tau.$$

Therefore, the Cauchy-Schwarz inequality and (8) yield

$$\frac{1}{U}\int\limits_{T_0+u}^{T+u} w(t-u)|f_2(\sigma+it) - g_n(\sigma+it)|\mathrm{d}t \ll$$

$$\int\limits_{-\infty}^{\infty} |l_n(\sigma_3 - \sigma + i\tau)| \frac{1}{U} \int\limits_{T_0+u+\tau}^{T+u+\tau} w(t-u-\tau)|f_2(\sigma_3+it)|\mathrm{d}t\mathrm{d}\tau \ll$$

$$\int\limits_{-\infty}^{\infty} |l_n(\sigma_3-\sigma+i\tau)| \frac{1}{\sqrt{U}} \left(\int\limits_{T_0+u+\tau}^{T+u+\tau} w(t-u-\tau)|f_2(\sigma_3+it)|^2 \mathrm{d}t \right)^{\frac{1}{2}} \mathrm{d}\tau \ll$$

$$\int\limits_{-\infty}^{\infty} |l_n(\sigma_3-\sigma+i\tau)| \left(1+|u|+|\tau|\right)^{\frac{1}{2}} \mathrm{d}\tau \ll \int\limits_{-\infty}^{\infty} |l_n(\sigma_3-\sigma+i\tau)| \left(1+|\tau|\right) \mathrm{d}\tau,$$

because u is bounded by a constant. This together with (10) show that

$$\frac{1}{U}\int\limits_{T_0}^{T} w(\tau) \sup_{s\in K} |f_2(s+i\tau)-g_n(s+i\tau)|\mathrm{d}\tau \ll$$

$$\sup_{\sigma\leq-\frac{\varepsilon}{4}} \int\limits_{-\infty}^{\infty} |l_n(\sigma+it)|(1+|t|)\mathrm{d}t = \mathrm{o}(1)$$

as $n\to\infty$. □

Theorem 10. *Suppose that the system $\{\lambda_m : m\in\mathbb{N}\}$ is linearly independent over the field of rational numbers. Let K be a compact subset of the half-plane D. Then, for almost all $\omega\in\Omega$,*

$$\lim_{n\to\infty} \limsup_{T\to\infty} \frac{1}{U}\int\limits_{T_0}^{T} w(\tau) \sup_{s\in K} |f_2(s+i\tau,\omega)-g_n(s+i\tau,\omega)|\mathrm{d}\tau = 0.$$

Proof. Let $a_\tau=\{\mathrm{e}^{-i\tau\lambda_m} : m\in\mathbb{N}\}$, $\tau\in\mathbb{R}$, and define $\varphi_\tau(\omega)=a_\tau\omega$ for $\omega\in\Omega$. Then $\{\varphi_\tau : \tau\in\mathbb{R}\}$ is a one-parameter group of measurable measure preserving transformations on Ω. In [6], Lemma 5, it was proved that the one-parameter group $\{\varphi_\tau : \tau\in\mathbb{R}\}$ is ergodic. From this and (6), similarly to the proof of Lemma 12 of [6] we obtain that, for $\sigma>\sigma_1$,

$$\int\limits_{T_0+v}^{T+v} w(t-v)|f_2(\sigma+it,\omega)|^2\mathrm{d}t \ll U(1+|v|)^{\alpha}$$

for almost all $\omega\in\Omega$ and all $v\in\mathbb{R}$. Using this estimate instead of (8) and repeating the arguments of the proof of Theorem 9, we obtain the assertion of the theorem. □

4. Limit theorems for the function $f_2(s)$

This section is devoted to limit theorems in the space of analytic functions for the function $f_2(s)$. On $(H(D)), \mathcal{B}(H(D))$, define two probability measures

$$P_{T,f_2,w}(A) = \frac{1}{U}\int\limits_{T_0}^{T} w(\tau) I_{\{\tau: f_2(s+i\tau)\in A\}} \mathrm{d}\tau$$

and, for $\omega \in \Omega$,

$$\hat{P}_{T,f_2,w}(A) = \frac{1}{U}\int\limits_{T_0}^{T} w(\tau) I_{\{\tau: f_2(s+i\tau,\omega)\in A\}} \mathrm{d}\tau.$$

Theorem 11. *Suppose that the hypotheses of Theorem 5 are satisfied. Then on $(H(D), \mathcal{B}(H(D)))$, there exists a probability measure $P_{f_2,w}$ such that the measure $P_{T,f_2,w}$ converges weakly to $P_{f_2,w}$ as $T \to \infty$.*

