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STAR EXTREME DISCREPANCY OF GENERALIZED TWO-DIMENSIONAL HAMMERSLEY POINT SETS

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ABSTRACT. We generalize to arbitrary bases recent results on the star extreme discrepancy of digitally shifted two-dimensional Hammersley point sets in base 2 by Kritzer, Larcher and Pillichshammer. The key idea is to link our fundamental formula for the discrepancy function of generalized van der Corput sequences to the corresponding quantity for generalized two-dimensional Hammersley point sets. In that way, we can derive precise formulas for the star extreme discrepancy of these point sets and obtain simple generalizations which are the best presently known with regard to low star extreme discrepancy. This study is parallel to the recent one by F. Pillichshammer and the author on the (star) L_p discrepancy ($p < \infty$) of the same point sets (to appear in Monatsh. Math.).

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1. Introduction

For a point set $\mathcal{P} = \{\boldsymbol{x}_1, \ldots, \boldsymbol{x}_N\}$ of $N \geq 1$ points in the s-dimensional unit-square $[0,1]^s$ and a subinterval J of $[0,1]^s$, the *discrepancy function* is defined as

$$E(J;\mathcal{P}) = A(J;\mathcal{P}) - NV(J)$$

where $A(J;\mathcal{P}) = \#\{M; 1 \leq M \leq N, \boldsymbol{x}_M \in J\}$ and $V(J)$ is the volume of J.

In this paper, we are interested in the measures of irregularity of distribution obtained with the supremum (L_∞) norm of the discrepancy function which measures are usually called *extreme discrepancies.* Classically, one considers the supremum either on all intervals or only on intervals anchored at the origin and

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defines the *(extreme) discrepancy* D and the *star (extreme) discrepancy* D^* of $\mathcal{P}$ as

$$D(\mathcal{P}) = \sup_J |E(J;\mathcal{P})| \quad \text{and} \quad D^*(\mathcal{P}) = \sup_{J^*} |E(J^*;\mathcal{P})|$$

where J (resp. J^*) is in the shape of $\prod_{i=1}^s [y_i, z_i)$ (resp. $\prod_{i=1}^s [0, y_i)$). Recall it is not difficult to prove that we always have

$$D^*(\mathcal{P}) \leq D(\mathcal{P}) \leq 2^s D^*(\mathcal{P}).$$

Various monographs have been devoted to the theory of discrepancy in the past decades; we refer to [4] and [19] among the last; see also [20] and [23] for an overview.

For an infinite sequence $\mathcal{X}$, we write respectively

$$E(J;N;\mathcal{X}) = A(J;N;\mathcal{X}) - NV(J),$$

$D^*(N,\mathcal{X})$ and $D(N,\mathcal{X})$ for the discrepancy function, the star discrepancy and the discrepancy of its first N points. Also, we do not normalize by the number of points N and keep the original definition which is better adapted to our statements.

In the following, we deal only with dimensions 1 or 2. In these two dimensions, in order to simplify the writing, we use the notations

$$E(x, y, \mathcal{P}) := E([0, x) \times [0, y); \mathcal{P}) \text{ and } E(\alpha; N; \mathcal{X}) := E([0, \alpha); N; \mathcal{X})$$

for the discrepancy function relative to intervals anchored at the origin.

Relation between sequences and point sets. A general principle (also valid in any dimension) states the link between one-dimensional sequences and two-dimentional point sets deduced from them [6, 20]: let $\mathcal{X} = (X_N)_{N\geq 1}$ be an infinite sequence taking its values in $[0, 1]$ and let $\mathcal{P}$ be the two-dimentional point set

$$\mathcal{P} = \left\{ \left(X_M, \frac{M-1}{N} \right) \; ; \; 1 \leq M \leq N \right\} \subset [0,1]^2. \text{ Then}$$

$$\max_{1\leq M\leq N} D^*(M,\mathcal{X}) \leq D^*(\mathcal{P}) \leq \max_{1\leq M\leq N} D^*(M,\mathcal{X}) + 1. \tag{1}$$

Roth was the first to use such point sets in base 2, deduced from the original van der Corput sequence, in his famous paper [21] on the L_2 discrepancy. Then, their utilization with bases $b \geq 2$ and in arbitrary dimension has been proposed by Hammersley for numerical integration [14] just before the introduction by Halton of his well-known sequences [12], which can be seen as multi-dimensional versions of van der Corput sequences. In this paper, we will consider generalized two-dimensional Hammersley point sets deduced from the generalized van der Corput sequences introduced in [5]:

Definition 1 (generalized van der Corput sequence)**.** Let $b \geq 2$ be an integer and let $\Sigma = (\sigma_r)_{r\geq 0}$ be a sequence of permutations of $\{0, 1, \ldots, b-1\}$. For any integers n and N with $n \geq 0$ and $1 \leq N \leq b^n$, write $N - 1 = \sum_{r=0}^{\infty} a_r(N)\ b^r$ in the b–adic system (so that $a_r(N) = 0$ if $r \geq n$). Then the *generalized van der Corput sequence* S_b^Σ *in base* b *associated to* Σ is defined by

$$S_b^\Sigma(N) := \sum_{r=0}^{\infty} \frac{\sigma_r\big(a_r(N)\big)}{b^{r+1}}, \quad \text{for all } N \geq 1\cdot$$

If $(\sigma_r) = (\sigma)$ is constant, we write $S_b^\Sigma = S_b^\sigma$. The *original van der Corput sequence* in base b, S_b^{id}, is obtained with the identical permutation id.

