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uniform distribution theory

ON WEIGHTED UNIFORM DENSITY

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ABSTRACT. Uniform density (also known as Banach density) was often used in various branches of mathematics, in particular in number theory and ergodic theory. Several characterizations of uniform density were given in the paper [Gáliková, Z. – László, B. – Šalát, T.:*Remarks on uniform density of sets of integers*, Acta Acad. Paed. Agriensis, 2002].

The notion of uniform density was recently generalized to weighted uniform density in the paper [Giuliano Antonini, R. – Grekos, G.: *Weighted uniform densities,* Journal de théorie des nombres de Bordeaux, 2007]. Some sufficient conditions for the existence of upper and lower weighted uniform density for every subset of the set of integers have been obtained in this paper.

We show that for positive weights the upper and lower weighted uniform density always exist. We also show that the alternative characterizations of the uniform density remain valid for the weighted uniform density as well. Moreover, we investigate related characterizations of the upper and lower weighted uniform density.

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1. Introduction

The notion of uniform density was introduced in [5]. It coincides with the notion of Banach density and was used in various parts of mathematics, in particular in number theory and ergodic theory, see for example [3, 6, 9, 21, 22, 23]. Several equivalent characterizations of the uniform density can be found in [10].

Georges Grekos and Rita Giuliano Antonini generalized this notion in the paper [11] to weighted uniform density (in a similar way as the asymptotic density was generalized to weighted density, see for example [1, 18, 12, 17, 20]).

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In this paper they have proved sufficient conditions for the existence of upper and lower weighted uniform density for every subset A of the set of integers $\mathbb{N}$.

In Section 3 of this paper we show that for positive weights the upper and lower weighted uniform density always exist. This extends the results obtained in [11].

The definition of weighted uniform density from [11] follows closely the definition of uniform density from [5, 6]. As we have already mentioned, the upper and lower uniform density is the same notion as the upper and lower Banach density ([9, p.72, Definition 3.7], [22]). For the detailed proof of the equivalence of these two notions we refer to [15]. Using the results from Section 3 we show that, for weight sequences with infinite sum, the lower and upper uniform density can be understood as limit superior and inferior of some double sequence, which simplifies the original definition. From this we get that a generalization, which would mimic the definition of Banach density, is in fact equivalent to the definition from [11].

In Section 5 we give several characterizations of upper and lower weighted uniform density generalizing some results from [10]. These generalizations are not true for arbitrary subsets of $\mathbb{N}$ and arbitrary weight sequences, but we give several sufficient conditions when they hold. In Section 6 we include several examples showing that the assumptions used in our results cannot be completely omitted.

2. Preliminaries

Let $c = (c_i)_{i=1}^{\infty}$ be any sequence of positive real numbers. For any set $A \subseteq \mathbb{N}$ and any interval I in $\mathbb{N}$ (ordered with the usual linear order) we define the *weight of the set A in the interval I* by

$$A_c(I) = \sum_{i \in I} c_i \chi_A(i).$$

The ratio

$$\gamma(A, I, c) := \frac{A_c(I)}{\mathbb{N}_c(I)}$$

is called the *relative weight of the set A in the interval I* with respect to the weight sequence c. In particular, when $I = (n, n+h]$ we will denote

$$A_c(n, n+h] := \sum_{i=n+1}^{n+h} c_i \chi_A(i) =: s_{n,h}(A),$$

$$\mathbb{N}_c(n, n+h] := \sum_{i=n+1}^{n+h} c_i =: S_{n,h}\,,$$

$$\gamma_{n,h}(A) := \frac{s_{n,h}(A)}{S_{n,h}}\,,$$

$$l_h(A) = \liminf_{n\to\infty} \gamma_{n,h}(A), \quad L_h(A) = \limsup_{n\to\infty} \gamma_{n,h}(A),$$

$$\lambda_h(A) = \inf_{n\in\mathbb{N}} \gamma_{n,h}(A), \qquad \Lambda_h(A) = \sup_{n\in\mathbb{N}} \gamma_{n,h}(A).$$

We will often omit the set A in the above notation if it is understood from the context.

Upper and *lower weighted uniform density* were defined in [11] as

$$\overline{u}_c(A) = \lim_{h\to\infty} L_h(A), \qquad \underline{u}_c(A) = \lim_{h\to\infty} l_h(A).$$

If $\overline{u}_c(A) = \underline{u}_c(A) =: u_c(A)$ the common value $u_c(A)$ is called *weighted uniform density of the set A* with respect to the weight sequence c or briefly *c-weighted uniform density of A*.

If $c_i = 1$ for each $i \in \mathbb{N}$, we obtain the (lower and upper) *uniform density*. It is denoted by $\underline{u}(A)$, $\overline{u}(A)$ and $u(A)$, respectively.

We will need the following lemmas. An easy proof of Lemma 2.1 is omitted.

Lemma 2.1. *Let $a_1, a_2, \dots a_k$ be real numbers and $b_1, b_2, \dots, b_k$ be positive real numbers. Then*

$$\min \frac{a_i}{b_i} \le \frac{a_1 + a_2 + \dots + a_k}{b_1 + b_2 + \dots + b_k} \le \max \frac{a_i}{b_i}.$$

Moreover, the equalities hold if and only if all quotients a_i/b_i are equal.

Lemma 2.2. *Let s, S, T, U and V be real numbers such that $0 < T < S$ and*

$$0 \le \frac{s-T}{S-T} < U < V < \frac{s}{S} \le 1.$$

Then

$$0 < S\frac{V-U}{1-U} < T.$$

Proof. From the obvious inequalities $VS < s$ and $s - T < U(S - T)$ we get $VS - T < US - UT$ and the last inequality is equivalent with $S\frac{V-U}{1-U} < T$. □

The next lemma gives some monotonicity conditions which we will use in the sequel.

Lemma 2.3. *Let $h, h' \in \mathbb{N}$ and $A \subseteq \mathbb{N}$. Then*

$$\Lambda_{h+h'} \leq \max\{\Lambda_h, \Lambda_{h'}\}, \tag{2.1}$$

$$L_{h+h'} \leq \max\{L_h, L_{h'}\}, \tag{2.2}$$

$$h|h' \Rightarrow \Lambda_{h'} \leq \Lambda_h\,, \tag{2.3}$$

$$h|h' \Rightarrow L_{h'} \leq L_h\,. \tag{2.4}$$

Proof. Inequalities (2.1) and (2.2) follow from the fact that $s_{n,h+h'}(A) = s_{n,h}(A) + s_{n+h,h'}(A)$ and Lemma 2.1. Properties (2.3) and (2.4) follow from (2.1) and (2.2), respectively. □

3. The existence of the upper and lower weighted uniform density

In this section we show that the upper and lower weighted uniform density exist for any weight sequence $(c_i)_{i=1}^{\infty}$ of positive integers.

Theorem 3.1. *Let $A \subseteq \mathbb{N}$. Then*

$$\lim_{h\to\infty} L_h(A) = \inf_{h\in\mathbb{N}} L_h(A), \tag{3.1}$$

$$\lim_{h\to\infty} l_h(A) = \sup_{h\in\mathbb{N}} l_h(A). \tag{3.2}$$

Proof. As $l_h(A) = 1 - L_h(\mathbb{N} \setminus A)$ it suffices to prove (3.1). We will do it by contradiction.

Let us assume that (3.1) does not hold and let $L = \inf_{h\in\mathbb{N}} L_h$. Then there exists an $\varepsilon > 0$ such that the set $H = \{h \in \mathbb{N};\ L_h > L + \varepsilon\}$ is infinite. Moreover, there exists an h_0 such $L_{h_0} < L + \varepsilon/2$. Put $U = L + \varepsilon/2$ and $V = L + \varepsilon$.

Let $d \in \mathbb{N}$ and $h = dh_0 + 1$. For any $h' > hh_0$ there exist nonnegative integers e and f such that $h' = eh + fh_0$. By Lemma 2.3 we have $L_{h'} \leq \max\{L_h, L_{h_0}\}$. As H is infinite, necessarily $h \in H$, that is, $L_h > V$.

By the properties of limes superior, there exists $n = n_d \in \mathbb{N}$ such that

$$\frac{s_{n,h}(A)}{S_{n,h}} > V \quad \text{and} \quad \forall j \geq 0 : \frac{s_{n+j,h_0}(A)}{S_{n+j,h_0}} < U.$$

By Lemma 2.1 also $s_{n,dh_0}(A)/S_{n,dh_0} < U$, hence $n + h = n + dh_0 + 1 \in A$ and by Lemma 2.2 we have $c_{n+dh_0+1} > (V - U)S_{n,h}/(1 - U)$.

Similarly, for any $i \in \{0, 1, \ldots, d\}$ we have $c_{n+ih_0+1} \in A$,

$$\frac{s_{n,h}(A) - c_{n+ih_0+1}}{S_{n,h} - c_{n+ih_0+1}} = \frac{\sum_{k=0}^{i-1} s_{n+kh_0,h_0}(A) + \sum_{k=i}^{d-1} s_{n+kh_0+1,h_0}(A)}{\sum_{k=0}^{i-1} S_{n+kh_0,h_0} + \sum_{k=i}^{d-1} S_{n+kh_0+1,h_0}} < U,$$

and by Lemma 2.2

$$\forall i \in \{0, 1, \ldots, d\} : \; c_{n+ih_0+1} > \frac{V - U}{1 - U} S_{n,h} \, .$$

Therefore,

$$S_{n,h} \geq \sum_{i=0}^{d} c_{n+ih_0+1} > (d + 1)\frac{V - U}{1 - U} S_{n,h} \, ,$$

that is $d + 1 < (1 - U)/(V - U)$. As d was arbitrary, this is a contradiction. □

By minor modifications of the above argument it is easy to prove also the following result.