P r o o f. By Theorem 7, we have that $P_{T,n,w}$ converges weakly to some probability measure $P_{n,w}$ on $(H(D), \mathcal{B}(H(D)))$ as $T \to \infty$. Let $X_{n,w}(s)$ be a $H(D)$-valued random element having the distribution $P_{n,w}$, and $\xrightarrow{\mathcal{D}}$ denote the convergence in distribution. Moreover, let θ_T be a random variable defined on a certain probability space $(\hat{\Omega}, \mathcal{B}(\hat{\Omega}), \mathbb{P})$ such that

$$\mathbb{P}(\theta_T \in A) = \frac{1}{U}\int\limits_{T_0}^{T} w(\tau) I_A \mathrm{d}\tau, \quad A \in \mathcal{B}(\mathbb{R}).$$

Define

$$X_{T,n,w}(s) = g_n(s + i\theta_T).$$

Then we have from Theorem 7 that

$$X_{T,n,w}(s) \xrightarrow[T\to\infty]{\mathcal{D}} X_{n,w}(s). \tag{11}$$

In the next step we prove that the family of probability measures $\{P_{n,w} : n \in \mathbb{N}\}$ is tight. It is well known, see, for example, Lemma 1.7.1 of [7], that there exists a sequence $\{K_l : l \in \mathbb{N}\}$ of compact subsets of D such that $D = \bigcup\limits_{l=1}^{\infty} K_l$, $K_l \subset K_{l+1}$, and if K is a compact subset of D, then $K \subseteq K_l$ for some l. Then

$$\varrho(g_1, g_2) = \sum_{l=1}^{\infty} 2^{-l} \frac{\sup\limits_{s\in K_l} |g_1(s) - g_2(s)|}{1 + \sup\limits_{s\in K_l} |g_1(s) - g_2(s)|}, \quad g_1, g_2 \in H(D),$$

is a metric on $H(D)$ which induces its topology of uniform convergence on compacta.

For every $M_l > 0$, $l \in \mathbb{N}$, we have

$$\frac{1}{U}\int_{T_0}^{T} w(\tau) I_{\{\tau:\, \sup_{s\in K_l} |g_n(s+i\tau)|>M_l\}} \mathrm{d}\tau \leq$$

$$\leq \frac{1}{M_l U}\int_{T_0}^{T} w(\tau) \sup_{s\in K_l} |g_n(s+i\tau)| \mathrm{d}\tau. \tag{12}$$

Moreover, by Theorem 9 and Cauchy integral formula

$$\sup_{n\in\mathbb{N}} \limsup_{T\to\infty} \frac{1}{U}\int_{T_0}^{T} w(\tau) \sup_{s\in K_l} |g_n(s+i\tau)| \mathrm{d}\tau \leq$$

$$\leq \sup_{n\in\mathbb{N}} \limsup_{T\to\infty} \frac{1}{U}\int_{T_0}^{T} w(\tau) \sup_{s\in K_l} |f_2(s+i\tau) - g_n(s+i\tau)| \mathrm{d}\tau +$$

$$+ \sup_{n\in\mathbb{N}} \limsup_{T\to\infty} \frac{1}{U}\int_{T_0}^{T} w(\tau) \sup_{s\in K_l} |f_2(s+i\tau)| \mathrm{d}\tau \ll$$

$$\ll 1 + \limsup_{T\to\infty} \frac{1}{U}\left(\sqrt{U}\left(\int_{-T}^{T} w(t)|f_2(\sigma+it)|^2 \mathrm{d}t\right)^{\frac{1}{2}}\right)$$

with some $\sigma > \sigma_1$. Therefore, this and (8) show that

$$\sup_{n\in\mathbb{N}} \limsup_{T\to\infty} \frac{1}{U}\int_{T_0}^{T} w(\tau) \sup_{s\in K_l} |g_n(s+i\tau)| \mathrm{d}\tau \leq R_l < \infty, \tag{13}$$

$l \in \mathbb{N}$. Now we take $M_l = M_{l,\varepsilon} = R_l 2^l \varepsilon^{-1}$, where $\varepsilon > 0$ is arbitrary number. Then, in view of (12) and (13),

$$\limsup_{T\to\infty} \mathbb{P}\left(\sup_{s\in K_l} |X_{T,n,w}(s)| > M_l\right) =$$

$$= \frac{1}{U}\int_{T_0}^{T} w(\tau) I_{\{\tau:\, \sup_{s\in K_l} |g_n(s+i\tau)|>M_l\}} \mathrm{d}\tau \leq \frac{\varepsilon}{2^l}, \quad l \in \mathbb{N}, \quad n \in \mathbb{N}. \tag{14}$$