Definition 2 (generalized Hammersley point set)**.** Let $b \geq 2$ be an integer, let S_b^Σ be a generalized van der Corput sequence in base b and let $n \geq 0$ be an integer. Then the *generalized two-dimensional Hammersley point set in base* b consisting of b^n points associated to Σ is defined by

$$\mathcal{H}_{b,n}^\Sigma := \left\{ \left(S_b^\Sigma(N), \frac{N-1}{b^n} \right); 1 \leq N \leq b^n \right\}.$$

In order to match with the traditional definition of arbitrary (shifted or not) Hammersley point sets which are "n-bits" (i.e., whose b-adic expansions do not exceed n bits), we restrict the infinite sequence of permutations Σ to permutations such that $\sigma_r(0) = 0$ for all $r \geq n$, for instance

$$\Sigma = (\sigma_0, \ldots, \sigma_{n-1}, id, id, id, \ldots).$$

Hence, the behavior of $\mathcal{H}_{b,n}^\Sigma$ will only depend on the finite sequence of n permutations $\boldsymbol{\sigma} = (\sigma_0, \ldots, \sigma_{n-1})$ and we will write $\mathcal{H}_{b,n}^\Sigma =: \mathcal{H}_{b,n}^{\boldsymbol{\sigma}}$.

Again, if we choose in the above definition $\sigma_j = id$ for all permutations, then we obtain the classical two-dimensional Hammersley point set in base b, $\mathcal{H}_{b,n}^{id}$. In base 2, this point set is known as Roth point set since it was first considered by Roth to give a complement to his lower bound on L_2 discrepancy (in fact Roth obtains the order $\log N$ for the star discrepancy of that point set and deduces his complement from it [21, (3) p.74]).

A profound result of Schmidt [22] (improved by Béjian [1] ten years after) states that $D^*(\mathcal{P}) > c \log N$ for any two-dimensional point set $\mathcal{P}$ of N points with an absolute constant $c \geq 0.06$. This lower bound is best possible, apart from the constant c, thanks to Roth and Hammersley point sets. Hence, the goal is to improve the leading constant term in the estimate $\mathcal{O}(\log N)$. Such improvements are achieved with generalized Hammersley point sets.

The main results concerning the star discrepancy of two-dimensional Hammersley point sets are of two kinds: some give exact formulas including complementary terms and the other give formulas for the leading terms within an error not computable, usually lower than a small additive constant. We recall them without going into the details of the formulas. If necessary more information will be given at the right place in the paper.

Three papers belong to the first kind:

First, Halton and Zaremba [13, 1969] obtained exact formulas both for the star extreme and L_2 discrepancies of the Roth point set $\mathcal{H}_{2,n}^{\boldsymbol{id}}$ and for a modification of this set which, in our terminology, reads as $\mathcal{H}_{2,n}^{\boldsymbol{\sigma}}$ with the sequence $\boldsymbol{\sigma} = (id, \tau, id, \tau, \ldots, id, \tau)$, where τ is the transposition ($\tau(k) = k + 1 \pmod 2$) usually called *digital shift*. The leading terms in their formulas for $D^*(\mathcal{H}_{2,n}^{\boldsymbol{id}})$ and $D^*(\mathcal{H}_{2,n}^{\boldsymbol{\sigma}})$ are respectively $\frac{1}{3}n$ and $\frac{1}{5}n$.

Next, L. De Clerck, in a remarkable paper [3, 1986] summing up her thesis (in Flemish), has been able to compute exactly the star extreme discrepancy of the classical Hammersley point set $\mathcal{H}_{b,n}^{\boldsymbol{id}}$ in an arbitrary base b, giving at the same time the intervals for which the discrepancy is reached (sometimes called extremal, see [18] p. 380). The leading term in her formula for $D^*(\mathcal{H}_{b,n}^{\boldsymbol{id}})$ is $\frac{b-1}{4}n$ if b is odd and $\frac{b^2}{4(b+1)}n$ if b is even.

Finally, Larcher and Pillichshammer, as a by-product of their study of $(0, m, 2)$ -nets in base 2 [18, 2003], gave a new proof of the exact formula found by Halton and Zaremba (and De Clerck) for $D^*(\mathcal{H}_{2,n}^{\boldsymbol{id}})$. There is an interesting discussion in this paper concerning the *discrete discrepancy* (instead of reals, the supremum is taken over n-bits numbers): the authors observe that "it differs from D^* at most by the almost negligible quantity 2 and that it seems for nets to be the more natural measure for the irregularities of distribtution". Moreover, their proofs of the exact formulas for these two discrepancies of $\mathcal{H}_{2,n}^{\boldsymbol{id}}$ "clearly illustrate that the discrete discrepancy is much easier to handle than D^*, in spite of the minimal difference between them" [18, Section 4 p. 399].

The other papers belong to the second kind:

First, the author [6, 1986], in parallel with the work of De Clerck, deduced approximate formulas from his study of generalized van der Corput sequences in arbitrary bases: general upper bounds valid for generalized Hammersley point sets $\mathcal{H}_{b,n}^{\Sigma}$ not necessarily n-bits nor limited to b^n points (Theorems 1 and 2) and more specific estimates for $D^*(\mathcal{H}_{b,n}^{\boldsymbol{id}})$ giving the results of De Clerck within an additive term $c_n \in [0, 2]$ (Theorem 3). As a consequence of Theorem 2, we got the smallest star discrepancy for finite two-dimensional point sets with the leading constant $0.223\ldots$ in base 12 (compare with the lower bound 0.06), but with a complementary term in $\mathcal{O}(\sqrt{\log N})$.

