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uniform distribution theory

ON THE SUM OF BOUNDED MULTIPLICATIVE FUNCTIONS OVER SOME SPECIAL SUBSETS OF INTEGERS

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ABSTRACT. Let $J_1, \dots, J_k \subseteq [0,1)$ be finite unions of intervals, $P_1(x), \dots, P_k(x) \in R[x]$ of degree at least one, $Q_{m_1,\dots,m_k}(x) = m_1 P_1(x) + \dots + m_k P_k(x)$, $m_1, \dots, m_k \in \mathbb{Z}$.

Assume that $Q_{m_1,\dots,m_k}(x) - Q_{m_1,\dots,m_k}(0)$ has at least one irrational coefficient for every $(m_1, \dots, m_k) \neq (0, \dots, 0)$.

Let $S := \{n \mid n \in \mathbb{N},\ \{P_l(n)\} \in J_l,\ l = 1, \dots, k\}$, $\lambda =$ Lebesgue measure. We shall prove the following theorem.

Under the conditions stated above

$$\sup_{g \in \mathcal{M}_1} \left| \frac{1}{x} \sum_{\substack{n \le x \\ n \in S}} g(n) - \frac{\lambda(J_1) \dots (\lambda(J_k)}{x} \sum_{n \le x} g(n) \right| = \tau_x \to 0$$

as $x \to \infty$. Here $\mathcal{M}_1$ is the set of complex valued multiplicative functions g satisfying $|g(n)| \le 1$ $(n \in \mathbb{N})$.

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1. Introduction

Let $c, c_1, c_2, \dots,\ K, K_1, K_2, \dots$ be positive constants, not necessarily the same at every occurrence.

Let $\mathcal{M}$ be the set of complex valued multiplicative functions, and $\mathcal{M}_1 \subseteq \mathcal{M}$ be the set of those $g \in \mathcal{M}$ for which $|g(n)| \le 1 \quad (n \in \mathbb{N})$ holds as well.

Let $e(\alpha) := e^{2\pi i \alpha}$.

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A famous theorem of H. Daboussi published in his joint paper with H. Delange [2] asserts that

$$\sup_{f\in\mathcal{M}_1}\left|\frac{1}{x}\sum_{n\le x} f(n)e(n\alpha)\right| = \varrho_x \to 0 \quad (x\to\infty) \tag{1.1}$$

whenever α is an irrational number. Here ϱ_x may depend on α.

The author proved that

$$\sup_{f\in\mathcal{M}_1}\left|\frac{1}{x}\sum_{n\le x} f(n)e(P(n))\right| = \varrho_x(P) \to 0 \quad (x\to\infty) \tag{1.2}$$

for every polynomial $P(u) = \alpha_k u^k + \dots + \alpha_1 u$, where $\alpha_k, \dots, \alpha_1 \in \mathbb{R}$, and at least one coefficient is irrational [4].

Let l_1 be the set of those arithmetical functions g for which

$$\|g\|_1 := \limsup \frac{1}{x}\sum_{n\le x} |g(n)|.$$

The function class $B^1 \subseteq l_1$ is defined as follows. We say that $g \in B^1$(= set of almost periodic functions) if for every $\varepsilon > 0$ there is a trigonometric polynomial $\sum_{\nu=0}^{N} a_\nu e(\alpha_\nu n) = t_\varepsilon(n)$ with real α_ν $(\nu = 0,\dots,N)$ such that $\|g - t_\varepsilon\|_1 < \varepsilon$.

Assume that $g \in B^1$. We say that $\alpha \in \mathbb{R}$ belongs to the spectrum of g if

$$\limsup_{x\to\infty} \frac{1}{x}\left|\sum_{n\le x} g(n)e(-\alpha n)\right| > 0.$$

It is known that the spectrum contains at most countable many numbers. On the other hand, $\sigma(f)$ can be empty. Furthermore

$$\hat{g}(\alpha) := \lim_{x\to\infty}\frac{1}{x}\sum_{n\le x} g(n)e(-\alpha n)$$

exists for all $\alpha \in \mathbb{R}$. Let $\alpha_1, \alpha_2, \dots, \in \sigma(f)$ be the spectrum points of g arranged so that $|\alpha_1| \le |\alpha_2| \le \dots$, and let $c_\nu = \hat{g}(\alpha_\nu)$.