Relation (11) implies

$$\sup_{s\in K_l} |X_{T,n,w}(s)| \xrightarrow[T\to\infty]{\mathcal{D}} \sup_{s\in K_l} |X_{n,w}(s)|, \quad l\in\mathbb{N}, \quad n\in\mathbb{N}.$$

Therefore, by (14),

$$\mathbb{P}\left(\sup_{s\in K_l} |X_{n,w}(s)| > M_l\right) \leq \frac{\varepsilon}{2^l}, \quad l\in\mathbb{N}, \quad n\in\mathbb{N}. \tag{15}$$

Define

$$H_\varepsilon = \{g\in H(D) : \sup_{s\in K_l} |g(s)| \leq M_{l,\varepsilon}, \quad l\in\mathbb{N}\}.$$

Since the set H_ε is uniformly bounded on every compact of D, it is a compact subset of $H(D)$. Moreover, in view of (15),

$$\mathbb{P}\left(X_{n,w}(s)\in H_\varepsilon\right) \geq 1-\varepsilon\sum_{l=1}^{\infty}\frac{1}{2^l} = 1-\varepsilon, \quad n\in\mathbb{N}.$$

This shows that the family $\{P_{n,w} : n\in\mathbb{N}\}$ is tight, therefore, by the Prokhorov theorem, see, for example, [1], Theorem 6.1, it is relatively compact. Hence, there exists a sequence $\{P_{n_k,w}\}\subset\{P_{n,w}\}$ such that $P_{n_k,w}$ converges weakly to a certain probability measure $P_{f_2,w}$ on $(H(D),\mathcal{B}(H(D)))$ as $k\to\infty$. In other words,

$$X_{n_k,w}(s) \xrightarrow[k\to\infty]{\mathcal{D}} P_{f_2,w}. \tag{16}$$

Define once one $H(D)$-valued random element $X_{T,w}(s)$ by

$$X_{T,w}(s) = f_2(s+i\theta_T).$$

Then, by Theorem 9, for every $\varepsilon>0$,

$$\lim_{n\to\infty}\limsup_{T\to\infty}\mathbb{P}\left(\rho(X_{T,w}(s),X_{T,n,w}(s))\geq\varepsilon\right)\leq$$

$$\leq \lim_{n\to\infty}\limsup_{T\to\infty}\frac{1}{U\varepsilon}\int_{T_0}^{T} w(\tau)\rho(f_2(s+i\tau),g_n(s+i\tau))\mathrm{d}\tau = 0.$$

This, (11), (16) and Theorem 4.2 of [1] show that

$$X_{T,w}(s) \xrightarrow[T\to\infty]{\mathcal{D}} P_{f_2,w}, \tag{17}$$

and the theorem is proved. □

Theorem 12. *Suppose that the hypotheses of Theorem* 6 *are valid. Then the probability measures $P_{T,f_2,w}$ and $\hat{P}_{T,f_2,w}$ both converge weakly to the same probability measure on $(H(D),\mathcal{B}(H(D)))$ as $T\to\infty$.*

P r o o f. In view of Theorem 11, it remains to show that the measure $\hat{P}_{T,f_2,w}$ also converges weakly to the measure $P_{f_2,w}$ as $T \to \infty$. We also preserve the notation of the proof of Theorem 11.

By Theorem 8, the measures $P_{T,n,w}$ and $\hat{P}_{T,n,w}$ converge weakly to some probability measure $P_{n,w}$ on $(H(D), \mathcal{B}(H(D)))$ as $T \to \infty$. Let

$$\hat{X}_{T,n,w}(s) = g_n(s + i\theta_T, \omega).$$

Then we have that

$$\hat{X}_{T,n,w}(s) \xrightarrow[T\to\infty]{\mathcal{D}} X_{n,w}(s), \tag{18}$$

where the $H(D)$-valued random element $X_{n,w}(s)$ was defined in the proof of Theorem 11. The relation (17) shows that the measure $P_{f_2,w}$ is independent of the choice of the sequence $\{P_{n_k,w}\}$. This and the relative compactness of the family $\{P_{n,w}\}$ imply the relation