Theorem 3.2. *Let $A \subseteq \mathbb{N}$. Then*

$$\lim_{h\to\infty} \Lambda_h(A) = \inf_{h\in\mathbb{N}} \Lambda_h(A), \tag{3.3}$$

$$\lim_{h\to\infty} \lambda_h(A) = \sup_{h\in\mathbb{N}} \lambda_h(A). \tag{3.4}$$

In the case of the (standard) uniform density we have $\inf L_h(A) = \inf \Lambda_h(A)$ for any set $A \subseteq \mathbb{N}$. As the following theorem shows, this property holds whenever $\sum_{n=1}^{\infty} c_n$ is divergent.

Theorem 3.3. *Let $A \subseteq \mathbb{N}$ and $\sum_{n=1}^{\infty} c_n = +\infty$. Then*

$$\inf_{h\in\mathbb{N}} L_h(A) = \inf_{h\in\mathbb{N}} \Lambda_h(A), \tag{3.5}$$

$$\sup_{h\in\mathbb{N}} l_h(A) = \sup_{h\in\mathbb{N}} \lambda_h(A). \tag{3.6}$$

P r o o f. Let $L = \inf_{h\in\mathbb{N}} L_h$. As $L_h \leq \Lambda_h$, it suffices to show that

$$\forall \varepsilon > 0 \, \exists h \in \mathbb{N} : \; \Lambda_h \leq L + \varepsilon.$$

Let $0 < \varepsilon < 1 - L$ be fixed and put $U = L + \varepsilon/2$, and $V = L + \varepsilon$. Then

$$\exists h_0, n_0 \in \mathbb{N} \, \forall n > n_0 : \; \frac{s_{n,h_0}(A)}{S_{n,h_0}} < L + \varepsilon/2 = U. \tag{3.7}$$

Without the loss of generality we can assume $h_0 | n_0$.

As $\sum_{n=0}^{\infty} c_n = +\infty$ we have

$$\forall\delta > 0\ \exists n_\delta \in \mathbb{N}\ \forall i \in \{1, 2, \dots, n_0\} :\ S_{i,n_0} < \delta S_{n_0+i,n_\delta}.$$

Again, we can assume $h_0|n_\delta$.

Let $h = n_0 + n_\delta$. As $h_0|h$ by Lemma 2.3 and (3.7) for any $n > n_0$ we have $s_{n,h}(A)/S_{n,h} < U$. Therefore $\Lambda_h > V$ if and only if there exists $i \in \{1, 2, \dots, n_0\}$ such that $s_{i,h}(A)/S_{i,h} > V$. Put $s = S_{i,n_0} + s_{n_0+i,n_\delta}(A)$, $S = S_{i,h}$ and $T = S_{i,n_0}$. Then $s/S > V$ and, as $h_0|n_0$, $(s - T)/(S - T) < U$. Therefore, by Lemma 2.2 we have $T > (V - U)S/(1 - U)$, that is,

$$S_{i,n_0} > \frac{V - U}{1 - U} S_{n_0+i,n_\delta}.$$

However, by the choice of n_δ for $\delta < (V-U)/(1-U)$ the above condition cannot hold. Hence for any such δ we have $\Lambda_h < V$. ☐

Remark 3.4. If $\sum_{n=1}^{\infty} c_n = C$, then for $A = \{1\}$ we have $\inf_{h\in\mathbb{N}} L_h(A) = 0$ while $\inf_{h\in\mathbb{N}} \Lambda_h(A) = c_1/C$. Therefore the condition $\sum_{n=1}^{\infty} c_n = +\infty$ is in fact necessary for (3.5), (3.6) to hold for every set $A \subseteq \mathbb{N}$.

4. Weighted uniform densities as uniform limits of relative weights

In this section we give a characterization of upper and lower weighted uniform densities as the uniform limit superior and limit inferior of the double sequence of relative weights $\gamma_{n,h}$.

We observe first that the uniform convergence of a double sequence $a_{s,t}$ for $t \to \infty$ uniformly in s, which we will denote by $\lim\limits_{t\rightrightarrows\infty} a_{s,t}$, is precisely the convergence of this double sequence considered as a net on the directed set $D = (\mathbb{N} \times \mathbb{N}, \prec)$, $(s,t) \prec (s',t') \Leftrightarrow t \le t'$. For any net $(x_\alpha)_{\alpha\in I}$ of real numbers defined on a general directed set $(I, \succeq)$ limit superior of the net is defined as

$$\limsup x_\alpha = \lim_{\alpha\in I} \sup_{\beta\succeq\alpha} x_\beta = \inf_{\alpha\in I} \sup_{\beta\succeq\alpha} x_\beta.$$

It can be defined equivalently as the largest limit point of the net.

In the case of double sequence defined on the directed set D we obtain that $\overline{L} = \limsup\limits_{t\to\infty} a_{s,t}$ uniformly in s if and only if

$$\lim_{t\to\infty} (\sup_{\substack{t'\ge t\\ s\in\mathbb{N}}} a_{s,t'}) = \inf_t (\sup_{\substack{t'\ge t\\ s\in\mathbb{N}}} a_{s,t'}) = \overline{L}.$$

We will use notation

$$\limsup_{t \rightrightarrows \infty} a_{s,t} = \overline{L}.$$

Similar notation is used when $\underline{L} = \liminf_{t\to\infty} a_{s,t}$ uniformly in s and we denote it by

$$\liminf_{t \rightrightarrows \infty} a_{s,t} = \underline{L}.$$

More about limes superior and inferior of nets can be found for example in [2, p. 32], [19, p. 217], [24, 7.43–7.47]. We will use several times the fact that the limit superior of a subnet is less or equal to the limit superior of a net. The dual inequality holds for limit inferior.

Now we can give the characterization of the upper and lower weighted uniform density in terms of the uniform convergence of relative weights.

From now on we will always assume that the weight sequence $(c_i)_{i=1}^\infty$ fulfills the conditions $c_i > 0$ and $\sum_{i=0}^\infty c_i = +\infty$. (If we allow zero weights of elements then we potentially allow arbitrarily long intervals with zero weight. In the case that the sum of the weights is finite, some finite sets have non-zero upper density. Hence both these conditions are quite natural and working with sequences not fulfilling them would lead to unnecessary complications.)

Theorem 4.1. *Let $c = (c_i)_{i=1}^\infty$ be a weight sequence. Then for any $A \subseteq \mathbb{N}$ the equalities hold*

$$\overline{u}_c(A) = \limsup_{h \rightrightarrows \infty} \gamma_{n,h}(A),$$
$$\underline{u}_c(A) = \liminf_{h \rightrightarrows \infty} \gamma_{n,h}(A).$$

Proof. We give the proof of the first equality, the second one can be proved by an easy modification of it. Let

$$\alpha = \limsup_{h \rightrightarrows \infty} \gamma_{n,h} = \lim_{h\to\infty} \sup_{h' \geq h, n \in \mathbb{N}} \gamma_{n,h'}.$$

Recall that for $\Lambda_h = \sup_{n\in\mathbb{N}} \gamma_{n,h}$ we have by Theorems 3.1 and 3.3 that

$$\overline{u}_c(A) = \lim_{h\to\infty} \Lambda_h.$$

From the obvious equality

$$\sup_{h' \geq h, n \in \mathbb{N}} \gamma_{n,h'} = \sup_{h' \geq h} \left(\sup_{n\in\mathbb{N}} \gamma_{n,h'} \right)$$

we get

$$\alpha = \lim_{h\to\infty} \sup_{h' \geq h} \Lambda_{h'} = \limsup_{h\to\infty} \Lambda_h.$$

By Theorem 3.2 the sequence $(\Lambda_h)_{h=1}^\infty$ has a limit, thus

$$\limsup_{h\to\infty} \Lambda_h = \lim_{h\to\infty} \Lambda_h \,,$$

that is, $\alpha = \overline{u}_c(A)$. □

As a direct consequence of the above result we obtain the following generalization of [10, Theorems 1.1 and 1.2].

Corollary 4.2. *Let $(c_i)_{i=1}^\infty$ be a weight sequence. Then for any $A \subseteq \mathbb{N}$*

$$\lim_{h\rightrightarrows\infty} \gamma_{n,h}(A) = u_c(A)$$

in the sense that $u_c(A)$ exists if and only if the limit on the left hand side exists and they have the same value.

This result is interesting for two reasons. Firstly, we have obtained an equivalent definition of the upper and lower weighted uniform density that is similar to the definition of the weighted density – it is defined using (some kind of) lim sup and lim inf. Secondly, as we have already mentioned, the upper Banach density is usually defined in this way. Thus the generalization of upper and lower Banach density to the case of weighted densities would lead precisely to the values indicated in the theorem.

