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COMPARISON BETWEEN LOWER AND UPPER $\alpha-$DENSITIES AND LOWER AND UPPER $\alpha-$ANALYTIC DENSITIES

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ABSTRACT. Let α be a real number, with $\alpha \geq -1$. We prove a general inequality between the upper (resp. lower) $\alpha-$analytic density and the upper (resp. lower) $\alpha-$density of a subset A of $\mathbb{N}^*$ (Proposition 2.1). Moreover, we prove by an example that the upper and the lower α–densities and the lower and upper α–analytic densities of A do not coincide in general (*i.e.*, the inequalities proved in (2.1) may be strict). On the other hand, we identify a class of subsets of $\mathbb{N}^*$ for which these values do coincide in the case $\alpha > -1$.

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1. Introduction

In the whole text α is a real number greater than or equal to -1.

Let $A \subseteq \mathbb{N}^*$ be a set of integers. We shall denote by 1_A the *characteristic function* of A. Put

$$\mathbb{N}^*_\alpha(n) = \sum_{k \leq n} k^\alpha \tag{1.1}$$

and

$$D_{A,\alpha}(n) := \frac{\sum_{k\leq n} k^\alpha 1_A(k)}{\mathbb{N}^*_\alpha(n)}, \quad \Delta_{A,\alpha}(t) := t \sum_{k\geq 1} k^\alpha e^{-t\mathbb{N}^*_\alpha(k)} 1_A(k). \tag{1.2}$$

Define the $\alpha-$density and the $\alpha-$analytic density of A, respectively as

$$d_\alpha(A) := \lim_{n\to\infty} D_{A,\alpha}(n), \quad \delta_\alpha(A) := \lim_{t\to 0^+} \Delta_{A,\alpha}(t),$$

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of course if these limits exist. Notice that $d_{-1}(A)$ and $\delta_{-1}(A)$ are, respectively, the classical logarithmic and analytic densities of A (see [6, pp. 272–273]).

The following theorem (established in [2] in a more general context) links the two concepts of density introduced above (the case $\alpha = -1$ is well known, see for instance [5, p.274]).

Theorem 1.3. *Let $A \subseteq \mathbb{N}^*$ be a set of integers. For every real number $\ell \in [0,1]$, the two following conditions are equivalent:*

(a) *A has α–density $d_\alpha(A) = \ell$.*

(b) *A has α–analytic density $\delta_\alpha(A) = \ell$.*

When the $\alpha-$density (resp. the $\alpha-$analytic density) of A doesn't exist, one can consider the upper and lower densities, defined as follows.

Definition 1.4. The *lower and upper α–densities* of A are defined respectively as

$$\underline{d}_\alpha(A) := \liminf_{n\to\infty} D_{A,\alpha}(n), \qquad \overline{d}_\alpha(A) := \limsup_{n\to\infty} D_{A,\alpha}(n).$$

When $\underline{d}_\alpha(A) = \overline{d}_\alpha(A)$ (*i.e.,* when the limit

$$d_\alpha(A) := \lim_{n\to\infty} D_{A,\alpha}(n)$$

exists), we say that A has *α–density* equal to $d_\alpha(A)$.

Definition 1.5. The *lower and upper α–analytic densities* of A are defined respectively as

$$\underline{\delta}_\alpha(A) := \liminf_{t\to 0^+} \Delta_{A,\alpha}(t), \qquad \overline{\delta}_\alpha(A) := \limsup_{t\to 0^+} \Delta_{A,\alpha}(t).$$

When $\underline{\delta}_\alpha(A) = \overline{\delta}_\alpha(A)$ (*i.e.*, when the limit

$$\delta_\alpha(A) := \lim_{t\to 0^+} \Delta_{A,\alpha}(t)$$

exists), we say that A has *α–analytic density* equal to $\delta_\alpha(A)$.

In this paper we outface the problem of the relation between lower and upper $\alpha-$densities and lower and upper $\alpha-$analytic densities. In Section 2 we prove that, in general, the upper $\alpha-$analytic density of A is not greater than the upper $\alpha-$density of A (concerning the lower densities, the inequality is obviously reversed). This is done in Proposition 2.1.

A natural question is whether the same kind of result as Theorem 1.3 can be stated also for lower and upper $\alpha-$densities and lower and upper $\alpha-$analytic densities, *i.e.*, whether the upper and the lower α–densities and the lower and upper α–analytic densities of A coincide. In Section 3 of this paper we prove that the answer is negative in general: see Theorem 3.1. On the other hand, in Section 4 we identify a class of subsets of $\mathbb{N}^*$ for which the question can be answered affirmatively if $\alpha > -1$.