Next, Kritzer [15, 2006], extending the fundamental Theorem 1 of [18], proved that $\mathcal{H}_{2,n}^{\boldsymbol{id}}$ is the worst distributed net among shifted digital $(0, m, 2)-$nets in base 2 and also that digitally shifted Hammersley nets in base 2 are still better than $\mathcal{H}_{2,n}^{\boldsymbol{id}}$. Among them, with the net $\mathcal{H}_{2,n}^{\boldsymbol{i\tau}}$, where $\boldsymbol{i\tau} = (id, \ldots, id, \tau, \ldots, \tau)$ (half and half about), he obtained $D^*(\mathcal{H}_{2,n}^{\boldsymbol{i\tau}}) \leq \frac{n}{6} + c$ with $c < \frac{4}{3}$, which gives the leading constant $\frac{1}{6 \log 2} = 0.240 \ldots$.

Next, Kritzer, Larcher and Pillichshammer [16, 2007], continuing the preceding paper, made a very detailed investigation of sums of distances to the nearest integer and deduced an explicit formula (Theorem 3) for the star discrepancy of shifted Hammersley point sets in base 2, within an additive term $c_n \in [0, 2]$ (in fact, they computed exactly the discrete discrepancy of these sets, see above). Using that formula, they were able to generalize the result of Kritzer with the sequence $\boldsymbol{\sigma} = (\underbrace{id, \ldots, id}_{m}, \underbrace{\tau, \ldots, \tau}_{m}, \ldots, \underbrace{id, \ldots, id}_{m}, \underbrace{\tau, \ldots, \tau}_{m})$ (the same number m of permutations in each block). This formula also allows them to recover two results they proved before in the paper with less accuracy (Theorems 1 and 2). We will generalize these results to arbitrary bases in Section 4 of the present paper.

Finally, at the end of our paper [10, 2008], we derive improved results for generalized Hammersley point sets from improvements on van der Corput sequences, in the same way as in [6]. For instance, we show that the (so-called) linear digit scramblings give very good leading terms (around 0.3) in any given prime base. But the best constants in bases 2, 3 and 12 are still obtained with a complementary term in $\mathcal{O}(\sqrt{\log N})$ due to the definition of the related van der Corput sequences [10, Section 5.2]. It was one of the aims of the present paper to remove this restriction, as announced in the last lines of [10].

The paper is organized as follows:

Section 2 introduces the necessary material for the proofs of theorems of Sections 3 to 5. In Section 3, we prove a general formula for generalized two-dimensional Hammersley point sets in arbitrary base b which is the analog of [16, Lemma 1] in base 2. Section 4 is devoted to results obtained with the identical permutation: generalizations to base b of [15, Theorem 3.2] and [16, Theorems 1 and 2]. Finally, in Section 5 we deal with arbitrary permutations in base b and obtain simple generalizations of two-dimensional Hammersley point sets which are the best presently known with regard to low star discrepancy, hence carrying out the proposal at the end of [10].

2. Prerequisites

In this section, we recall the main results we will need for the proofs of our theorems. They concern generalized van der Corput sequences and come from [5, 2], in French. Another reference is [9] in English in which, even if the subject is different, many definitions are the same and formulas are quite similar (see also [8]).

2.1. Functions $\varphi_{b,h}^{\sigma}$ related to a pair (b, σ)

Let σ be a permutation of $\{0, 1, \ldots, b-1\}$ and let $\mathcal{Z}_b^{\sigma} = \Big(\dfrac{\sigma(0)}{b}, \cdots, \dfrac{\sigma(b-1)}{b}\Big)$. For $h \in \{0, 1, \ldots, b-1\}$ and $x \in [(k-1)/b, k/b)$, where $k \in \{1, \ldots, b\}$, we define

$$\varphi_{b,h}^{\sigma}(x) = \begin{cases} A\Big(\Big[0, \dfrac{h}{b}\Big); k; \mathcal{Z}_b^{\sigma}\Big) - hx & \text{if } 0 \leq h \leq \sigma(k-1), \\ (b-h)x - A\Big(\Big[\dfrac{h}{b}, 1\Big); k; \mathcal{Z}_b^{\sigma}\Big) & \text{if } \sigma(k-1) < h < b. \end{cases}$$

Further, the function $\varphi_{b,h}^{\sigma}$ is extended to $\mathbb{R}$ by periodicity. For future use in Lemma 2, $(\varphi_{b,h}^{\sigma})'$ denotes its right derivative. Note that $\varphi_{b,0}^{\sigma} = 0$ for any σ and that $\varphi_{b,h}^{\sigma}(0) = 0$ for any σ and any h.

As a matter of fact, the functions $\varphi_{b,h}^{\sigma}$ are linearizations of remainders related to $\mathcal{Z}_b^{\sigma}$. They are the fundamental tool for the study of irregularities of distribution of sequences S_b^{Σ}. Actually, they give rise to other functions, depending only on (b, σ), according to the notion of discrepancy we are dealing with. Presently, we will need

$$\psi_b^{\sigma,+} = \max_{0 \leq h \leq b-1} (\varphi_{b,h}^{\sigma}), \ \psi_b^{\sigma,-} = \max_{0 \leq h \leq b-1} (-\varphi_{b,h}^{\sigma}) \text{ and}$$

$$\psi_b^{\sigma} = \psi_b^{\sigma,+} + \psi_b^{\sigma,-} = \max_{0 \leq h < h' \leq b-1} |\varphi_{b,h'}^{\sigma} - \varphi_{b,h}^{\sigma}|.$$

Many properties of ψ functions are given in [5, 3.2]. Recall for future use that they are continuous and convex on intervals $\big[\frac{k-1}{b}, \frac{k}{b}\big]$ $(1 \leq k \leq b)$.