5. An alternative characterization of the weighted uniform density

In this section we would like to generalize the following result from [10, Theorem 3.1]:

$$\lim_{p\rightrightarrows\infty} \frac{p}{a_{k+p} - a_{k+1}} = u(A) \tag{5.1}$$

in the sense that $u(A)$ exists if and only if the limit in (5.1) exists and they have the same value.

The above characterization makes sense only for infinite sets. Since we always assume that $\sum_{n=0}^\infty c_n = +\infty$, every finite set has zero uniform weighted density, thus we can assume that A is an infinite set. Therefore all limits superior and inferior defined bellow will be automatically considered zero for finite sets.

As we have already seen, the upper weighted uniform density is equal to

$$\limsup_{|I|\to\infty} \frac{A_c(I)}{\mathbb{N}_c(I)} \,,$$

that is, if we denote $\Lambda_h = \sup_{|I|=h} \frac{A_c(I)}{\mathbb{N}_c(I)}$ for $h \in \mathbb{N}$, then $\overline{u}_c(A) = \limsup \Lambda_h$, (in fact, it is equal to $\lim_{h\to\infty} \Lambda_h$). In a similar way, corresponding to (5.1), we will consider the intervals containing the same number of elements of A instead of the intervals of fixed length. Thus we will investigate

$$\limsup_{|I\cap A|\to\infty} \frac{A_c(I)}{\mathbb{N}_c(I)} \qquad \textit{and} \qquad \liminf_{|I\cap A|\to\infty} \frac{A_c(I)}{\mathbb{N}_c(I)}\,.$$

If we look at all the intervals whose intersection with the set A is

$$\{a_p, \dots, a_{p+k}\}$$

and compare maximal and minimal possible value of the fraction $\frac{A_c(I)}{\mathbb{N}_c(I)}$ for such intervals, we see that these values are equal to

$$\overline{\gamma}_{p,k}(A) = \frac{A_c[a_p, a_{p+k}]}{\mathbb{N}_c[a_p, a_{p+k}]}\,,$$

$$\underline{\gamma}_{p,k}(A) = \frac{A_c(a_{p-1}, a_{p+k+1})}{\mathbb{N}_c(a_{p-1}, a_{p+k+1})}\,,$$

where we put $a_0 = 0$ by definition.

The above intervals are the maximal and minimal ones with the property $A \cap I = \{a_p, \dots, a_{p+k}\}$. Among the remaining intervals with this property also the intervals $(a_{p-1}, a_{p+k}]$ and $[a_p, a_{p+k+1})$ are in some sense extremal. Thus it is natural to study also the following fractions

$$\gamma'_{p,k}(A) = \frac{A_c(a_{p-1}, a_{p+k}]}{\mathbb{N}_c(a_p, a_{p+k}]}\,,$$

$$\gamma^*_{p,k}(A) = \frac{A_c[a_p, a_{p+k+1})}{\mathbb{N}_c[a_p, a_{p+k})}\,.$$

As a generalization of (5.1) we will investigate whether the following equalities hold.

$$\overline{u}_c(A) = \limsup_{k\rightrightarrows\infty} \overline{\gamma}_{p,k}(A), \tag{$\overline{\text{S}}$}$$

$$\overline{u}_c(A) = \limsup_{k\rightrightarrows\infty} \underline{\gamma}_{p,k}(A), \tag{$\underline{\text{S}}$}$$

$$\overline{u}_c(A) = \limsup_{k\rightrightarrows\infty} \gamma'_{p,k}(A), \tag{S'}$$

$$\overline{u}_c(A) = \limsup_{k\rightrightarrows\infty} \gamma^*_{p,k}(A). \tag{S*}$$

We will also investigate the analogous equalities for the lower weighted uniform density.

$$\underline{u}_c(A) = \liminf_{k\rightrightarrows\infty} \overline{\gamma}_{p,k}(A), \tag{$\bar{\mathrm{I}}$}$$

$$\underline{u}_c(A) = \liminf_{k\rightrightarrows\infty} \underline{\gamma}_{p,k}(A), \tag{$\underline{\mathrm{I}}$}$$

$$\underline{u}_c(A) = \liminf_{k\rightrightarrows\infty} \gamma'_{p,k}(A), \tag{I'}$$

$$\underline{u}_c(A) = \liminf_{k\rightrightarrows\infty} \gamma^*_{p,k}(A). \tag{I*}$$

The following inequalities are almost immediate (when comparing

$$\limsup_{k\rightrightarrows\infty} \overline{\gamma}_{p,k}(A) \qquad \text{and} \qquad \limsup_{h\rightrightarrows\infty} \gamma_{n,h}(A),$$

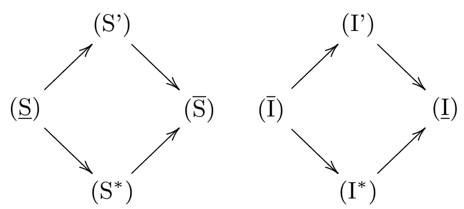
we use the fact that $(\overline{\gamma}_{p,k}(A))_{(p,k)\in\mathbb{N}\times\mathbb{N}}$ is a subnet of $(\gamma_{n,h}(A))_{(n,h)\in\mathbb{N}\times\mathbb{N}}$).

$$\underline{u}_c(A) = \liminf_{h\rightrightarrows\infty} \gamma_{n,h}(A) \le \liminf_{k\rightrightarrows\infty} \underline{\gamma}_{p,k}(A) \le \limsup_{k\rightrightarrows\infty} \overline{\gamma}_{p,k}(A) \le \limsup_{h\rightrightarrows\infty} \gamma_{n,h}(A) = \overline{u}_c(A).$$

$$\liminf_{k\rightrightarrows\infty} \underline{\gamma}_{p,k}(A) \le \liminf_{k\rightrightarrows\infty} \gamma'_{p,k}(A) \le \limsup_{k\rightrightarrows\infty} \gamma'_{p,k}(A) \le \limsup_{k\rightrightarrows\infty} \overline{\gamma}_{p,k}(A),$$

$$\liminf_{k\rightrightarrows\infty} \underline{\gamma}_{p,k}(A) \le \liminf_{k\rightrightarrows\infty} \gamma^*_{p,k}(A) \le \limsup_{k\rightrightarrows\infty} \gamma^*_{p,k}(A) \le \limsup_{k\rightrightarrows\infty} \overline{\gamma}_{p,k}(A).$$

This yields the following implications denoted by arrows in the diagram



Remark 5.1. On some occasions we will use the fact that in the definition of the upper (lower) uniform density we can use $\gamma_{n,h}$ for $n \ge n_0$ instead of $n \in \mathbb{N}$ and that the same fact is true of $\overline{\gamma}_{p,k}$, that is, for each $n_0, p_0 \in \mathbb{N}$

$$\overline{u}_c(A) = \limsup_{h\to\infty}(\sup_{\substack{n\ge n_0\\ h'\ge h}} \gamma_{n,h'}(A)) = \limsup_{h\rightrightarrows\infty} \gamma_{n+n_0,h}(A)\,,$$

$$\limsup_{p\rightrightarrows\infty} \overline{\gamma}_{p,k}(A) = \limsup_{k\to\infty}(\sup_{\substack{p\ge p_0\\ k'\ge k}} \overline{\gamma}_{p,k'}(A)) = \limsup_{p\rightrightarrows\infty} \overline{\gamma}_{p+p_0,k}(A).$$

To see this it is enough to note that $u_c(F) = \lim_{p \rightrightarrows \infty} \overline{\gamma}_{p,k}(F) = 0$ for every finite set. Note that here we use again the assumption $\sum_{i=1}^{\infty} c_i = +\infty$.

5.1. Conditions on weights

As can be seen from the examples in Section 6 neither of the equalities mentioned above is true for arbitrary sets and arbitrary weight sequences. Therefore, we will try to find some sufficient or necessary conditions on weights or sets such that these equalities are true.

Let us start by introducing some conditions on the sequence $(c_k)_{k=1}^{\infty}$, that are not too restrictive but, in some cases, they will be sufficient to obtain the desired results.

In the paper [18] Alexander's densities were studied — this is a weighted analogue of the asymptotic density. The authors have shown that the following condition

$$\lim_{n\to\infty} \frac{c_n}{\sum_{k=1}^{n} c_k} = 0 \tag{M}$$

is necessary and sufficient for the associated weighted density to have the Darboux property. In the paper [20, Theorem 2.3] the authors have shown that the weighted density is a compact submeasure if and only if the weight sequence has the property (M). Let us note that in [18] it is also shown that (M) is equivalent to

$$\lim_{n\to\infty} \frac{\max\{c_k; k = 1, \dots, n\}}{\sum_{k=1}^{n} c_k} = 0$$

for every sequence $(c_k)_{k=1}^{\infty}$ such that $\sum\limits_{k=1}^{\infty} c_k = +\infty$.

Motivated by these facts we tried to find a uniform analogy to the condition (M). The following conditions seem to be convenient.

$$\lim_{h\rightrightarrows\infty} \frac{\max\{c_t; t \in \mathbb{N} \cap (n, n+h]\}}{\mathbb{N}_c(n, n+h]} = 0, \tag{U^m}$$

$$\lim_{h\rightrightarrows\infty} \frac{c_{n+h}}{\mathbb{N}_c(n, n+h]} = 0, \tag{U^e}$$

$$\lim_{h\rightrightarrows\infty} \frac{c_n}{\mathbb{N}_c[n, n+h]} = 0. \tag{U^b}$$

The following condition has been used in [11].