The case $\alpha = -1$ is open.

2. A first result

In this section we prove the following proposition.

Proposition 2.1. *For every $A \subseteq \mathbb{N}^*$, the following inequalities hold:*

$$\underline{d}_\alpha(A) \le \underline{\delta}_\alpha(A) \le \overline{\delta}_\alpha(A) \le \overline{d}_\alpha(A).$$

P r o o f. Let $A \subseteq \mathbb{N}^*$ and $\alpha \ge -1$ be fixed. For $x \ge 1$, put

$$A_\alpha(x) = \sum_{k \le x} k^\alpha 1_A(k), \qquad S_\alpha(x) = \begin{cases} \frac{x^{\alpha+1}}{\alpha+1} & \text{for } \alpha > -1, \\ \log x & \text{for } \alpha = -1. \end{cases}$$

At first we shall prove the following lemma.

Lemma 2.2. *The following relation holds:*

$$\sum_{k \in \mathbb{N}^*} k^\alpha 1_A(k) e^{-tS_\alpha(k)} = t \int_1^\infty A_\alpha(x)\, x^\alpha e^{-tS_\alpha(x)} \mathrm{d}x.$$

P r o o f. We simplify the notations and write $A(x)$ and $S(x)$ instead of $A_\alpha(x)$ and $S_\alpha(x)$ respectively. For every integer $m > 1$ we have

$$\begin{aligned} t \int_1^m A(x)\, x^\alpha e^{-tS(x)} \mathrm{d}x &= t \sum_{k=1}^{m-1} \int_k^{k+1} A(x)\, x^\alpha e^{-tS(x)} \mathrm{d}x \\ &= \sum_{k=1}^{m-1} A(k) \int_k^{k+1} t x^\alpha e^{-tS(x)} \mathrm{d}x. \end{aligned}$$

By performing the change of variable $y = tS(x)$ in the integral, the above quantity is transformed into

$$\sum_{k=1}^{m-1} A(k) \int_{tS(k)}^{tS(k+1)} e^{-y}\mathrm{d}y = \sum_{k=1}^{m-1} A(k)\Big[e^{-tS(k)} - e^{-tS(k+1)}\Big]$$
$$= A(1)e^{-tS(1)} + \sum_{k=2}^{m-1} A(k)e^{-tS(k)} - \sum_{k=1}^{m-2} A(k)e^{-tS(k+1)} - A(m-1)e^{-tS(m)}$$
$$= A(1)e^{-tS(1)} + \sum_{k=2}^{m-1} \Big(A(k) - A(k-1)\Big)e^{-tS(k)} - A(m-1)e^{-tS(m)}$$
$$= \sum_{k=1}^{m-1} k^\alpha 1_A(k)e^{-tS(k)} - A(m-1)e^{-tS(m)},$$

since $A(1) = 1^\alpha 1_A(1)$ and, for $k \geq 2$, $A(k) - A(k-1) = k^\alpha 1_A(k)$. We obtain the statement by passing to the limit as $m \to \infty$, since

$$0 \leq A(m-1) \leq \mathbb{N}_\alpha^*(m) \sim S(m), \qquad m \to \infty. \tag{2.3}$$

□

P r o o f o f P r o p o s i t i o n 2.1 c o n t i n u e d. We use the (simplified) notations of Lemma 2.2 and prove only the last inequality: the first one follows since $\underline{d}_\alpha(A) = 1 - \overline{d}_\alpha(\mathbb{N}^* \setminus A)$ and $\underline{\delta}_\alpha(A) = 1 - \overline{\delta}_\alpha(\mathbb{N}^* \setminus A)$; the second one is obvious. For every $\epsilon > 0$, there exists an integer ν such that, for every $n \geq \nu$, we have

$$\frac{A(n)}{S(n)} \leq \overline{d}_\alpha(A) + \epsilon. \tag{2.4}$$

Hence, by Lemma 2.2, we have

$$t\sum_k k^\alpha 1_A(k)e^{-tS(k)} = t^2 \int_1^\infty A(x)\, x^\alpha e^{-tS(x)}\mathrm{d}x = t^2 \int_1^\infty A([x])\, x^\alpha e^{-tS(x)}\mathrm{d}x$$
$$\leq t^2 \left(\int_1^\nu A([x])\, x^\alpha e^{-tS(x)}\mathrm{d}x + (\overline{d}_\alpha(A) + \epsilon) \int_\nu^\infty S([x])\, x^\alpha e^{-tS(x)}\mathrm{d}x\right)$$
$$\leq t^2 \left(\int_1^\nu A([x])\, x^\alpha e^{-tS(x)}\mathrm{d}x + (\overline{d}_\alpha(A) + \epsilon) \int_\nu^\infty S(x)\, x^\alpha e^{-tS(x)}\mathrm{d}x\right). \tag{2.5}$$