An important permutation which plays a leading part in constructions improving discrepancy [5, 6, 10, 11] is the so-called *swapping permutation* τ defined by $\tau(k) = b - k - 1$ $(0 \leq k \leq b-1)$. For a given permutation σ, τ swaps $\psi_b^{\sigma,+}$ and $\psi_b^{\sigma,-}$ which means that $\psi_b^{\tau\circ\sigma,+} = \psi_b^{\sigma,-}$ and $\psi_b^{\tau\circ\sigma,-} = \psi_b^{\sigma,+}$ (see [5, 4.4.1] and also [11, Lemma 1] when $\sigma = id$). We will use it in Sections 4 and 5.

In the special case $b = 2$, we only have two permutations which give either $\varphi_{2,1}^{\sigma} = \|\cdot\|$ if $\sigma = id$ or $\varphi_{2,1}^{\sigma} = -\|\cdot\|$ if $\sigma = \tau$, where $\|\cdot\|$ is the distance to the nearest integer. Naturally, this function takes a central place in the studies in base 2 of Kritzer, Larcher and Pillichshammer, especially [18, 15, 16] for the star discrepancy. Recall that in base 2, the swap τ is usually called *digital shift* because $\tau(k) = k+1$ in $\mathbb{F}_2$.

2.2. Exact formulas for the discrepancies of S_b^{Σ}

For any infinite sequence X, we define the positive and negative discrepancies as

$$D^+(N,\mathcal{X}) := \sup_{0\leq\alpha\leq 1} E(\alpha,N,\mathcal{X}) \text{ and } D^-(N,\mathcal{X}) := \sup_{0\leq\alpha\leq 1} (-E(\alpha,N,\mathcal{X})).$$

Then [5, Section 3.3.6], for all integers $n \geq 1$ and N with $1 \leq N \leq b^n$, we have

$$D^+(N,S_b^{\Sigma}) = \sum_{j=1}^{n} \psi_b^{\sigma_{j-1},+}\left(\frac{N}{b^j}\right) + \frac{N}{b^n} - N \sum_{j=n+1}^{\infty} \frac{\sigma_{j-1}(0)}{b^j} \tag{2}$$

$$\text{and } D^-(N,S_b^{\Sigma}) = \sum_{j=1}^{n} \psi_b^{\sigma_{j-1},-}\left(\frac{N}{b^j}\right) + N \sum_{j=n+1}^{\infty} \frac{\sigma_{j-1}(0)}{b^j}. \tag{3}$$

Recall that $D^* = \max(D^+, D^-)$ and $D = D^+ + D^-$, so that we also have exact formulas for $D^*(N,S_b^{\Sigma})$ and $D(N,S_b^{\Sigma})$.

In our paper [5], formulas (2) and (3) needed a lot of lemmas to be proved because, at this time, we did not have an exact formula for the discrepancy function $E(\alpha,N,S_b^{\Sigma})$. Indeed, they can be obtained more simply with the help of such a formula (see Section 2.4 below) which also will play a leading part in the study of generalized Hammersley point sets.

2.3. Estimates for sums of ψ functions

In order to compare generalized Hammersley point sets and find the best possible with respect to low discrepancy, we will need estimations of sums of ψ functions which appear in (2) and (3). We collect the necessary information in the following lemma, in which the notation ψ stands either for ψ_b^{σ} or $\psi_b^{\sigma,+}$ or $\psi_b^{\sigma,-}$.

Lemma 1. *(i) For any integer $n \geq 1$, let $d_n := \max_{x\in\mathbb{R}} \sum_{j=1}^{n} \psi\left(\frac{x}{b^j}\right)$ and $\alpha := \inf_{n\geq 1} \frac{d_n}{n}$. Then $\alpha = \lim_{n\to\infty} \frac{d_n}{n}$ and there exists $\beta_n \in [0,1]$ such that $d_n = \alpha n + \beta_n$ for all $n \geq 1$.*

(ii) For any $x \in [0,1]$ and any $n \geq 1$, let $F_n(x) := \sum_{k=0}^{n-1} \psi(xb^k)$. Then $d_n = \max_{x\in[0,1]} F_n(x)$ and for all integers ν, a with $1 \leq a \leq b^\nu$, we have

$$\frac{1}{\nu} F_\nu \left(\frac{a}{b^\nu - 1} \right) \leq \alpha.$$

P r o o f. The proof is scattered in [5, 4.2.2 and 5.2.1] (see also [2, 6.3.1-2]). Practically, the exact computation of α is possible in some specific cases, for instance for small b or for the identical permutation. In that cases, the notion of dominating interval is very useful to reduce the problem and to formulate an induction hypothesis for the value of d_n (see [5, 5.3] and also [8, 4.4]). But in any case, it is easy to obtain upper and lower bounds with small values of n and ν in the formulas $\alpha = \inf_{n\geq 1} \frac{d_n}{n}$ and $\frac{1}{\nu} F_\nu \left(\frac{a}{b^\nu - 1} \right) \leq \alpha$ above. □

2.4. Descent Lemma

This is the fundamental Lemma 5.2 of [2] and, in the adaptation to $(0,1)-$sequences, Lemma 6.2 of [9]. The foremost version was found for the precise study of the L_2 discrepancy of the van der Corput sequence and its symmetrisized version in base two [7].