$$X_{n,w}(s) \xrightarrow[n\to\infty]{\mathcal{D}} P_{f_2,w}. \tag{19}$$

Let

$$\hat{X}_{T,w}(s) = f_2(s + i\theta_T, \omega).$$

Then, by Theorem 10, for every $\varepsilon > 0$ and almost all $\omega \in \Omega$,

$$\lim_{n\to\infty} \limsup_{T\to\infty} \mathbb{P}\left(\rho(\hat{X}_{T,w}(s), \hat{X}_{T,n,w}(s)) \geq \varepsilon\right) \leq$$

$$\leq \lim_{n\to\infty} \limsup_{T\to\infty} \frac{1}{U\varepsilon} \int_{T_0}^{T} w(\tau)\rho(f_2(s+i\tau,\omega), g_n(s+i\tau,\omega))\mathrm{d}\tau = 0.$$

This, (18) and (19) together with Theorem 4.2 of [1] again imply the weak convergence of $\hat{P}_{T,f_2,w}$ to $P_{f_2,w}$ as $T \to \infty$. $\square$

Denote by P_{f_2} the distribution of the random element $f_2(s, \omega)$.

Theorem 13. *Suppose that the hypotheses of Theorem 6 are valid. Then the probability measure $P_{T,f_2,w}$ converges weakly to P_{f_2} as $T \to \infty$.*

P r o o f. In view of Theorem 12, it suffices to show that the measure $P_{f_2,w}$ coincides with P_{f_2}.

Let $A \in \mathcal{B}(H(D))$ be an arbitrary fixed continuity set of the limit measure $P_{f_2,w}$ in Theorem 12. Then Theorem 12 and an equivalent of weak convergence of probability measures, see Theorem 2.1 of [1], imply

$$\lim_{T\to\infty} \frac{1}{U} \int_{T_0}^{T} w(\tau) I_{\{\tau : f_2(s+i\tau,\omega)\in A\}} \mathrm{d}\tau = P_{f_2,w}(A). \tag{20}$$

On the probability space $(\Omega, \mathcal{B}(\Omega), m_H)$, define the random variable θ by

$$\theta(\omega) = \begin{cases} 1 & \text{if } f_2(s,\omega) \in A, \\ 0 & \text{if } f_2(s,\omega) \notin A. \end{cases}$$

Then, obviously,

$$\mathbb{E}\theta = \int\limits_{\Omega} \theta(\omega)\mathrm{d}m_H = m_H\left(\omega \in \Omega : f_2(s,\omega) \in A\right) = P_{f_2}(A). \tag{21}$$

Since the one-parameter group $\{\varphi_\tau : \tau \in \mathbb{R}\}$ is ergodic, the random process $\theta(\varphi_\tau(\omega))$ is ergodic, too. Hence, using the hypothesis (6) with $v = 0$, we have that, for almost all $\omega \in \Omega$,

$$\lim_{T\to\infty} \frac{1}{U} \int\limits_{T_0}^{T} w(\tau)\theta(\varphi_\tau(\omega))\mathrm{d}\tau = \mathbb{E}\theta. \tag{22}$$

From the definitions of θ and $\varphi_\tau(\omega)$ it follows that

$$\frac{1}{U} \int\limits_{T_0}^{T} w(\tau)\theta(\varphi_\tau(\omega))\mathrm{d}\tau = \frac{1}{U} \int\limits_{T_0}^{T} w(\tau) I_{\{\tau : f_2(s,\varphi_\tau(\omega)) \in A\}}\mathrm{d}\tau =$$

$$= \frac{1}{U} \int\limits_{T_0}^{T} w(\tau) I_{\{\tau : f_2(s+i\tau,\omega) \in A\}}\mathrm{d}\tau.$$

Therefore, taking into account (21) and (22), we obtain that, for almost all $\omega \in \Omega$,

$$\lim_{T\to\infty} \frac{1}{U} \int\limits_{T_0}^{T} w(\tau) I_{\{\tau : f_2(s+i\tau,\omega) \in A\}}\mathrm{d}\tau = P_{f_2}(A).$$

Thus, by (20), for all continuity sets A of $P_{f_2,w}$,

$$P_{f_2,w}(A) = P_{f_2}(A).$$

Hence, $P_{f_2,w}(A) = P_{f_2}(A)$ for all $A \in \mathcal{B}(H(D))$. The theorem is proved. $\square$