Now

$$0 \leq t^2 \int_1^\nu A([x])\, x^\alpha e^{-tS(x)}\,\mathrm{d}x \leq t^2 \int_1^\nu A([x])\, x^\alpha\,\mathrm{d}x \leq t^2 A(\nu)\nu^\alpha(\nu-1), \tag{2.6}$$

while

$$t^2 \int_\nu^\infty S(x)\, x^\alpha e^{-tS(x)} \mathrm{d}x = \int_\nu^\infty (tS(x))\,(tx^\alpha) e^{-tS(x)} \mathrm{d}x$$
$$\le \int_0^\infty (tS(x))\,(tx^\alpha) e^{-tS(x)} \mathrm{d}x = \int_0^\infty y e^{-y} \mathrm{d}y = 1. \tag{2.7}$$

By using (2.6) and (2.7) and passing to the limsup as $t \to 0$ in (2.5) we obtain

$$\overline{\delta}_\alpha(A) = \limsup_{t\to 0} t \sum_k k^\alpha 1_A(k) e^{-tS(k)} \le \overline{d}_\alpha(A) + \epsilon,$$

and the statement follows by the arbitrariness of ϵ. □

3. A second result

In general the upper and the lower α–densities and the lower and upper α–analytic densities of A do not coincide (*i.e.*, the inequalities of Proposition 2.1 may be strict). We prove it hereafter for $\alpha = 0$.

Theorem 3.1. *There is a subset A of $\mathbb{N}^*$ such that $\overline{\delta}_0(A) < \overline{d}_0(A)$.*

Proof. Let r be a real number, with $0 < r \le 1$, and put $p_n = 10^n$, $q_n = \lfloor 10^{n+r} \rfloor$. We consider the set

$$A = \bigcup_{n\ge 1} \big([p_n, q_n[\cap \mathbb{N}^*\big).$$

It is not difficult to calculate $\underline{d}_0(A)$ and $\overline{d}_0(A)$ using the following proposition, proved in [2] in a more general context.

Proposition 3.2. *Let A be a subset of $\mathbb{N}^*$, given in the form*

$$A = \bigcup_{n\ge 1} \big([p_n, q_n[\cap \mathbb{N}^*\big).$$

Put

$$\rho_k = q_k - p_k, \quad \sigma_k = q_k - q_{k-1}, \quad k \ge 1 \quad (q_0 = 1).$$

Then

$$\overline{d}_0(A) = \limsup_{n\to\infty} \frac{\sum_{k=1}^n \rho_k}{\sum_{k=1}^n \sigma_k}, \quad \underline{d}_0(A) = \liminf_{n\to\infty} \frac{q_{n-1}}{p_n} \frac{\sum_{k=1}^{n-1} \rho_k}{\sum_{k=1}^{n-1} \sigma_k}.$$

For the set A in (3.1), from the relations

$$x \le \lfloor x \rfloor \le x + 1 \tag{3.3}$$

we find easily

$$\sum_{k=1}^{n} \sigma_k = q_n - 1 = \lfloor 10^{n+r} \rfloor - 1 \sim 10^{n+r}. \tag{3.4}$$

On the other hand

$$\sum_{k=1}^{n} \rho_k = \sum_{k=1}^{n} \big(\lfloor 10^{k+r} \rfloor - 10^k\big)$$

and, using (3.3) again, we have

$$\sum_{k=1}^{n} \big(10^{k+r} - 10^k\big) - n \le \sum_{k=1}^{n} \rho_k \le \sum_{k=1}^{n} \big(10^{k+r} - 10^k\big). \tag{3.5}$$

Since

$$\sum_{k=1}^{n} \big(10^{k+r} - 10^k\big) = \big(10^r - 1\big) \sum_{k=1}^{n} 10^k = \frac{10}{9}\big(10^r - 1\big)\big(10^n - 1\big),$$

from (3.5) we get

$$\sum_{k=1}^{n} \rho_k \sim \frac{10}{9}\big(10^r - 1\big)\big(10^n - 1\big), \tag{3.6}$$

and finally, from (3.4) and (3.6), we conclude that

$$\overline{d}_0(A) = \lim_{n\to\infty} \frac{\sum_{k=1}^{n} \rho_k}{\sum_{k=1}^{n} \sigma_k} = \frac{10}{9}\big(1 - 10^{-r}\big). \tag{3.7}$$