Lemma 2. *Let $n \geq 1, N$ and λ be integers with $1 \leq N \leq b^n$ and $1 \leq \lambda < b^n$ and let $\lambda = \lambda_1 b^{n-1} + \cdots + \lambda_{n-1} b + \lambda_n$ be the b-adic expansion of λ. Then, the discrepancy function of the first N points of S_b^Σ on $[0, \frac{\lambda}{b^n}[$ satisfies*

$$E\Big(\frac{\lambda}{b^n}; N; S_b^\Sigma\Big) = \sum_{j=1}^{n} \varphi_{b,\varepsilon_j}^{\sigma_{j-1}} \Big(\frac{N}{b^j}\Big), \tag{4}$$

the ε_j's being defined step by step as follows: $\varepsilon_n := \eta_n := \lambda_n$ and,

$$\text{for } 1 \leq j < n, \quad \eta_j := \lambda_j + \frac{\eta_{j+1}}{b} + \frac{1}{b} \left(\varphi_{b,\varepsilon_{j+1}}^{\sigma_{j-1}} \right)' \Big(\frac{N}{b^{j+1}} \Big), \tag{5}$$
$$\varepsilon_j := \eta_j \quad \text{if} \quad 0 \leq \eta_j < b \quad \text{and} \quad \varepsilon_j := 0 \quad \text{if} \quad \eta_j = b.$$

Remark 1. For $\lambda = b^n$ the formula is trivially true with $\varepsilon_j = 0$ for all $1 \leq j \leq n$ (since $E\left(1; N; S_b^\Sigma\right) = N - N = 0$) and for $N = b^n$ also since again both terms are naught ($E\left(\frac{\lambda}{b^n}; b^n; S_b^\Sigma\right) = \lambda - \lambda = 0$ and the functions φ are zero on integers).

P r o o f. We refer to [2] for the proof. The analog for $(0,1)-$sequences in [9], in English, is more complicated but follows the same framework: step by step, we obtain the discrepancy function $E(\frac{\lambda}{b^n}; N; S_b^\Sigma)$ by means of discrepancy functions with more and more rough intervals and with less and less points; at each step, the difference between the discrepancy functions is under control with the help of the functions $\varphi_{b,h}^{\sigma}$ while the relation between the intervals depends on the right derivatives of these functions. At the end of the descent, we obtain Equation (4). □

3. A general formula for $D^*(\mathcal{H}_{b,n}^{\boldsymbol{\sigma}})$

Now, we deal with generalized Hammersley points sets $\mathcal{H}_{b,n}^{\boldsymbol{\sigma}}$ and first link the discrepancy function of $\mathcal{H}_{b,n}^{\boldsymbol{\sigma}}$ on the two-dimensional interval $[0, \frac{\lambda}{b^n}) \times [0, \frac{N}{b^n})$ to the discrepancy function of the first N points of S_b^{Σ} on $[0, \frac{\lambda}{b^n})$ (recall from the definition of $\mathcal{H}_{b,n}^{\boldsymbol{\sigma}}$ that $\Sigma = (\sigma_0, \ldots, \sigma_{n-1}, id, id, \ldots)$ and $\boldsymbol{\sigma} = (\sigma_0, \ldots, \sigma_{n-1})$).

Lemma 3. *Let $n \geq 1, N$ and λ be integers with $1 \leq N \leq b^n$ and $1 \leq \lambda < b^n$. Then,*

$$E\left(\frac{\lambda}{b^n}, \frac{N}{b^n}, \mathcal{H}_{b,n}^{\boldsymbol{\sigma}}\right) = E\left(\frac{\lambda}{b^n}; N; S_b^{\Sigma}\right)$$

P r o o f. The proof is almost immediate: from the definition of $\mathcal{H}_{b,n}^{\boldsymbol{\sigma}}$ the counting functions $A\left(\frac{\lambda}{b^n}, \frac{N}{b^n}, \mathcal{H}_{b,n}^{\boldsymbol{\sigma}}\right)$ and $A\left(\frac{\lambda}{b^n}; N; S_b^{\Sigma}\right)$ take the same value and on the other hand, $b^n \dfrac{\lambda}{b^n} \dfrac{N}{b^n} = N \dfrac{\lambda}{b^n}$, so that the discrepancy functions are equal. □

Note that Lemma 3 together with Lemma 2 is very close to [11, Lemma 1] (only the form of digits ε_j in Lemma 2 differs). Next, we prove an exact formula for the so-called (star) discrete discrepancy (see [18, Section 4]):

Lemma 4. *Let $n \geq 1$ be an integer. Then, the star discrete discrepancy of $\mathcal{H}_{b,n}^{\boldsymbol{\sigma}}$,*

$$\begin{aligned}
&\max_{1 \leq \lambda, N \leq b^n} \left| E\left(\frac{\lambda}{b^n}, \frac{N}{b^n}, \mathcal{H}_{b,n}^{\boldsymbol{\sigma}}\right) \right| \\
&= \max\left(\max_{1 \leq N \leq b^n} \sum_{j=1}^{n} \psi_b^{\sigma_{j-1},+}\left(\frac{N}{b^j}\right), \max_{1 \leq N \leq b^n} \sum_{j=1}^{n} \psi_b^{\sigma_{j-1},-}\left(\frac{N}{b^j}\right) \right). \qquad (6)
\end{aligned}$$

P r o o f. From Lemmas 2 and 3 we first have (the ε_j being defined by the formulas (5)):

$$E\left(\frac{\lambda}{b^n}, \frac{N}{b^n}, \mathcal{H}_{b,n}^{\boldsymbol{\sigma}}\right) = \sum_{j=1}^{n} \varphi_{b,\varepsilon_j}^{\sigma_{j-1}}\left(\frac{N}{b^j}\right) \quad \text{for all } 1 \leq \lambda, N \leq b^n.$$

Now, from the definitions of $\psi_b^{\sigma,+}$ and $\psi_b^{\sigma,-}$, we deduce that

$$E\left(\frac{\lambda}{b^n}, \frac{N}{b^n}, \mathcal{H}_{b,n}^{\boldsymbol{\sigma}}\right) \le \sum_{j=1}^{n} \psi_b^{\sigma_{j-1},+}\Big(\frac{N}{b^j}\Big),\ -E\left(\frac{\lambda}{b^n}, \frac{N}{b^n}, \mathcal{H}_{b,n}^{\boldsymbol{\sigma}}\right) \le \sum_{j=1}^{n} \psi_b^{\sigma_{j-1},-}\Big(\frac{N}{b^j}\Big).$$