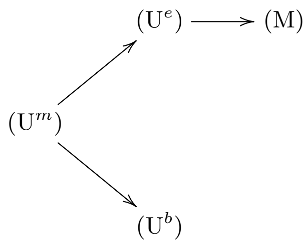
$$\lim_{r\to\infty} \sup_{n\in\mathbb{N}} \frac{S_{n+rq,q}}{S_{n,rq}} = 0 \qquad \text{for every fixed integer} \quad q \geq 1. \tag{5.2}$$

It can be shown the condition is equivalent to (M). Let us start by describing the relationships between these conditions.

Clearly, (U^m) $\Rightarrow$ (U^e) $\Rightarrow$ (M) and (U^m) $\Rightarrow$ (U^b). Every non-decreasing sequence fulfills (U^b) and every non-increasing sequence fulfills (U^e).

The weight sequences used in Examples 6.1, 6.3 show that (U^b) $\not\Rightarrow$ (M). The fact that (U^e) $\not\Rightarrow$ (U^m) is exemplified by the sequence from Example 6.2.

This can be expressed in the form of the following diagram where none of the implications indicated in the diagram by an arrow can be reversed and, if there is no path between two conditions, the corresponding implication does not hold.



Proposition 5.2. *A sequence* $(c_n)_{n=1}^{\infty}$ *fulfills* (U^m) *if and only if it fulfils both* (U^e) *and* (U^b).

Proof. Clearly, (U^m) implies both (U^e) and (U^b).

Now, suppose that $(c_n)_{n=1}^{\infty}$ fulfills both (U^e) and (U^b). Thus there exists an h_0 such that

$$\frac{c_{n+1}}{\mathbb{N}_c(n, n+l]} < \varepsilon \qquad \text{and} \qquad \frac{c_{n+l}}{\mathbb{N}_c(n, n+l]} < \varepsilon$$

whenever $l \geq h_0$.

Now, suppose that $h \geq 2h_0$. We will show that

$$\frac{\max\{c_t; t \in (n, n+h]\}}{\mathbb{N}_c(n, n+h]} < \varepsilon.$$

If $t \geq n + h_0$, then

$$\frac{c_t}{\mathbb{N}_c(n, n+h]} \leq \frac{c_t}{\mathbb{N}_c(t - h_0, t]} < \varepsilon$$

(using (U^e)). In the remaining case we have $t + h_0 < n + 2h_0 = n + h$. Thus

$$\frac{c_t}{\mathbb{N}_c(n, n+h]} \leq \frac{c_t}{\mathbb{N}_c[t, t+h_0]} < \varepsilon.$$

□

We next show how the conditions (U^m), (U^e) and (U^b) influence the relations between the equalities $(\underline{\mathrm{S}})$ through $(\overline{\mathrm{S}})$.

We have

$$\frac{A_c(a_p, a_{p+k}]}{\mathbb{N}_c(a_p, a_{p+k}]} = \frac{A_c[a_p, a_{p+k}] - c_{a_p}}{\mathbb{N}_c[a_p, a_{p+k}] - c_{a_p}} = \frac{A_c[a_p, a_{p+k}] - c_{a_p}}{\mathbb{N}_c[a_p, a_{p+k}]} \cdot \frac{\mathbb{N}_c[a_p, a_{p+k}]}{\mathbb{N}_c[a_p, a_{p+k}] - c_{a_p}}.$$

If we assume that $(c_n)_{n=1}^{\infty}$ fulfills (U^b), then

$$\lim_{k \rightrightarrows \infty} \frac{\mathbb{N}_c[a_p, a_{p+k}]}{\mathbb{N}_c[a_p, a_{p+k}] - c_{a_p}} = 1,$$

therefore

$$\limsup_{k\rightrightarrows\infty} \frac{A_c(a_p, a_{p+k}]}{\mathbb{N}_c(a_p, a_{p+k}]} = \limsup_{k\rightrightarrows\infty} \frac{A_c[a_p, a_{p+k}] - c_{a_p}}{\mathbb{N}_c[a_p, a_{p+k}]}.$$

Using (U^b) once again, we get that the right hand side is equal to $\limsup\limits_{k\rightrightarrows\infty} \frac{A_c[a_p,a_{p+k}]}{\mathbb{N}_c[a_p,a_{p+k}]}$.
Thus (U^b) implies

$$\limsup_{k\rightrightarrows\infty} \gamma'_{p,k}(A) = \limsup_{k\rightrightarrows\infty} \overline{\gamma}_{p,k}(A).$$

The same reasoning works for $\liminf$ instead of $\limsup$.

Note that we have used $(a_p, a_{p+k}]$ instead of $(a_{p-1}, a_{p+k}]$ when expressing $\gamma'_{p,k}(A)$. This is justified by Remark 5.1.

Under the assumption (U^e), an analogous derivation works for $\gamma^*_{p,k}(A)$ and $\overline{\gamma}_{p,k}(A)$. (The only difference is subtracting $c_{a_{p+k}}$ instead of c_{a_p}.)

In a similar manner we can establish the equality of the remaining pairs.

Summarizing this we get

Proposition 5.3. *If the weight sequence $(c_n)_{n=1}^\infty$ fulfills (U^b), then*

$$\limsup_{k\rightrightarrows\infty} \gamma'_{p,k}(A) = \limsup_{k\rightrightarrows\infty} \overline{\gamma}_{p,k}(A) \quad \text{and} \quad \limsup_{k\rightrightarrows\infty} \gamma^*_{p,k}(A) = \limsup_{k\rightrightarrows\infty} \underline{\gamma}_{p,k}(A).$$

If the weight sequence $(c_n)_{n=1}^\infty$ fulfills (U^e), then

$$\limsup_{k\rightrightarrows\infty} \gamma^*_{p,k}(A) = \limsup_{k\rightrightarrows\infty} \overline{\gamma}_{p,k}(A) \quad \text{and} \quad \limsup_{k\rightrightarrows\infty} \gamma'_{p,k}(A) = \limsup_{k\rightrightarrows\infty} \underline{\gamma}_{p,k}(A).$$

The analogous equalities hold for limit inferior.

Consequently, if $(c_n)_{n=1}^\infty$ fulfills (U^m), all four values of limit superior (limit inferior) are the same.

The above result implies the implications between the studied conditions indicated in the diagrams on Fig. 1. In particular, (U^m) implies the equivalence of all four conditions.

5.2. Syndetic sets

An alternative approach to putting conditions on the weight sequences is to find systems of subsets of $\mathbb{N}$ for which the studied equalities are satisfied without any restriction on weights. Now we will consider a class of subsets of $\mathbb{N}$ for which the equalities ($\overline{\mathrm{S}}$) and ($\underline{\mathrm{I}}$) hold without any additional assumptions on $(c_n)_{n=1}^\infty$.

Definition 5.4. A set $A = \{a_1 < a_2 < \dots < a_n < \dots\} \subseteq \mathbb{N}$ is called a *syndetic set* if

$$\limsup_{n\to\infty}(a_{n+1} - a_n) < \infty,$$

that is, if the set A has "bounded gaps", see for example [9, 21]. Such sets are sometimes also called *relatively dense* [4].

If a set $A \subseteq \mathbb{N}$ is not syndetic (that is, $\limsup\limits_{n\to\infty}(a_{n+1} - a_n) = +\infty$) we say that A has *arbitrarily large gaps.*

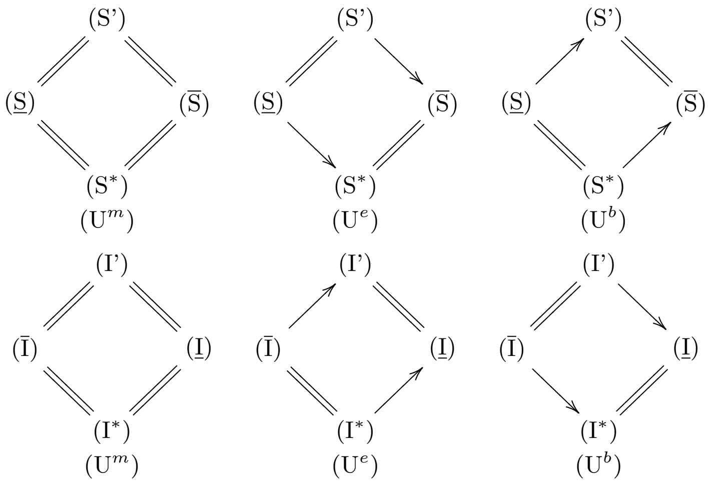


FIGURE 1.

Note that a set A has arbitrarily large gaps if and only if $\underline{u}(A) = 0$.

Suppose we are given a set $A \subseteq \mathbb{N}$. For any n, h we put

$$k := A(n, n+h] - 1. \tag{5.3}$$

Moreover, if $A \cap (n, n+h] \neq \emptyset$, then there exists an integer p such that

$$a_p := \min A \cap (n, n+h], \tag{5.4}$$

that is, a_p is the first and a_{p+k} is the last element of $A \cap (n, n+h]$.

As p and k depend only on n, h and the set A, which is fixed, we will denote them by $p(n,h)$ and $k(n,h)$.