Remark 3.8. With the same technique we can calculate also $\underline{d}_0(A)$, obtaining

$$\underline{d}_0(A) = \frac{10^r - 1}{9}.$$

We now show that, for sufficiently small values of r, the lower and upper 0–analytic densities of A are different from the above calculated values. We have

$$t \sum_{k\in\mathbb{N}^*} e^{-tk} 1_A(k) = t \sum_{n\ge 1} \Big(\sum_{k=p_n}^{q_n - 1} e^{-tk} \Big) = \frac{t}{1 - e^{-t}} \sum_{n\ge 1} \big(e^{-tp_n} - e^{-tq_n}\big). \tag{3.9}$$

Since $t/(1 - e^{-t}) \to 1$ as $t \to 0$, the two functions

$$t \mapsto t \sum_{k\in\mathbb{N}^*} e^{-tk} 1_A(k); \qquad t \mapsto \sum_{n\geq 1} \left(e^{-tp_n} - e^{-tq_n}\right)$$

have the same limit points as $t \to 0^+$.

It follows that the values of $\overline{\delta}_0(A)$ and $\underline{\delta}_0(A)$ are given respectively by

$$\limsup_{t\to 0^+} \sum_{n\geq 1} \left(e^{-tp_n} - e^{-tq_n}\right); \quad \liminf_{t\to 0^+} \sum_{n\geq 1} \left(e^{-tp_n} - e^{-tq_n}\right),$$

and in the sequel we shall estimate these two quantities. Obviously

$$\sum_{k\in\mathbb{N}^*} \left(e^{-tp_k} - e^{-tq_k}\right) \leq \sum_{k\in\mathbb{N}^*} \left(e^{-t10^k} - e^{-t10^{k+r}}\right), \tag{3.10}$$

and we are going to evaluate each term in the second sum above. By Lagrange's Theorem, for every integer k there exists a real number x_k, with $k \leq x_k \leq k+r$, such that

$$\begin{aligned}\left(e^{-t10^k} - e^{-t10^{k+r}}\right) &= (-r)\frac{d}{dx}\left(e^{-t10^x}\right)\Big|_{x=x_k} = (\log 10)\, rt\, 10^{x_k}\, e^{-t10^{x_k}} \\ &\leq (\log 10)\, rt\, 10^{k+r}\, e^{-t10^k} = (\log 10)\, rt\, 10^r \left(10^k e^{-t10^k}\right).\end{aligned} \tag{3.11}$$

From (3.10) and (3.11) we get

$$\sum_{k\in\mathbb{N}^*} \left(e^{-tk} - e^{-t(k+1)}\right) 1_A(k) \leq (\log 10)\, rt\, 10^r \sum_{k\in\mathbb{N}^*} 10^k e^{-t10^k}, \tag{3.12}$$

and now we want to find an upper bound for the last series. We can confine ourselves to the case $t \in (0,1)$. Put $x_t = -\log_{10} t$; then it is easy to see that the function

$$x \mapsto 10^x e^{-t10^x}$$

has a unique absolute maximum in $x = x_t = -\log_{10} t$, which means that it is strictly increasing for $x \in (0, x_t]$ and strictly decreasing for $x \in [x_t, +\infty)$. Then, applying twice the theorem of comparison of a sum and an integral for monotone functions (see Theorem 2, p. 4 of [6]) we get, for every integer $n > x_t$ and for two suitable numbers θ_1 and θ_2 in $[0,1]$,

$$\sum_{0<k\le n} 10^k e^{-t10^k} = \sum_{0<k\le \lfloor x_t\rfloor} 10^k e^{-t10^k} + \sum_{\lfloor x_t\rfloor<k\le n} 10^k e^{-t10^k}$$
$$= \int_0^{\lfloor x_t\rfloor} 10^x e^{-t10^x}\,\mathrm{d}x + \theta_1\big(10^{\lfloor x_t\rfloor} e^{-t10^{\lfloor x_t\rfloor}} - e^{-t}\big)$$
$$+ \int_{\lfloor x_t\rfloor}^{n} 10^x e^{-t10^x}\,\mathrm{d}x + \theta_2\big(10^{n} e^{-t10^{n}} - 10^{\lfloor x_t\rfloor} e^{-t10^{\lfloor x_t\rfloor}}\big)$$
$$= \int_0^{n} 10^x e^{-t10^x}\,\mathrm{d}x + \theta_1\big(10^{\lfloor x_t\rfloor} e^{-t10^{\lfloor x_t\rfloor}} - e^{-t}\big)$$
$$+ \theta_2\big(10^{n} e^{-t10^{n}} - 10^{\lfloor x_t\rfloor} e^{-t10^{\lfloor x_t\rfloor}}\big)$$
$$\le \int_0^{n} 10^x e^{-t10^x}\,\mathrm{d}x + 10^{\lfloor x_t\rfloor} e^{-t10^{\lfloor x_t\rfloor}} + 10^{n} e^{-t10^{n}}$$
$$\le \int_0^{n} 10^x e^{-t10^x}\,\mathrm{d}x + 10^{x_t} e^{-t10^{x_t}} + 10^{n} e^{-t10^{n}}$$
$$= \int_0^{n} 10^x e^{-t10^x}\,\mathrm{d}x + \frac{1}{et} + 10^{n} e^{-t10^{n}}.$$