We say that the formal series

$$\sum_{\nu=0}^{\infty} c_\nu e(\alpha_\nu n)$$

is the Fourier expansion of g. It is known that

$$\|g - r_N\|_1 \to 0 \ (N \to \infty),$$

where $r_N = \sum_{\nu=0}^{N} c_\nu e(\alpha_\nu n)$.

These assertions can be obtained from more general theorems in functional analysis. It is treated in J.L. Mauclarie [5], A.G. Postnikov [6], W. Schwarz [7].

Daboussi and Delange proved in [3] the following assertion:

Let S be an arithmetical function satisfying the following conditions:

(i) S is almost-periodic B^1,

(ii) the Fourier series of S is $\lambda + \sum \lambda_\nu e(\alpha_\nu n)$

where all the α_ν are irrational.
Then, as x tends to infinity, we have

$$\sup_{f \in \mathcal{M}_1} \left| \frac{1}{x} \sum_{n \le x} f(n)S(n) - \frac{1}{\lambda} \sum_{n \le x} f(n) \right| \le \varrho_x(S), \quad \varrho_x(S) \to 0 \quad \text{as} \quad (x \to \infty). \tag{1.3}$$

In [1] among other results a special case of (1.3) is rediscovered, by proving (1.3) for the set

$$S(n) = S_{\alpha,\beta}(n) := \begin{cases} 1 & \text{if} \quad n \in \{[\alpha m + \beta] \mid m \in \mathbb{N}\} \\ 0 & \text{otherwise} \end{cases}$$

$\alpha > 0$ be irrational.

2. Formulation of the theorem

Let $k \ge 1$ be fixed, $J_1, \dots, J_k \subseteq [0,1)$ be such sets which are the union of finitely many intervals. Let $P_1(x), \dots, P_k(x)$ be non-constant real valued polynomials,

$$Q_{m_1,\dots,m_k}(x) = m_1 P_1(x) + \dots + m_k P_k(x) \tag{2.1}$$

for $m_1, \dots, m_k \in \mathbb{Z}$.

Assume that $Q_{m_1,\dots,m_k}(x) - Q_{m_1,\dots,m_k}(0)$ has an irrational coefficient for every $m_1, \dots, m_k \in \mathbb{Z}$, except when $m_1 = \dots = m_k = 0$.

Let

$$S := \{n \mid n \in \mathbb{N}, \quad \{P_l(n)\} \in J_l, \quad l = 1, \dots, k\}. \tag{2.2}$$

Let λ be the Lebesgue measure.

We shall prove the following

Theorem. *Under the conditions stated for* $P_1, \dots, P_k, J_1, \dots, J_k$ *we have*

$$\sup_{g\in\mathcal{M}_1}\left|\frac{1}{x}\sum_{\substack{n\le x\\ n\in S}} g(n) - \frac{\lambda(J_1)\cdots\lambda(J_k)}{x}\sum_{n\le x} g(n)\right| = \tau_x, \tag{2.3}$$

$\tau_x \to 0$ *as* $(x\to\infty)$.

Remark. The relation (2.3) in the special case

$$S(n) = \begin{cases} 1, & \text{if } \ n\in\{[\alpha m] \mid m\in\mathbb{N}\} \\ 0 & \text{otherwise} \end{cases}$$

has been proved earlier in [3], section 3.2.1.

3. Proof of the theorem

Let $f_l(x)$ be defined by

$$f_l(x) = \begin{cases} 1, & \text{if } \ x\in J_l \\ 0, & \text{if } \ x\in[0,1)\setminus J_l \end{cases} \tag{3.1}$$

and be extended to $\mathbb{R}$ as a function periodic $\bmod 1$. One verifies that its Fourier series $\sum\limits_{m\in\mathbb{Z}} c_m^{(l)} e(m\,x)$ satisfies

$$|c_m^{(l)}| \le \frac{K_l}{|m|} \quad (m\ne 0), \quad c_0^{(l)} = \lambda(J_l),$$

K_l may depend on J_l.

Let $\Delta > 0$ be a small constant,

$$g_l(x) = \frac{1}{2\Delta}\int\limits_{-\Delta}^{\Delta} f_l(x+y)\mathrm{d}y. \tag{3.2}$$

It is clear that if $x\in[0,1)$, and $f_l(x)\ne g_l(x)$, then the interval $[x-\Delta, x+\Delta]$ intersects both of the sets J_l and $[0,1)\setminus J_l$.