For such p and k we get

$$A_c(n, n+h] = A_c[a_p, a_{p+k}] = A_c(a_{p-1}, a_{p+k+1}), \tag{5.5}$$

$$\overline{\gamma}_{p,k}(A) = \frac{A_c[a_p, a_{p+k}]}{\mathbb{N}_c[a_p, a_{p+k}]} \geq \frac{A_c(n, n+h]}{\mathbb{N}_c(n, n+h]} = \gamma_{n,h}(A), \tag{5.6}$$

$$\underline{\gamma}_{p,k}(A) = \frac{A_c(a_{p-1}, a_{p+k+1})}{\mathbb{N}_c(a_{p-1}, a_{p+k+1})} \leq \frac{A_c(n, n+h]}{\mathbb{N}_c(n, n+h]} = \gamma_{n,h}(A). \tag{5.7}$$

The above estimation yields immediately:

THEOREM 5.5. *Let $(c_i)_{i=1}^\infty$ be a weight sequence. Let $A \subseteq \mathbb{N}$ be a syndetic set. Then*

$$\overline{u}_c(A) = \limsup_{k \rightrightarrows \infty} \overline{\gamma}_{p,k}(A),$$

$$\underline{u}_c(A) = \liminf_{k \rightrightarrows \infty} \underline{\gamma}_{p,k}(A).$$

P r o o f. The inequalities (5.6), (5.7) imply that it suffices to show

$$\lim_{h\rightrightarrows\infty} k(n,h) = +\infty.$$

If $\sup(a_{n+1}-a_n) = M$, then we have $k(n,h) \geq \left\lfloor \frac{h}{M} \right\rfloor - 2$, thus this is indeed true. (In fact, the condition that A is syndetic is equivalent to $\lim\limits_{h\rightrightarrows\infty} k(n,h) = +\infty$.) □

Hence for any syndetic set both $(\overline{\mathrm{S}})$ and $(\underline{\mathrm{I}})$ hold.

If $(c_i)_{i=1}^\infty$ fulfills (U^m), then also the equalities $(\underline{\mathrm{S}})$ and $(\overline{\mathrm{I}})$ hold for every syndetic set by Proposition 5.3. Examples 6.2, 6.3 show that the condition (U^m) can be weakened neither to (U^e) nor to (U^b).

5.3. Upper and lower weighted uniform density for arbitrary sets

We first show that if a weight sequence $(c_i)_{i=1}^\infty$ fulfills (U^m), then $(\overline{\mathrm{S}})$ holds for every $A \subseteq \mathbb{N}$.

Theorem 5.6. *Let a weight sequence $(c_i)_{i=1}^\infty$ fulfill (U^m). Then, for any $A \subseteq \mathbb{N}$,*

$$\limsup_{k\rightrightarrows\infty} \overline{\gamma}_{p,k}(A) = \limsup_{k\rightrightarrows\infty} \frac{A_c[a_p, a_{p+k}]}{\mathbb{N}_c[a_p, a_{p+k}]} = \overline{u}_c(A). \tag{5.8}$$

In the other words, if $(c_i)_{i=1}^\infty$ fulfills (U^m) then $(\overline{\mathrm{S}})$ holds for every $A \subseteq \mathbb{N}$.

P r o o f. The condition (U^m) means that for any given $\varepsilon_1 > 0$ there exists h_1 such that for $h > h_1$ and $n \in \mathbb{N}$

$$\frac{c_t}{\mathbb{N}_c(n, n+h]} < \varepsilon_1$$

holds for any $t \in (n, n+h]$. Note that this implies

$$\gamma_{n,h} = \frac{A_c(n,n+h]}{\mathbb{N}_c(n,n+h]} \leq \varepsilon_1 A(n,n+h] \tag{5.9}$$

whenever $h > h_1$. (The choice of ε_1 and all other ε's will be specified later.)

Let us denote $\overline{u}_c(A) = \alpha$. Without loss of generality we can assume $\alpha > 0$.

Let us consider the upper density first. By Theorem 4.1 the equality $\overline{u}_c(A) = \alpha$ implies that

$$\sup_{\substack{h\geq h_2\\ n\in\mathbb{N}}} \gamma_{n,h} \geq \alpha$$

for each h_2, that is, for any given $\varepsilon_2 > 0$ and arbitrary h_2 there exist $h \geq h_2$ and $n \in \mathbb{N}$ with

$$\gamma_{n,h} = \frac{A_c(n,n+h]}{\mathbb{N}_c(n,n+h]} \geq \alpha - \varepsilon_2. \tag{5.10}$$

From now on let $h > h_1$.

Now let $p := p(n,h)$ and $k := k(n,h)$ (see (5.4), (5.3)) for some n, h fulfilling (5.10) (note that, since $A_c(n,n+h] > 0$ by (5.10), there exists at least one element of the set A in this interval.) For this choice of p and k we have (see (5.5))

$$\overline{\gamma}_{p,k}(A) = \frac{A_c[a_p,a_{p+k}]}{\mathbb{N}_c[a_p,a_{p+k}]} = \frac{A_c(n,n+h]}{\mathbb{N}_c[a_p,a_{p+k}]} \geq \frac{A_c(n,n+h]}{\mathbb{N}_c(n,n+h]} \geq \alpha - \varepsilon_2.$$

Suppose $\varepsilon_1 \geq \varepsilon_2$. Then we have $k+1 = A(n,n+h] \geq \frac{\alpha-\varepsilon_2}{\varepsilon_1} \geq \frac{\alpha}{\varepsilon_1} - 1$, hence by choosing ε_1 small enough, we can get $k > k_0$ for any given k_0.

This implies

$$\limsup_{k\rightrightarrows\infty} \overline{\gamma}_{p,k} \geq \alpha - \varepsilon_2$$

and, since $\varepsilon_2 > 0$ can be chosen arbitrarily small,

$$\limsup_{k\rightrightarrows\infty} \overline{\gamma}_{p,k} \geq \alpha = \overline{u}_c(A).$$

The opposite inequality is obvious. □

Example 6.2 shows that (U^m) cannot be weakened to (U^e). Example 6.4 shows that it cannot be weakened to (U^b).

Next we show that if a weight sequence $(c_i)_{i=1}^{\infty}$ fulfills either (U^e) or (U^b), then the condition ($\underline{\mathrm{I}}$) is fulfilled for every set $A \subseteq \mathbb{N}$.

Theorem 5.7. *If the weight sequence fulfills* (U^b) *then* ($\underline{\mathrm{I}}$) *holds for each* $A \subseteq \mathbb{N}$.

P r o o f. If the set A is syndetic then the claim follows from Theorem 5.5. So it remains to consider sets with arbitrarily large gaps. Since for any such set A we have $\underline{u}_c(A) = 0$, it suffices to prove

$$\liminf_{k\rightrightarrows\infty} \underline{\gamma}_{p,k}(A) = 0.$$

Suppose we are given a set $A \subseteq \mathbb{N}$ with arbitrarily large gaps, a positive real number ε and a positive integer k_0. We want to show the existence of $k \geq k_0$ and p with

$$\underline{\gamma}_{p,k} < \varepsilon.$$

Let $k := k_0$. By the condition (U^b), there exists h_0 such that

$$\frac{c_n}{\mathbb{N}_c[n,n+h]} < \frac{\varepsilon}{k}, \qquad \text{whenever} h > h_0.$$

Since A has arbitrarily large gaps, for any $h > h_0$ there exists $n > a_{k+1}$ such that $A \cap [n, n+h] = \emptyset$. This also implies $A_c[n, n+h] = 0$.

For such n and h, let p be positive integer such that

$$a_{p+k+1} := \min A \cap (n+h, \infty).$$

Now

$$\begin{aligned} \frac{A_c(a_{p-1}, a_{p+k+1})}{\mathbb{N}_c(a_{p-1}, a_{p+k+1})} &= \frac{c_{a_p} + \cdots + c_{a_{p+k}}}{\mathbb{N}_c(a_{p-1}, a_{p+k+1})}, \\ \frac{c_{a_{p+t}}}{\mathbb{N}_c(a_{p-1}, a_{p+k+1})} &\leq \frac{c_{a_{p+t}}}{\mathbb{N}_c[a_{p+t}, n+h]} < \frac{\varepsilon}{k} \end{aligned}$$

for $t = 0, \dots, k$, since $a_{p+t} < n$, $n + h - a_{p+t} > h$. Consequently,

$$\frac{A_c(a_{p-1}, a_{p+k+1})}{\mathbb{N}_c(a_{p-1}, a_{p+k+1})} = \frac{c_{a_p} + \cdots + c_{a_{p+k}}}{\mathbb{N}_c(a_{p-1}, a_{p+k+1})} < k\frac{\varepsilon}{k} = \varepsilon.$$

Thus

$$\liminf_{k\rightrightarrows\infty} \underline{\gamma}_{p,k}(A) = 0.$$

□

By a minor modification of the above proof it is possible to show

Theorem 5.8. *If the weight sequence fulfills* (U^e) *then* ($\underline{\mathrm{I}}$) *holds for each* $A \subseteq \mathbb{N}$.

The same examples which we have used for syndetic sets show that ($\bar{\mathrm{I}}$) need not be true if only one of the conditions (U^e), (U^b) is fulfilled.