But these upper bounds are reached for some λ's by using the reverse algorithm of the construction (5) of the ε_j from the λ_j as we did in the proof of Theorem 1 of [9, 6.4]. We recall the proof for the sake of completeness: for $1 \le j \le n$, let p_j ($0 \le p_j < b$) be an integer such that $\psi_b^{\sigma_{j-1},+}(\frac{N}{b^j}) = \varphi_{b,p_j}^{\sigma_{j-1}}(\frac{N}{b^j})$, so that we have a fixed sequence $\varepsilon_j = p_j$ from which we can deduce an integer $\lambda = \lambda_1 b^{n-1} + \cdots + \lambda_n$ satisfying $E(\frac{\lambda}{b^n}, \frac{N}{b^n}, \mathcal{H}_{b,n}^{\boldsymbol{\sigma}}) = \sum_{j=1}^n \varphi_{b,p_j}^{\sigma_{j-1}}(\frac{N}{b^j}) = \sum_{j=1}^n \psi_b^{\sigma_{j-1},+}(\frac{N}{b^j})$. Indeed, using the formulas of (5) for η_j and ε_j, we build λ step by step, first by setting $\lambda_n := \eta_n := p_n$ and then, if η_j and λ_j are achieved, by using the reverse algorithm:

if $\frac{\eta_j}{b} + \frac{1}{b}(\varphi_{b,p_j}^{\sigma_{j-1}})'(\frac{N}{b^j}) = 0$ then $\lambda_{j-1} := \eta_{j-1} := p_{j-1}$
else (i.e. $\frac{\eta_j}{b} + \frac{1}{b}(\varphi_{b,p_j}^{\sigma_{j-1}})'(\frac{N}{b^j}) = 1$) then
  if $p_{j-1} \ge 1$ then $\eta_{j-1} := p_{j-1}$ and $\lambda_{j-1} := \eta_{j-1} - 1$
  else (i.e. $p_{j-1} = 0$) then $\eta_{j-1} := b$ and $\lambda_{j-1} := \eta_{j-1} - 1$.

The same is valid for $\psi_b^{\sigma,-}$ and therefore we have proved that

$$\max_{1\le\lambda\le b^n} E\left(\frac{\lambda}{b^n}, \frac{N}{b^n}, \mathcal{H}_{b,n}^{\boldsymbol{\sigma}}\right) = \sum_{j=1}^{n} \psi_b^{\sigma_{j-1},+}\left(\frac{N}{b^j}\right) =: D_n^+(N, S_b^\Sigma) \text{ and}$$

$$\max_{1\le\lambda\le b^n} \left(-E\left(\frac{\lambda}{b^n}, \frac{N}{b^n}, \mathcal{H}_{b,n}^{\boldsymbol{\sigma}}\right)\right) = \sum_{j=1}^{n} \psi_b^{\sigma_{j-1},-}\left(\frac{N}{b^j}\right) =: D_n^-(N, S_b^\Sigma), \text{ which implies}$$

$$\max_{1\le\lambda\le b^n} \left|E\left(\frac{\lambda}{b^n}, \frac{N}{b^n}, \mathcal{H}_{b,n}^{\boldsymbol{\sigma}}\right)\right| = \max\left(D_n^+(N, S_b^\Sigma), D_n^-(N, S_b^\Sigma)\right), \text{ so that}$$

$$\max_{1\le\lambda,N\le b^n} \left|E\left(\frac{\lambda}{b^n}, \frac{N}{b^n}, \mathcal{H}_{b,n}^{\boldsymbol{\sigma}}\right)\right| = \max_{1\le N\le b^n} \left(\max\left(D_n^+(N, S_b^\Sigma), D_n^-(N, S_b^\Sigma)\right)\right).$$

($D_n^+(N, S_b^\Sigma)$ and $D_n^-(N, S_b^\Sigma)$ are called the discrete positive and negative discrepancies of S_b^Σ). Finally, discussing the two possibilities for the integer N_0 which achieves the value of $\max_{1\le N\le b^n}\left(\max\left(D_n^+(N, S_b^\Sigma), D_n^-(N, S_b^\Sigma)\right)\right)$, we see that

$$\max_{1\le N\le b^n}\left(\max\left(D_n^+(N, S_b^\Sigma), D_n^-(N, S_b^\Sigma)\right)\right) = \max\left(\max_{1\le N\le b^n}\left(D_n^+(N, S_b^\Sigma)\right), \max_{1\le N\le b^n}\left(D_n^-(N, S_b^\Sigma)\right)\right)$$

and Lemma 4 follows. $\square$

Theorem 1. *For any integer $n \geq 1$ and any $\boldsymbol{\sigma} = (\sigma_0, \ldots, \sigma_{n-1})$ we have, with some $c_n \in [0, 2]$,*

$$D^*(\mathcal{H}_{b,n}^{\boldsymbol{\sigma}}) = \max\left(\max_{1\leq N\leq b^n} \sum_{j=1}^{n} \psi_b^{\sigma_{j-1},+}\left(\frac{N}{b^j}\right)\ ,\ \max_{1\leq N\leq b^n} \sum_{j=1}^{n} \psi_b^{\sigma_{j-1},-}\left(\frac{N}{b^j}\right)\right) + c_n.$$