5. Proof of theorems 5 and 6

First we observe that

$$f_1(s) = \prod_{j=1}^{r} \left(1 - \mathrm{e}^{\lambda_1(s_j - s)}\right) = \sum_{m=0}^{r} b_m \mathrm{e}^{-\lambda_1 m s}$$

is a Dirichlet polynomial with some coefficients b_m and exponents $m\lambda_1$. Therefore, an application of Lemma 1 shows that the probability measure

$$\frac{1}{U}\int_{T_0}^{T} w(\tau) I_{\{\tau : f_1(s+i\tau)\in A\}} \mathrm{d}\tau, \quad A \in \mathcal{B}(H(D)),$$

converges weakly to the distribution P_{f_1} the $H(D)$-valued random element

$$f_1(s,\omega) = \prod_{j=1}^{r}\left(1-\omega(1)\mathrm{e}^{\lambda_1(s_j-s)}\right) = \sum_{m=0}^{r} b_m \omega^m(1)\mathrm{e}^{-\lambda_1 ms}$$

as $T\to\infty$.

Now let $H^2(D) = H(D)\times H(D)$, and

$$P_{T,f_1,f_2,w}(A) = \frac{1}{U}\int_{T_0}^{T} w(\tau) I_{\{\tau : (f_1(s+i\tau), f_2(s+i\tau))\in A\}} \mathrm{d}\tau, \quad A \in \mathcal{B}(H^2(D)),$$

Lemma 2. *Suppose that the hypotheses of Theorem* 5 *are satisfied. Then on* $(H^2(D), \mathcal{B}(H^2(D)))$ *there exists a probability measure* $P_{f_1,f_2,w}$ *such that* $P_{T,f_1,f_2,w}$ *converges weakly to* P_{f_1,f_2} *as* $T\to\infty$.

P r o o f. Let, for $\underline{g_j} = (g_{j1}, g_{j2}) \in H^2(D)$, $j = 1, 2$,

$$\underline{\rho}(\underline{g_1}, \underline{g_2}) = \max_{1\le j\le 2} \left(\rho(g_{11}, g_{21}), \rho(g_{12}, g_{22})\right).$$

Then $\underline{\rho}$ is a metric on $H^2(D)$ inducing its topology. Using this metric and repeating the arguments of the proof of Theorem 11 with obvious changes, we obtain the statement of the lemma. □

Define the $H^2(D)$-valued random element $F(s,\omega)$ by

$$F(s,\omega) = (f_1(s,\omega), f_2(s,\omega)),$$

and denote by P_F its distribution.

Lemma 3. *Suppose that the hypotheses of Theorem 6 are valid. Then the measure* $P_{T,f_1,f_2,w}$ *converges weakly to* P_F *as* $T\to\infty$.

P r o o f. The lemma is obtained in the same way as Theorem 13 with using the metric $\underline{\rho}$. □

Proof of Theorem 5. Define the function $h : H^2(D) \to M(D)$, by the formula

$$h(g_1, g_2) = \frac{g_2}{g_1}, \quad g_1, g_2 \in H(D).$$

Since the metric d satisfies

$$d(g_1, g_2) = d\left(\frac{1}{g_1}, \frac{1}{g_2}\right),$$

the function h is continuous. Therefore, by Lemma 2 and Theorem 5.1 of [1], the measure $P_{T,w} = P_{T,f_1,f_2,w}h^{-1}$ converges weakly to the measure $P_{f_1,f_2,w}h^{-1}$ as $T \to \infty$. □

Proof of Theorem 6. Similarly to the proof of Theorem 5, using Lemma 3, we find that the measure $P_{f_1,f_2,w}$ converges weakly to the measure $P_F h^{-1}$ as $T \to \infty$. However,

$$P_F h^{-1}(A) = m_H\left(\omega \in \Omega : \frac{f_2(s,\omega)}{f_1(s,\omega)} \in A\right), A \in \mathcal{B}(M(D)).$$

Since

$$f_2(s,\omega) = \sum_{j=0}^{r}\sum_{m=1}^{\infty} a_{m,j}\omega^j(1)\omega(m)\mathrm{e}^{-(\lambda_m + j\lambda_1)s} =$$

$$= \prod_{j=1}^{r}\left(1 - \omega(1)\mathrm{e}^{\lambda_1(s_j - s)}\right)\sum_{m=1}^{\infty} a_m\omega(m)\mathrm{e}^{-\lambda_m s},$$

hence we have that $P_{T,w}$ converges weakly to the distribution of the random element $f(s,\omega)$ as $T \to \infty$. □

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