Of course, if both of them hold, then by Propositions 5.2 and 5.3 the equalities ($\bar{\mathrm{I}}$) and ($\underline{\mathrm{I}}$) are equivalent.

5.4. Weighted uniform density for arbitrary sets

If we want to establish a characterization similar to (5.1) with a sequence in A, the following condition is in a direct analogy with (5.1)

$$\lim_{k\rightrightarrows\infty} \gamma'_{p,k}(A) = u_c(A) \tag{L'}$$

or

$$\lim_{k\rightrightarrows\infty} \gamma^*_{p,k}(A) = u_c(A), \tag{L*}$$

in the sense that the set A has weighted uniform density if and only if the limit on the left hand side exists and they have the same value.

Another possible generalization of (5.1) is the following

$$u_c(A) = \lim_{k\rightrightarrows\infty} \overline{\gamma}_{p,k}(A) = \lim_{k\rightrightarrows\infty} \underline{\gamma}_{p,k}(A) \tag{$\overline{\mathrm{L}}$}$$

in the sense that $u_c(A)$ exists if and only if both limits on the right hand side exist and have the same value and, in this case, the value of u_c is the common value of these limits.

Note that all these conditions for the unit weights are equivalent to (5.1).

Clearly, if a set A has weighted uniform density, then all limits mentioned above are equal to $u_c(A)$.

The condition (L') implies ($\overline{\mathrm{L}}$) and the same is true for (L^*). All these conditions are equivalent for the weight sequences fulfilling (U^m). From Example 6.1 we see that ($\overline{\mathrm{L}}$) implies neither (L') nor (L^*).

From the results we obtained in the preceding section we get

Proposition 5.9. *The condition* ($\overline{\mathrm{L}}$) *holds for any syndetic subset of* $\mathbb{N}$.

If the weight sequence $(c_i)_{i=1}^{\infty}$ *fulfills* (U^m)*, then* ($\overline{\mathrm{L}}$) *holds for any subset of* $\mathbb{N}$.

In Example 6.2 we construct a syndetic set not fulfilling (L') and a syndetic set not fulfilling (L^*).

Also, from Example 6.2 we see that the assumption (U^m) cannot be weakened to (U^e). Example 6.4 shows that it cannot be weakened to (U^b).

5.5. Necessary conditions for (I') and ($\overline{\mathrm{S}}$)

In the preceding parts we have obtained some sufficient conditions for some of possible generalizations of the result from [10]. Next we include some partial results concerning the necessity of these conditions.

Let us first recall another notion characterizing smallness of sets which is related to the uniform density.

Definition 5.10. A set $A = \{a_1 < a_2 < \cdots < a_n < \dots\} \subseteq \mathbb{N}$ is called *lacunary* if $\lim\limits_{n\to\infty}(a_{n+1} - a_n) = +\infty$.

The reader should be warned that the term lacunary is also used with different meanings in literature. The same convention as here is used in [5, 10]. Lacunary sets are called (SC)-sets in [8].

Some authors use the term lacunary for the sets fulfilling $\limsup\limits_{n\to\infty} \frac{a_{n+1}}{a_n} > 1$, such sets are called *Hadamard lacunary* in [16], *almost thin* in [7]. These are precisely the sets having the gap density defined in [14] greater than 1 (see also [13]).

Theorem 5.11. *If* (M) *does not hold for a weight sequence* $(c_n)_{n=1}^{\infty}$*, that is,*

$$\limsup_{n\to\infty} \frac{c_n}{\mathbb{N}_c(0,n]} = \alpha > 0, \tag{5.11}$$

then there exists a lacunary set

$$A \subseteq \mathbb{N} \quad \textit{with} \quad \lim_{k\rightrightarrows\infty} \gamma'_{p,k} = \alpha\,. \tag{5.12}$$

As A is lacunary, we have thus $\underline{u}_c(A) = 0$.

This shows that (I') $\Rightarrow$ (M). We also see from this theorem that (L') implies (M).

Example 6.1 shows that ($\underline{\mathrm{I}}$) $\not\Rightarrow$ (M). From the same example we see that (L') $\not\Rightarrow$ (M).

P r o o f. Since $\limsup\limits_{n\to\infty} \frac{c_n}{\mathbb{N}_c(0,n]} = \alpha$, there exists a subsequence $(c_{b_n})_{n=1}^{\infty}$ with

$$\lim_{n\to\infty} \frac{c_{b_n}}{\mathbb{N}_c(0,b_n]} = \alpha\,. \tag{5.13}$$

Choose a subsequence $a_k = b_{n_k}$ of the sequence $(b_n)_{n=1}^{\infty}$ with the properties

$$\lim_{k\to\infty}(a_{k+1} - a_k) = +\infty \quad \text{and} \quad \lim_{k\to\infty} \frac{\mathbb{N}_c(0,a_k]}{\mathbb{N}_c(0,a_{k+1}]} = 0\,. \tag{5.14}$$

The existence of such a subsequence follows from the fact that $\sum_{k=1}^{\infty} c_k = +\infty$.

Being a subsequence of $(b_n)_{n=1}^{\infty}$, this sequence fulfills

$$\lim_{k\to\infty} \frac{c_{a_k}}{\mathbb{N}_c(0,a_k]} = \alpha > 0 \tag{5.15}$$

as well. Note that (5.14) implies

$$\lim_{k\to\infty} \frac{\mathbb{N}_c(a_k, a_{k+1}]}{\mathbb{N}_c(0, a_{k+1}]} = 1, \tag{5.16}$$

$$\lim_{k\to\infty} \frac{\mathbb{N}_c(0, a_k]}{c_{a_{k+1}}} = \lim_{k\to\infty} \frac{\mathbb{N}_c(0, a_k]}{\mathbb{N}_c(0, a_{k+1}]} \cdot \frac{\mathbb{N}_c(0, a_{k+1}]}{c_{a_{k+1}}} \overset{(5.14),(5.15)}{=} 0\,.$$

The set $A = \{a_1 < a_2 < \dots < a_k < \dots\}$ is lacunary. We will show that (5.12) holds for this set. First, note that

$$\gamma'_{p,k} = \frac{c_{a_{p+1}} + \dots + c_{a_{p+k}}}{\mathbb{N}_c(a_p, a_{p+k}]} \geq \frac{c_{a_{p+k}}}{\mathbb{N}_c(0, a_{p+k}]}\,.$$

The condition (5.15) implies that the right hand side goes to α uniformly in p as k tends to infinity. Thus we get

$$\liminf_{k\rightrightarrows\infty} \gamma'_{p,k} \geq \alpha.$$

On the other hand,

$$\gamma'_{p,k} = \frac{c_{a_{p+1}} + \dots + c_{a_{p+k}}}{\mathbb{N}_c(a_p, a_{p+k}]} = \frac{c_{a_{p+1}} + \dots + c_{a_{p+k}}}{c_{a_{p+k}}} \cdot \frac{\mathbb{N}_c(a_{p+k-1}, a_{p+k}]}{\mathbb{N}_c(a_p, a_{p+k}]} \cdot \frac{c_{a_{p+k}}}{\mathbb{N}_c(a_{p+k-1}, a_{p+k}]}\,.$$

For the first fraction we have the estimate

$$\frac{c_{a_{p+1}} + \dots + c_{a_{p+k}}}{c_{a_{p+k}}} \leq \frac{\mathbb{N}_c(0, a_{p+k-1}] + c_{a_{p+k}}}{c_{a_{p+k}}} = 1 + \frac{\mathbb{N}_c(0, a_{p+k-1}]}{c_{a_{p+k}}}$$

and the right hand side tends to 1 uniformly in p as k tends to infinity by (5.14).

The limit of the second fraction $\frac{\mathbb{N}_c(a_{p+k-1}, a_{p+k}]}{\mathbb{N}_c(a_p, a_{p+k}]}$ for $k \to \infty$ is 1 by (5.16). The limit of the third fraction $\frac{c_{a_{p+k}}}{\mathbb{N}_c(a_{p+k-1}, a_{p+k}]}$ is α by (5.15). In both cases, the convergence is uniform in p.

Together we get that the limit of the right hand side is α and

$$\limsup_{k\rightrightarrows\infty} \gamma'_{p,k} \leq \alpha.$$

Thus we have shown that

$$\lim_{k\to\infty} \gamma'_{p,k} = \alpha \quad \text{uniformly in } p\,.$$

□

Theorem 5.12. *Suppose that a weight sequence* $(c_n)_{n=1}^{\infty}$ *fulfills* (M) *but it doest not fulfill* (U^m)*. Then there exists a set* $A \subseteq \mathbb{N}$ *with* $\lim\limits_{k\rightrightarrows\infty} \overline{\gamma}_{p,k}(A) = 0$ *and* $\overline{u}_c(A) > 0$.

This shows that $(\overline{\mathrm{S}}) \Rightarrow (\mathrm{U}^m) \vee \neg(\mathrm{M})$. In the other words, if we restrict ourselves to the sequences fulfilling (M), then (U^m) is necessary for $(\overline{\mathrm{S}})$ to hold.

We also see from the above theorem that $(\overline{\mathrm{L}}) \Rightarrow (\mathrm{U}^m) \vee \neg(\mathrm{M})$.