By passing to the limit with respect to n, we conclude that

$$\sum_{k\in\mathbb{N}^*} 10^k e^{-t10^k} \le \int_0^{\infty} 10^x e^{-t10^x}\,\mathrm{d}x + \frac{1}{et} = \frac{1}{\log 10}\int_1^{\infty} e^{-ty}\mathrm{d}y + \frac{1}{et}$$
$$\le \frac{1}{\log 10}\int_0^{\infty} e^{-ty}\,\mathrm{d}y + \frac{1}{et} = \frac{1}{t}\left(\frac{1}{\log 10} + \frac{1}{e}\right). \tag{3.13}$$

Going back to relation (3.12), and using (3.13), we find that

$$\sum_{k\in\mathbb{N}^*} \Big(e^{-tk} - e^{-t(k+1)}\Big) 1_A(k) \le r10^r \left(1 + \frac{\log 10}{e}\right),$$

hence also

$$\overline{\delta}_0(A) = \limsup_{t\to 0^+} \sum_{k\in\mathbb{N}^*} \Big(e^{-tk} - e^{-t(k+1)}\Big) 1_A(k) \le r10^r \left(1 + \frac{\log 10}{e}\right). \tag{3.14}$$

Now we prove that, for sufficiently small r, we have $\overline{\delta}_0(A) \ne \overline{d}_0(A)$. Recalling relations (3.14) and (3.7), it will be enough to prove that

$$r10^r \left(1 + \frac{\log 10}{e}\right) < \frac{10}{9}\big(1 - 10^{-r}\big)$$

for sufficiently small r, and this is clearly true since

$$\lim_{r\to 0}\frac{10\big(1-10^{-r}\big)}{9\,r10^{r}} = \frac{10}{9}\log 10 \approx 2.55, \qquad \text{while} \qquad \left(1+\frac{\log 10}{e}\right) \approx 1.847.$$

□

Remark 3.15. For the lower densities of A, the same technique gives

$$\underline{\delta}_0(A) \geq r10^{-r} > \frac{10^r - 1}{9} = \underline{d}_0(A),$$

for sufficiently small r.

4. The case of regular sets

In this Section we shall exhibit a class of subsets of $\mathbb{N}^*$ for which the values of the lower and upper $\alpha-$densities and of the lower and upper $\alpha-$analytic densities coincide. Recall that the *counting function* of A is the function defined as

$$A(x) := \operatorname{card}\{k \in A : k \leq x\} = \sum_{k\leq x} 1_A(k), \quad x \in \mathbb{R}.$$

Obviously

$$1_A(n) = A(n) - A(n-1), \quad n \in \mathbb{N}^*.$$

Definition 4.1. A subset A of $\mathbb{N}^*$ is *regular* if its counting function $A(x)$ has the form

$$A(x) = L(x)x^{\gamma},$$

for a suitable positive slowly varying function L and a suitable real number $\gamma \in [0,1]$. A function $L\ :]0,+\infty[\mapsto]0,+\infty[$ is called *slowly varying* if for any $t > 0$, $\frac{L(tx)}{L(x)}$ tends to 1, as x tends to $+\infty$. γ is called the *regularity exponent* of A (see [5]).

If $\gamma < 1$, then obviously A has 0–density and $d_0(A) = 0$. Moreover, by Cor. 6.9 and Ex. 6.10 p. 271 of [2], A has $\alpha-$density for every $\alpha > -1$ and $d_\alpha(A) = 0$.

If $\gamma = 1$ things go differently. In fact, in this case the regularity assumption doesn't imply that A has 0–density, as it is shown by an example constructed in [3] (Appendix, p. 190). By the above quoted results (*i.e.,* Cor. 6.9 and Ex. 6.10, p. 271 of [2]), such a set A cannot have $\alpha-$density for any $\alpha > -1$ either. These remarks motivates our main result, here below.