P r o o f. For $x \in [0,1]$, let $x(n) := \dfrac{\lceil x b^n \rceil}{b^n}$, so that $x(n) - \dfrac{1}{b^n} \leq x < x(n)$. Since all points of $\mathcal{H}_{b,n}^{\boldsymbol{\sigma}}$ have n-bit coordinates (i.e., are of the form α/b^n), we have

$$E(x, y, \mathcal{H}_{b,n}^{\boldsymbol{\sigma}}) = E(x(n), y(n), \mathcal{H}_{b,n}^{\boldsymbol{\sigma}}) + b^n(x(n)y(n) - xy).$$

From the definition of $x(n)$ we deduce that $0 < b^n(x(n)y(n) - xy) \leq x(n) + y(n) - \frac{1}{b^n} < 2$ and therefore there exists $c_n \in [0, 2]$ such that

$$D^*(\mathcal{H}_{b,n}^{\boldsymbol{\sigma}}) = \sup_{x,y\in[0,1]} |E(x, y, \mathcal{H}_{b,n}^{\boldsymbol{\sigma}})| = \max_{1\leq \lambda, N\leq b^n} \left| E\left(\frac{\lambda}{b^n}, \frac{N}{b^n}, \mathcal{H}_{b,n}^{\boldsymbol{\sigma}}\right)\right| + c_n.$$

By Lemma 4, the proof of Theorem 1 is complete. $\square$

Corollary 1. *For any integer $n \geq 1$ and any $\boldsymbol{\sigma} = (\sigma_0, \ldots, \sigma_{n-1})$ we have, with some $c_n \in [-2, 2]$,*

$$D^*(\mathcal{H}_{b,n}^{\boldsymbol{\sigma}}) \leq D^*(\mathcal{H}_{b,n}^{\boldsymbol{id}}) + c_n.$$

P r o o f. This corollary results from the following properties of ψ functions: $\psi_b^{\sigma,+} \leq \psi_b^{id}$, $\psi_b^{\sigma,-} \leq \psi_b^{id}$, $\psi_b^{id,+} = \psi_b^{id}$ and $\psi_b^{id,-} = 0$. The complementary term c_n could be removed with more effort in the context of $(0, m, 2)-$nets. $\square$

4. Swapping with the identical permutation

In this section, we consider generalized Hammersley point sets $\mathcal{H}_{b,n}^{\boldsymbol{\sigma}}$, where $\boldsymbol{\sigma} = (\sigma_0, \ldots, \sigma_{n-1}) \in \{id, \tau\}^n$, i.e. where the permutations σ_j are either id or $\tau \circ id = \tau$, τ being the *swapping* permutation defined by $\tau(k) = b - k - 1$ $(0 \leq k \leq b-1)$. The more general case with an arbitrary permutation σ instead of id will be handled in Section 5, but recall (see Section 2.1) that the interest of permutation τ is to swap $\psi_b^{\sigma,+}$ for $\psi_b^{\sigma,-}$ and $\psi_b^{\sigma,-}$ for $\psi_b^{\sigma,+}$ i.e. to give $\psi_b^{\tau\circ\sigma,+} = \psi_b^{\sigma,-}$ and $\psi_b^{\tau\circ\sigma,-} = \psi_b^{\sigma,+}$ which, when $\sigma = id$, reads as $\psi_b^{\tau,+} = \psi_b^{id,-} = 0$ and $\psi_b^{\tau,-} = \psi_b^{id,+} = \psi_b^{id}$.

First let us consider the sequence $\boldsymbol{i\tau} = (\overbrace{id, \dots, id}^{\frac{n}{2}}, \overbrace{\tau, \dots \tau}^{\frac{n}{2}})$, if n is even and $\boldsymbol{i\tau} = (\overbrace{id, \dots, id}^{\frac{n-1}{2}}, \overbrace{\tau, \dots \tau}^{\frac{n+1}{2}})$, if n is odd, like Kritzer did in base 2 [15]. Applying Theorem 1, we can easily extend his result [15, Theorem 3.2 and proposition 3.1] to arbitrary bases:

Theorem 2. *For any integer $n \geq 1$ we have, with some $c_n \in [0,3]$,*

$$\text{if } b \text{ is odd:} \quad D^*(\mathcal{H}_{b,n}^{\boldsymbol{i\tau}}) = \begin{cases} \dfrac{b-1}{8} n + c_n & \text{if } n \text{ is even} \\ \dfrac{b-1}{8}(n+1) + c_n & \text{if } n \text{ is odd} \end{cases}$$

$$\text{if } b \text{ is even:} \quad D^*(\mathcal{H}_{b,n}^{\boldsymbol{i\tau}}) = \begin{cases} \dfrac{b^2}{8(b+1)} n + c_n & \text{if } n \text{ is even} \\ \dfrac{b^2}{8(b+1)}(n+1) + c_n & \text{if } n \text{ is odd.} \end{cases}$$

P r o o f. We first deal with an even integer, say $n = 2m$, and apply Theorem 1 with $\psi_b^{\sigma_{j-1},+} = \psi_b^{id,+} = \psi_b^{id}$, $\psi_b^{\sigma_{j-1},-} = \psi_b^{id,-} = 0$ if $1 \leq j \leq m$ and $\psi_b^{\sigma_{j-1},+} = \psi_b^{\tau,+} = 0$, $\psi_b^{\sigma_{j-1},-} = \psi_b^{\tau,-} = \psi_b^{id}$ if $m+1 \leq j \leq 2m$. The two sums of n terms reduce to sums of m terms and we get with some $c_n' \in [0,2]$

$$D^*(\mathcal{H}_{b,n}^{\boldsymbol{i\tau}}) = \max\left(\max_{1 \leq N \leq b^n} \sum_{j=1}^{m} \psi_b^{id}\left(\frac{N}{b^j}\right), \max_{1 \leq N \leq b^n} \sum_{j=m+1}^{2m} \psi_b^{id}\left(\frac{N}{b^j}\right) \right) + c_n'.$$