Furthermore, $0 \le g_l(x) \le 1$ always holds. Let $g_l(x) = \sum_{m\in\mathbb{Z}} d_m^{(l)} e(m\ x)$. One can compute that $d_0^{(l)} = \lambda(J_l)$, and that $d_m^{(l)} = c_m^{(l)} \frac{\sin \pi m\Delta}{\pi m\Delta}$, whence

$$|d_m^{(l)}| \le K \min\left(\frac{1}{|m|}, \frac{1}{\Delta m^2}\right) \quad \text{if} \quad m \ne 0 \tag{3.3}$$

and K may depend only on $P_1, \dots, P_k, J_1, \dots, J_k$.

Since $P_j(x) - P_j(0)$ has an irrational coefficient, according to a well-known theorem of H. Weyl, $\{P_j(n)\}$ is uniformly distributed mod1, consequently

$$\sum_{n\le x} |f_l(P_l(n)) - g_l(P_l(n))| \le C_l \Delta x \tag{3.4}$$

where C_l is a constant, $C_l = C_l(P_l, J_l)$.

Let

$$\sigma(n) := f_1(P_1(n)) \cdots f_k(P_k(n)),$$
$$\kappa(n) := g_1(P_1(n)) \cdots g_k(P_k(n)).$$

We can observe that $n \in S$ if and only if $\sigma(n) = 1$, furthermore that

$$|\sigma(n) - \kappa(n)| \le c \sum_{n\le x} |f_l(P_l(n)) - g_l(P_l(n))|,$$

consequently

$$\frac{1}{x} \#\{n \le x \mid \sigma(n) \ne \kappa(n)\} \le c\Delta, \tag{3.5}$$

$c = c(P_1, \dots, P_k, J_1, \dots, J_k)$.

Let

$$E(x) := \sum_{\substack{n\le x\\ n\in S}} g(n) = \sum_{n\le x} g(n)\sigma(n),$$
$$K(x) := \sum_{n\le x} g(n)\kappa(n).$$

Then from (3.5)

$$|E(x) - K(x)| \le c\Delta x \tag{3.6}$$

furthermore

$$K(x) = \sum_{m_1,\dots,m_k} d_{m_1}^{(1)} \dots d_{m_k}^{(k)} \sum_{n\le x} g(n) e(Q_{m_1,\dots,m_k}(n)). \tag{3.7}$$

We have $d_0^{(1)} \dots d_0^{(k)} = \lambda(J_1) \dots \lambda(J_k)$. From (3.3) it follows that

$$\sum_{m_1,\dots,m_k} |d_{m_1}^{(1)} \dots d_{m_k}^{(k)}| < \infty.$$

Furthermore, taking into account (1.2) applied for $P = Q_{m_1,\ldots,m_k}$, $\quad f = g$, our theorem immediately follows.

4. Corollary

Let $u(n)$ be an additive function for which there exist $A(x), B(x)$ such that

$$\lim_{x\to\infty} \frac{1}{x} \#\{n \le x \mid \frac{u(n) - A(x)}{B(x)} < y\} = F(y) \tag{4.1}$$

exists for almost all $y \in \mathbb{R}$, and F is a distribution function.

Let $P_1, \ldots, P_k$, $J_1, \ldots, J_k$ and S be as in Part 2.
Then

$$\lim_{x\to\infty} \frac{1}{\lambda(J_1)\ldots\lambda(J_k)x} \#\{n \le x, n \in S \mid \frac{u(n) - A(x)}{B(x)} < y\} = F(y) \tag{4.2}$$

at every continuity point y of $F(y)$.

P r o o f. Let $h_x(n) := e^{i\frac{\tau u(n)}{B(x)}}$. From (4.1) we obtain that

$$\lim_{x\to\infty} \left| e^{-i\tau\frac{A(x)}{B(x)}} \cdot \frac{1}{x} \sum_{n\le x} h_x(n) - \varphi(\tau) \right| = 0,$$

φ is the characteristic function corresponding to F. From our theorem we obtain that

$$\lim_{x\to\infty} \left| e^{-i\tau\frac{A(x)}{B(x)}} \cdot \frac{1}{\lambda(J_1)\ldots\lambda(J_k)x} \sum_{\substack{n\le x\\ n\in S}} h_x(n) - \varphi(\tau) \right| = 0,$$

whence the assertion immediately follows. □

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