Theorem 4.2. *Let $\alpha > -1$; let A be a regular subset of $\mathbb{N}^*$, with regularity exponent $\gamma = 1$. For every real number $\ell \in [0,1]$, the following two conditions are equivalent:*

(4.2)(a) *ℓ is a cluster point for the sequence*

$$n \mapsto D_{A,\alpha}(n), \quad \textit{as} \quad n \to \infty;$$

(4.2)(b) *ℓ is a cluster point for the function*

$$t \mapsto \Delta_{A,\alpha}(t), \quad \textit{as} \quad t \to 0^+.$$

Terminology. ℓ is a *cluster point for the sequence*

$$n \mapsto f(n), \quad \text{as} \quad n \to \infty$$

if there is a sequence $n_1 < n_2 < \dots$ of integers such that $f(n_k)$ tends to ℓ, as k tends to ∞.
ℓ is a *cluster point for the function*

$$t \mapsto g(t), \quad \text{as} \quad t \to 0^+$$

if there is a real sequence (t_n) tending to zero, such that $g(t_n)$ tends to ℓ, as n tends to ∞.

Theorem 4.2 says that the set of cluster points for the sequence $D_{A,\alpha}(n)$ is equal to the set of cluster points on 0 for the function $\Delta_{A,\alpha}(t)$. By taking the supremum and the infimum of this common set of cluster points, we deduce the following corollary.

Corollary 4.3. *Let $\alpha > -1$; let A be a regular subset of $\mathbb{N}^*$, with regularity exponent $\gamma = 1$. Then $\overline{\delta}_0(A) = \overline{d}_0(A)$ and $\underline{\delta}_0(A) = \underline{d}_0(A)$.*

In order to give the proof of Theorem 4.2, we need some preliminaries. We start by recalling a well known tauberian theorem (proved in [1, pp. 443-444]).

Theorem 4.4. *Let F be a distribution function with Laplace transform ϕ, i.e.,*

$$\phi(t) = \int_0^\infty e^{-tx}\, F(dx).$$

We assume that $\phi(t)$ is finite for every t in a right neighborhood of 0. Let ρ be a real number, with $\rho \geq 0$. The two following conditions are equivalent:

$$\lim_{x\to\infty} \frac{F(xy)}{F(x)} = y^\rho\,; \tag{4.4)(a}$$

$$\lim_{t\to 0^+} \frac{\phi(t\lambda)}{\phi(t)} = \frac{1}{\lambda^\rho}\,\cdot \tag{4.4)(b}$$

Moreover, each of them implies that

$$\phi(t) \sim \Gamma(\rho+1)F(t^{-1}), \quad as \quad t \to 0^+ \,. \tag{4.5}$$

Remark 4.6. Condition (4.4)(a) (resp. (4.4)(b)) means that F (resp ϕ) is *regularly varying* with exponent ρ (resp. $-\rho$), as $x \to \infty$ (resp. as $t \to 0^+$).

An immediate consequence of Theorem 4.4 is the following.

Proposition 4.7. *Assume that either of the equivalent conditions of Theorem 4.4 holds, and let $\ell \geq 0$. The two following conditions are equivalent:*

(4.7)(a) *ℓ is a cluster point for the function*

$$x \mapsto \frac{F(x)}{x}, \quad as \quad x \to \infty;$$

(4.7)(b) *$\Gamma(\rho+1)\cdot\ell$ is a cluster point for the function*

$$t \mapsto t\phi(t), \quad as \quad t \to 0^+.$$

Proof of Proposition 4.7. (a)⇒(b). Let $(x_n)_{n\geq 1}$ be a sequence of real numbers such that

$$\lim_{n\to\infty} x_n = \infty; \quad \lim_{n\to\infty} \frac{F(x_n)}{x_n} = \ell.$$

Put $t_n = x_n^{-1}$. Then $\lim_{n\to\infty} t_n = 0$ and from relation (4.5) we deduce

$$\lim_{n\to\infty} t_n\phi(t_n) = \lim_{n\to\infty} t_nF(t_n^{-1})\Gamma(\rho+1) = \lim_{n\to\infty} \frac{F(x_n)}{x_n}\Gamma(\rho+1) = \Gamma(\rho+1)\cdot\ell.$$