P r o o f. Suppose that (U^m) is not true for $(c_n)_{n=1}^{\infty}$. This means that there exists $\varepsilon > 0$ such that for any given h_0 we can find n and h with $h > h_0$ and $\frac{c_t}{\mathbb{N}_c(n,n+h]} \geq \varepsilon$ for some $t \in (n, n+h]$.

Inductively we construct sequences $(h_k)_{k=1}^{\infty}$, $(n_k)_{k=1}^{\infty}$, $(m_k)_{k=1}^{\infty}$, $(a_k)_{k=1}^{\infty}$ such that

- $h_k \geq k$;
- $a_k \in (n_k, n_k + h_k]$;
- $\frac{c_{a_k}}{\mathbb{N}_c(n_k,n_k+h_k]} \geq \varepsilon$;
- $\frac{c_{a_k}}{\mathbb{N}_c(0,a_k]} \leq \frac{1}{2^k}$;
- $m_k > a_k$ and $\mathbb{N}_c(a_k, m_k] \geq 2^k \mathbb{N}_c(0, a_k]$ (the choice of such an m_k is possible, since $\sum_{k=1}^{\infty} c_k = +\infty$);
- $a_{k+1} > m_k$;
- $(a_k)_{k=1}^{\infty}$ is increasing (this follows from the last two conditions).

The only thing to be clarified is, whether it is possible to choose a_{k+1} large enough to get $a_{k+1} > m_k$ and $\frac{c_t}{\mathbb{N}_c(n,n+h]} \geq \varepsilon$.

More precisely, the condition (M) implies that there exists t_k such that

$$\frac{c_t}{\mathbb{N}_c(0,t]} \leq \frac{1}{2^k}, \qquad \text{whenever} \quad t > t_k.$$

By the assumption we know that, for any given $h_0 \geq k$, there exists $h > h_0$, n and $t \in (n, n+h]$ such that

$$\frac{c_t}{\mathbb{N}_c(n,n+h]} \geq \varepsilon.$$

We want to show that it is possible to choose such n, h and t with the additional property $t > N_k := \max\{m_k, t_k\}$. (Then we can put $a_{k+1} = t$.)

Let us choose $h_0 > N_k$, at first. Note that, whenever $n < t < N_k$, we have

$$\frac{c_t}{\mathbb{N}_c(n,n+h]} \leq \frac{\max\{c_1,\dots,c_{N_k}\}}{\mathbb{N}_c(N_k,h]}.$$

By choosing h_0 large enough, we can get

$$\mathbb{N}_c(N_k,h] > \frac{\max\{c_1,\dots,c_{N_k}\}}{\varepsilon}, \qquad \text{whenever} \quad h > h_0.$$

Such a choice of h_0 will enforce that $a_{k+1} > N_k$.

If we have $A = \{a_k; k \in \mathbb{N}\}$ constructed as above, then clearly $\overline{u}_c(A) \geq \varepsilon$ since

$$\frac{c_{a_k}}{\mathbb{N}_c(n_k,n_k+h_k]} \geq \varepsilon$$

and h_k tends to infinity. Thus $\overline{u}_c(A) \geq \varepsilon > 0$.

It remains to show that

$$\overline{\gamma}_{p,k}(A) = \frac{A_c[a_p, a_{p+k}]}{\mathbb{N}_c[a_p,a_{p+k}]} = \frac{c_{a_p} + \cdots + c_{a_{p+k-1}} + c_{a_{p+k}}}{\mathbb{N}_c[a_p,a_{p+k}]}$$

converges to 0 uniformly in p as k tends to infinity.

Now we have the estimate
$$\frac{c_{a_p}+\cdots+c_{a_{p+k-1}}}{\mathbb{N}_c[a_p,a_{p+k}]}\le\frac{\mathbb{N}_c(0,a_{p+k-1}]}{\mathbb{N}_c(a_{p+k-1},m_{p+k-1}]}\le\frac{1}{2^{p+k-1}}\,.$$
It remains to estimate the last term
$$\frac{c_{a_{p+k}}}{\mathbb{N}_c[a_p,a_{p+k}]}\,.$$
We have
$$\begin{aligned}\frac{c_{a_{p+k}}}{\mathbb{N}_c[a_p,a_{p+k}]}\le\frac{c_{a_{p+k}}}{\mathbb{N}_c(a_{p+k-1},a_{p+k}]}&=\frac{c_{a_{p+k}}}{\mathbb{N}_c(0,a_{p+k}]}\frac{\mathbb{N}_c(0,a_{p+k-1}]+\mathbb{N}_c(a_{p+k-1},a_{p+k}]}{\mathbb{N}_c(a_{p+k-1},a_{p+k}]}\\&\le\frac{1}{2^{p+k}}\Big(1+\frac{1}{2^{p+k-1}}\Big)\le\frac{2}{2^{p+k}}=\frac{1}{2^{p+k-1}}.\end{aligned}$$
Together we get
$$\overline{\gamma}_{p,k}(A)=\frac{A_c[a_p,a_{p+k}]}{\mathbb{N}_c[a_p,a_{p+k}]}\le\frac{2}{2^{p+k-1}}\le\frac{1}{2^{k-2}}$$
which implies
$$\lim_{k\to\infty}\overline{\gamma}_{p,k}(A)=0$$
uniformly in p. □

6. Examples

In this section we include several examples where we explicitly compute values of the weighted uniform densities of some sets, as well as the limits superior and limits inferior which appear in the equalities we have studied.

Most of the examples are pathological in some sense. For example, in Example 6.1 the value of the upper and lower weighted uniform density depends only on the finiteness of the set or its complement. However, the purpose of these examples is to provide counterexamples showing that the assumptions in some of the results of preceding section cannot be weakened. Also, Theorems 5.11 and 5.12 imply that the weight sequence cannot be "too nice" in order to obtain such a counterexample.

Example 6.1. Let us define the weight sequence by $c_n=2^{2^n}$. Note that it fulfills (U^b), since it is increasing. It fulfills neither (M) nor (U^m).

Obviously,
$$\overline{u}_c(A)=\limsup_{k\rightrightarrows\infty}\overline{\gamma}_{p,k}(A)=\begin{cases}0, & \text{if } A \text{ is finite},\\ 1, & \text{otherwise},\end{cases}$$
$$\underline{u}_c(A)=\liminf_{k\rightrightarrows\infty}\underline{\gamma}_{p,k}(A)=\begin{cases}1, & \text{if } A \text{ is cofinite},\\ 0, & \text{otherwise}.\end{cases}$$

On the other hand, we have also

$$\liminf_{k\rightrightarrows\infty} \overline{\gamma}_{p,k}(A) = \begin{cases} 0, & \text{if } A \text{ is finite,} \\ 1, & \text{otherwise,} \end{cases}$$

$$\limsup_{k\rightrightarrows\infty} \underline{\gamma}_{p,k}(A) = \begin{cases} 1, & \text{if } A \text{ is cofinite,} \\ 0, & \text{otherwise.} \end{cases}$$

Thus in this case ($\overline{\text{S}}$), ($\underline{\text{I}}$) and ($\overline{\text{L}}$) hold.

But ($\underline{\text{S}}$) and ($\overline{\text{I}}$) fail for the set A and the weight sequence given above.

Also, since the weight sequence does not fulfill (M), we get from Theorem 5.11 that (L') fails for this sequence. As the weight sequence fulfills (U^b), we have

$$\limsup_{k\rightrightarrows\infty} \underline{\gamma}_{p,k}(A) = \limsup_{k\rightrightarrows\infty} \gamma^*_{p,k}(A) \qquad \text{and} \qquad \liminf_{k\rightrightarrows\infty} \underline{\gamma}_{p,k}(A) = \liminf_{k\rightrightarrows\infty} \gamma^*_{p,k}(A).$$

From this we get that for every $A \subseteq \mathbb{N}$ the limit $\lim\limits_{k\rightrightarrows\infty} \gamma^*_{p,k}(A)$ exists. Therefore (L^*) is not valid in this case.

Example 6.2. We will use the following sequence:

$$c = \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \underbrace{\frac{1}{16}, \dots, \frac{1}{16}}_{16\text{-times}}, \dots, \underbrace{\frac{1}{2^{2^n}}, \dots, \frac{1}{2^{2^n}}}_{2^{2^n}\text{-times}}, \dots$$

As this sequence is non-increasing, it fulfills (U^e). Clearly, it does not fulfill (U^b) and, consequently, it does not fulfill (U^m).

We choose the set A as the set of all ends of the blocks of length 2^{2^n}. Thus we have $c_{a_k} = \frac{1}{2^{2^k}}$. Then

$$\frac{A_c[a_p, a_{p+k}]}{\mathbb{N}_c[a_p, a_{p+k}]} = \frac{\frac{1}{2^{2^p}} + \dots + \frac{1}{2^{2^{p+k}}}}{\frac{1}{2^{2^p}} + k} \le \frac{1 + \frac{1}{2} + \dots + \frac{1}{2^{2^{p+k}}}}{k} \le \frac{2}{k}$$

thus

$$\lim_{k\rightrightarrows\infty} \overline{\gamma}_{p,k}(A) = 0.$$

On the other hand, this set has the upper density $\overline{u}_c(A) = 1$. (Consider $\gamma_{n,h}$ for n, h such that $n+1 \in A$.) Thus neither ($\overline{\text{S}}$) nor ($\overline{\text{L}}$) is true in this case.