(b)⇒(a) is similar. □

Proof of Theorem 4.4. Now we are ready to prove Theorem 4.4. Consider the measure on $\mathbb{N}^*$ defined as

$$\mu = \sum_{k\in\mathbb{N}^*} k^\alpha 1_A(k)\epsilon_{\mathbb{N}^*_\alpha(k)},$$

where ϵ_x denoted the Dirac mass placed in the point x. The distribution function of μ is

$$F(x) = \sum_{k:\mathbb{N}^*_\alpha(k)\leq x} k^\alpha 1_A(k), \tag{4.8}$$

while its Laplace transform is given by the formula

$$\phi(t) = \sum_{k\in\mathbb{N}^*} k^\alpha 1_A(k)e^{-t\mathbb{N}^*_\alpha(k)}.$$

The abscissa of convergence σ_0 of the Dirichlet series defining $\phi(t)$ is given by the well known formula (see [4])

$$\sigma_0 = \limsup_{n\to\infty} \frac{\log\Big(\sum_{k=1}^n k^\alpha 1_A(k)\Big)}{\mathbb{N}_\alpha^*(n)},$$

which equals 0 since

$$0 \le \frac{\log\Big(\sum_{k=1}^n k^\alpha 1_A(k)\Big)}{\mathbb{N}_\alpha^*(n)} \le \frac{\log\Big(\sum_{k=1}^n k^\alpha\Big)}{\mathbb{N}_\alpha^*(n)} = \frac{\log \mathbb{N}_\alpha^*(n)}{\mathbb{N}_\alpha^*(n)}.$$

Hence ϕ is defined for every $t > 0$ and the first assumption of Theorem 4.4 is satisfied.

By Proposition 4.7, the Theorem will be proved if we show that

(i) F satisfies assumption (4.4) (a) with exponent $\rho = 1$;

(ii) assumption (4.2) (a) is equivalent to assumption (4.7)(a).

(i) By Abel's summation formula and (4.1) we have, for every integer n,

$$\begin{aligned}\sum_{k\le n} k^\alpha 1_A(k) &= A(n)n^\alpha - \sum_{k\le n-1} A(k)\big((k+1)^\alpha - k^\alpha\big) \\ &= n^{\alpha+1}L(n)(1 - M(n)) \qquad (4.9)\end{aligned}$$

where

$$M(n) = \frac{\sum_{k\le n-1} kL(k)\big((k+1)^\alpha - k^\alpha\big)}{n^{\alpha+1}L(n)}.$$

We prove that

$$\lim_{n\to\infty} M(n) = \frac{\alpha}{\alpha+1}. \qquad (4.10)$$

Put

$$R(n) = \frac{(n+1)^\alpha - n^\alpha}{\alpha n^{\alpha-1}}.$$

Then, by an application of Lagrange's Theorem it is easily seen that

$$\lim_{n\to\infty} R(n) = 1,$$

so that, for every $\epsilon > 0$, there exists an integer n_0 such that, for every integer $n > n_0$, we have $1 - \epsilon < R(n) < 1 + \epsilon$. Take $n > n_0 + 1$. Then we can write

$$\begin{aligned}M(n) &= \alpha \frac{\sum_{k\le n-1} k^\alpha L(k)R(k)}{n^{\alpha+1}L(n)} \\ &= \alpha \frac{\sum_{k\le n_0} k^\alpha L(k)R(k)}{n^{\alpha+1}L(n)} + \alpha \frac{\sum_{n_0<k\le n-1} k^\alpha L(k)R(k)}{n^{\alpha+1}L(n)}.\end{aligned}$$

The first fraction above goes to 0 as $n \to \infty$, since $\lim_{n\to\infty} n^{\alpha+1}L(n) = \infty$. As to the second one, we have

$$(1-\epsilon)\frac{\sum_{n_0<k\le n-1} k^\alpha L(k)}{n^{\alpha+1}L(n)} \le \frac{\sum_{n_0<k\le n-1} k^\alpha L(k)R(k)}{n^{\alpha+1}L(n)}$$
$$\le (1+\epsilon)\frac{\sum_{n_0<k\le n-1} k^\alpha L(k)}{n^{\alpha+1}L(n)}.$$

By Lemma 4.8, p. 180 of [3], the first and last fractions above go to $(\alpha+1)^{-1}$ as $n \to \infty$, and we conclude by the arbitrariness of ϵ .