Next we include two examples of syndetic sets.

For $B = 2\mathbb{N}+1$ we get $\gamma^*_{p,k}(B) = \frac{1}{2}$ (for each p, k), hence

$$\lim_{k\rightrightarrows\infty} \gamma^*_{p,k}(B) = \frac{1}{2},$$

while

$$\underline{u}_c(B) = 0, \qquad \overline{u}_c(B) = \frac{1}{2}.$$

This shows that (I*) does not hold for this sequence. By Proposition 5.3, this condition is equivalent to ($\overline{\mathrm{I}}$). Hence this example fails to fulfill ($\overline{\mathrm{I}}$), even for syndetic sets. We also see that (L*) is not true for this weight sequence.

Now, let us consider the set $C = 2\mathbb{N} = \mathbb{N} \setminus B$. Since it is the complement of B, we have immediately

$$\overline{u}_c(C) = 1, \qquad \underline{u}_c(C) = \frac{1}{2}\,.$$

In this case $\gamma'_{p,k}(C) = \frac{1}{2}$ for each p, k, hence

$$\lim_{k\rightrightarrows\infty} \gamma'_{p,k}(C) = \frac{1}{2}\,.$$

So neither (S') nor (L') is true for this set. Again, using Proposition 5.3, the condition (S') is equivalent to ($\underline{\mathrm{S}}$). Thus it fails even for syndetic sets.

Example 6.3. This example is, in a sense, similar to Example 6.2. Let us choose

$$c = 1, 1, 2, 2, \dots, 2^{2^n}, 2^{2^n}, \dots$$

This sequence fulfills (U^b), but not (U^e). Recall that by Proposition 5.3 the condition (S*) is equivalent to ($\underline{\mathrm{S}}$) in this case. Similarly, (I') is equivalent to ($\overline{\mathrm{I}}$).

For $A = 2\mathbb{N} + 1$ we have $\overline{u}_c(A) = 1$, $\underline{u}_c(A) = \frac{1}{2}$ and $\lim\limits_{k\rightrightarrows\infty} \gamma^*_{p,k}(A) = \frac{1}{2}$. Thus (S*) and ($\underline{\mathrm{S}}$) do not hold.

For $B = 2\mathbb{N}$ we have $\overline{u}_c(B) = \frac{1}{2}$, $\underline{u}_c(B) = 0$ and $\lim\limits_{k\rightrightarrows\infty} \gamma'_{p,k}(B) = \frac{1}{2}$. Thus (I') and ($\overline{\mathrm{I}}$) do not hold.

Both A and B are syndetic sets.

Example 6.4. Consider the sequence

$$c = 4, 4, 4, 4, \underbrace{16, \dots, 16}_{16\text{-times}}, \dots, \underbrace{2^{2^n}, \dots, 2^{2^n}}_{2^{2^n}\text{-times}}, \dots$$

It fulfills (U^b), since it is non-decreasing.

For the set A we chose the beginnings of blocks of length 2^{2^n}.

We have $\overline{u}_c(A) = 1$. (Consider $\gamma_{n,h}$ for n and h such that $n + h \in A$.)

On the other hand, we have

$$\overline{\gamma}_{p,k}(A) = \frac{A_c[a_p, a_{p+k}]}{\mathbb{N}_c[a_p, a_{p+k}]} = \frac{2^{2^p} + 2^{2^{p+1}} + \dots + 2^{2^{p+k-1}} + 2^{2^{p+k}}}{2^{2^{p+1}} + 2^{2^{p+2}} + \dots + 2^{2^{p+k}} + 2^{2^{p+k}}}\,.$$

Clearly,

$$\overline{\gamma}_{p,k}(A) \geq \frac{2^{2^p} + 2^{2^{p+k}}}{2^{2^{p+k}} + 2^{2^{p+k}}} \geq \frac{1}{2}\,.$$

On the other hand,

$$\overline{\gamma}_{p,k}(A) \leq \frac{2^{2^{p+k-1}+1} + 2^{2^{p+k}}}{2.2^{2^{p+k}}} = \frac{1}{2} + \frac{2^{2^{p+k-1}}}{2^{2^{p+k}}} = \frac{1}{2} + \frac{1}{2^{2^{p+k-1}}} \leq \frac{1}{2} + \frac{1}{2^{2^{k-1}}}\,,$$

hence,

$$\lim_{k\rightrightarrows\infty} \overline{\gamma}_{p,k}(A) = \frac{1}{2}\,.$$

Thus we have shown that this weight sequence does not fulfill the conditions $(\overline{\mathrm{S}})$ and $(\overline{\mathrm{L}})$.

If we modify this example by repeating the same value enough times before starting the next block, we can obtain an example with similar properties that, in addition, fulfills (M).

7. Conclusion

We have shown that for any positive weight sequence the upper weighted and lower weighted uniform density (as defined in [11]) exist. If, in addition, $\sum_{i=0}^{\infty} c_i = +\infty$, then they coincide with the weighted analogue of the upper and lower Banach density and they can be described in terms of uniform limit superior and inferior of the double sequence $\gamma_{n,k}(A)$ of relative weights.

We have also investigated several possibilities of extending an alternative characterization of the uniform density given in [10] to the case of the weighted uniform density. We have seen that these characterizations do not hold in general, but we succeeded to find sufficient conditions on the given set A or on the weight sequence $(c_i)_{i=1}^{\infty}$ such that they are true. Let us note that the equalities $(\overline{\mathrm{S}})$ and $(\underline{\mathrm{I}})$ seem to be the most important and most natural among several considered alternatives. However, in some cases they are equivalent to some of the remaining conditions and working with $\gamma'_{p,k}$ or $\gamma^*_{p,k}$ instead of $\underline{\gamma}_{p,k}$ and $\overline{\gamma}_{p,k}$ can simplify the computations. The results and counterexamples given in the preceding sections suggest also that the condition (U^m) seems to be appropriate, since under this condition the weighted uniform density behaves relatively well.

The overview of the obtained results is given in the following two tables, where the shortcut syn stands for syndetic sets. As far as the necessity of the assumptions is concerned, we have obtained only partial results.

$\boxed{\Rightarrow}$	$(\underline{\mathrm{S}})$	$(\overline{\mathrm{S}})$	$(\overline{\mathrm{I}})$	$(\underline{\mathrm{I}})$
syn	-	+	-	+
syn+(U^m)	+	+	+	+
syn+(U^e)	-	+	-	+
syn+(U^b)	-	+	-	+

$\boxed{\Rightarrow}$	$(\underline{\mathrm{S}})$	$(\overline{\mathrm{S}})$	$(\underline{\mathrm{I}})$	$(\overline{\mathrm{I}})$
(U^m)	+	+	+	+
(U^e)	-	-	-	+
(U^b)	-	-	-	+
(M)	-	-	-	-

Remark 7.1. Let us note that there is another possibility to generalize the (5.1), namely, fixing the sum instead of the number of elements in a given interval. In this case we would investigate the values

$$\limsup_{\substack{k\rightrightarrows\infty \\ A_c(I)\geq k}} \frac{A_c(I)}{\mathbb{N}_c(I)} \qquad \text{and} \qquad \liminf_{\substack{k\rightrightarrows\infty \\ A_c(I)\geq k}} \frac{A_c(I)}{\mathbb{N}_c(I)}\,. \tag{7.1}$$

(To be precise, the above expressions make sense only for $A_c(\mathbb{N}) = +\infty$. In the case that $A_c(\mathbb{N}) < \infty$ we consider both values to be 0.) This alternative could be subject of further study.

Remark 7.2. The motivation for expressing the uniform density in the form (5.1) is clear — this characterization resembles the analogous characterization of the asymptotic density $d(A) = \lim\limits_{n\to\infty} \frac{n}{a_n}$. A generalization of this result to the Alexander's densities is easy.

Let us recall that, using our notation, the upper and lower Alexander's density (also called weighted density) given by the weight sequence $(c_i)_{i=1}^{\infty}$ is

$$\overline{d}_c(A) = \limsup \frac{A_c(0,n]}{\mathbb{N}_c(0,n]} \qquad \text{and} \qquad \underline{d}_c(A) = \liminf \frac{A_c(0,n]}{\mathbb{N}_c(0,n]}\,.$$

It can be shown easily that

$$\overline{d}_c(A) = \limsup \frac{A_c(0,a_k]}{\mathbb{N}_c(0,a_k]} \qquad \text{and} \qquad \underline{d}_c(A) = \liminf \frac{A_c(0,a_{k+1})}{\mathbb{N}_c(0,a_{k+1})}\,.$$

The proof is almost the same as in the case of the asymptotic density. As far as we know, this result on Alexander's densities was not published elsewhere.

It was shown in [10, Theorem 4.2] that the uniform density has Darboux property. In the case of the Alexander's densities it was shown in [18] that the Alexander's density associated with some weight sequence c has Darboux property if and only if c fulfills (M). Moreover, the proof is similar to the proof of Darboux property for the asymptotic density.

The above mentioned proof in [10] for the uniform density strongly depends on the fact that all weights are the same and it cannot be easily generalized. Thus we propose the following problem.

Problem 7.3. Find necessary and sufficient conditions under which the weighted uniform density has Darboux property.

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