Now put

$$n_x = \max\{k \in \mathbb{N}^* : \mathbb{N}^*_\alpha(k) \le x\},$$

so that we can write

$$F(x) = \sum_{k:\mathbb{N}^*_\alpha(k)\le x} k^\alpha 1_A(k) = \sum_{k\le n_x} k^\alpha 1_A(k). \tag{4.11}$$

The two relations

$$\mathbb{N}^*_\alpha(n) \sim \mathbb{N}^*_\alpha(n+1), \qquad n\to\infty \qquad \text{and} \qquad \mathbb{N}^*_\alpha(n_x) \le x < \mathbb{N}^*_\alpha(n_x+1)$$

yield that

$$\mathbb{N}^*_\alpha(n_x) \sim x, \qquad x\to\infty. \tag{4.12}$$

On the other hand, from (4.12) and the relation

$$\mathbb{N}^*_\alpha(n) \sim \frac{n^{\alpha+1}}{\alpha+1},$$

we deduce that

$$\frac{{n_x}^{\alpha+1}}{\alpha+1} \sim x,$$

or, equivalently

$$n_x \sim \big((\alpha+1)x\big)^{(\alpha+1)^{-1}}. \tag{4.13}$$

Recalling (4.11), we can use (4.13) in (4.9) and, from (4.10) and the fact that L is slowly varying we get immediately

$$\lim_{t\to\infty} \frac{F(xt)}{F(x)} = x,$$

i.e., the claim of point (i).

(ii) (4.2)(a) $\Rightarrow$ (4.7)(a). Assume that ℓ is a limit point for the sequence

$$n \mapsto \frac{\sum_{k\le n} k^\alpha 1_A(k)}{\mathbb{N}^*_\alpha(n)}, \quad \text{as} \quad n\to\infty;$$

since (see (4.8))

$$\sum_{k\le n} k^\alpha 1_A(k) = \sum_{\mathbb{N}^*_\alpha(k)\le \mathbb{N}^*_\alpha(n)} k^\alpha 1_A(k) = F(\mathbb{N}^*_\alpha(n)),$$

the above statement is clearly equivalent to the statement that ℓ is a limit point for the sequence

$$n \mapsto \frac{F(\mathbb{N}^*_\alpha(n))}{\mathbb{N}^*_\alpha(n)}, \quad \text{as} \quad n\to\infty,$$

which in turn implies (4.7)(a).

(4.7)(a) $\Rightarrow$ (4.2)(a). Assume that ℓ is a limit point for the function

$$x \mapsto \frac{F(x)}{x}, \quad \text{as} \quad x\to\infty.$$

and let (x_k) be a sequence of numbers such that $\lim_{k\to\infty} x_k = +\infty$ and

$$\lim_{k\to\infty} \frac{F(x_k)}{x_k} = \ell.$$

In order to simplify the notations, we put $\mathbb{N}^*_\alpha(n) = S(n)$. Denote by the same symbol S the function defined on $\mathbb{R}^+$ as

$$S(x) = \begin{cases} S(n) & \text{for } x = n, \forall\, n \in \mathbb{N}^*; \\ \text{linear} & \text{elsewhere .} \end{cases}$$

Then S is strictly increasing, hence invertible; let S^{-1} its inverse and put $n_k = \lfloor S^{-1}(x_k)\rfloor$. We want to prove that

$$\lim_{k\to\infty} \frac{F(S(n_k))}{S(n_k)} = \ell. \tag{4.14}$$

We can write

$$\frac{F(S(n_k))}{S(n_k)} = \frac{F(S(n_k))}{S(n_k+1)} \cdot \frac{S(n_k+1)}{S(n_k)}\cdot \tag{4.15}$$

The second fraction above goes to 1 as $k\to\infty$ since

$$S(n) \sim \frac{n^\alpha}{\alpha+1} \sim \frac{(n+1)^\alpha}{\alpha+1} \sim S(n+1), \quad n\to\infty.$$

As to the first one, we observe that, since $n_k \le S^{-1}(x_k) < n_k+1$, we have

$$S(n_k) \le x_k \le S(n_k+1)$$

hence, since F is non–decreasing,

$$\limsup_{k\to\infty} \frac{F(S(n_k))}{S(n_k+1)} \le \limsup_{k\to\infty} \frac{F(x_k)}{x_k} = \ell. \tag{4.16}$$

From (4.15) and (4.16) we conclude that

$$\limsup_{k\to\infty} \frac{F(S(n_k))}{S(n_k)} \le \ell. \tag{4.17}$$

A similar argument shows that

$$\liminf_{k\to\infty} \frac{F(S(n_k))}{S(n_k)} \ge \ell, \tag{4.18}$$

hence, from (4.17) and (4.18)

$$\lim_{k\to\infty} \frac{\sum_{k\le n_k} k^\alpha 1_A(k)}{S(n_k)} = \lim_{k\to\infty} \frac{F(S(n_k))}{S(n_k)} = \ell,$$

which concludes the proof of (ii) and of Theorem 4.2. □

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