We see that the balance between the two sums of Theorem 1 is perfect, which will divide by 2 the discrepancy of the usual Hammersley net. Indeed, we know the behavior of sums of ψ functions, see Section 2.3, and since these functions are continuous, 1-periodic and convex on intervals $\left[\frac{k-1}{b}, \frac{k}{b}\right]$ $(1 \leq k \leq b)$, the maximum for $x \in \mathbb{R}$ in Lemma 1 (i) is reduced to the maximum for integers $N \in [1, b^n]$ only. Hence, applying Lemma 1 (i), we obtain (note that the two first equalities below hold as well from the same property):

$$\max_{1 \leq N \leq b^n} \sum_{j=1}^{m} \psi_b^{id}\left(\frac{N}{b^j}\right) = \max_{1 \leq N \leq b^n} \sum_{j=m+1}^{2m} \psi_b^{id}\left(\frac{N}{b^j}\right) = \max_{1 \leq N \leq b^m} \sum_{j=1}^{m} \psi_b^{id}\left(\frac{N}{b^j}\right)$$
$$= \alpha_b^{id} m + \beta_m$$

with $\beta_m \in [0,1]$. We have computed the constants α_b^{id} in [5, 5.5] and we found that $\alpha_b^{id} = \dfrac{b-1}{4}$, if b is odd and $\alpha_b^{id} = \dfrac{b^2}{4(b+1)}$, if b is even. This achieves the proof, when n is even (with $c_n = c_n' + \beta_m$).

In the case, where $n = 2m+1$ is odd, in the same way we obtain

$$D^*(\mathcal{H}_{b,n}^{\boldsymbol{i\tau}}) = \max\left(\max_{1\leq N\leq b^n} \sum_{j=1}^{m} \psi_b^{id}\left(\frac{N}{b^j}\right), \max_{1\leq N\leq b^n} \sum_{j=m+1}^{2m+1} \psi_b^{id}\left(\frac{N}{b^j}\right)\right) + c_n.$$

Here, the balance between the two sums of Theorem 1 differs by one term only, which explains the factor $n+1$ instead of n in the result for odd n. $\square$

Corollary 2. *Asymptotically, we get*

$$\lim_{n\to\infty} \frac{D^*(\mathcal{H}_{b,n}^{\boldsymbol{i\tau}})}{\log b^n} = \frac{b-1}{8\log b} \quad \textit{if } b \textit{ is odd and}$$

$$\lim_{n\to\infty} \frac{D^*(\mathcal{H}_{b,n}^{\boldsymbol{i\tau}})}{\log b^n} = \frac{b^2}{8(b+1)\log b} \quad \textit{if } b \textit{ is even.}$$

Remark 2. The interval for c_n could be reduced, in particular with the exact values for β_m. Of course, we recover the result of Kritzer [15] and Kritzer–Larcher–Pillichshammer [16], with the same sequence $\boldsymbol{i\tau}$, in the case of $b = 2$. The best constant is obtained for $b = 3$ with $\frac{1}{4\log 3} = 0.227\ldots$, whereas for $b = 2$ we only have $\frac{1}{6\log 2} = 0.240\ldots$.

Now, we will show that the choice of the sequence $\boldsymbol{i\tau}$ in the set $\{id, \tau\}^n$ of sequences $\boldsymbol{\sigma} = (\sigma_0, \ldots, \sigma_{n-1})$ is best possible in the sense that the leading terms in Theorem 2 cannot be made smaller whatever the $\sigma_{j-1} \in \{id, \tau\}$, $1 \leq j \leq n$, can be.

Theorem 3. *For any integer $n \geq 1$ and any $\boldsymbol{\sigma} \in \{id, \tau\}^n$ we have*

$$\lim_{n\to\infty} \frac{D^*(\mathcal{H}_{b,n}^{\boldsymbol{\sigma}})}{\log b^n} \geq \frac{b-1}{8\log b} \quad \textit{if } b \textit{ is odd and}$$

$$\lim_{n\to\infty} \frac{D^*(\mathcal{H}_{b,n}^{\boldsymbol{\sigma}})}{\log b^n} \geq \frac{b^2}{8(b+1)\log b} \quad \textit{if } b \textit{ is even.}$$

Proof. We use the properties of infinite sequences S_b^{Σ} together with the first inequality in relation (1) to derive the desired lower bound for $D^*(\mathcal{H}_{b,n}^{\boldsymbol{\sigma}})$.
Recall from Definition 2 that $\Sigma = (\sigma_0, \ldots, \sigma_{n-1}, id, id \ldots)$ with $\sigma_{j-1} \in \{id, \tau\}$. This implies that for all j, $1 \leq j \leq n$, we have $\psi_b^{\sigma_{j-1}} = \psi_b^{id}$ since $\psi_b^{\tau} = \psi_b^{id}$, so that, from formulas of Section 2.2, we deduce $D(N, S_b^{\Sigma}) = D(N, S_b^{id})$. Recall too that in one dimension we have $D \leq 2D^*$. Now, we can give the proof:

$$D^*(\mathcal{H}_{b,n}^{\boldsymbol{\sigma}}) \geq \max_{1\leq N\leq b^n} D^*(N, S_b^{\Sigma}) \geq \frac{1}{2} \max_{1\leq N\leq b^n} D(N, S_b^{id}) \geq \frac{1}{2}\alpha_b^{id} n$$

and the result follows from the reminders in the proof of Theorem 2. $\square$

In base 2, Theorem 3 has been shown in [16] by other arguments involving more computations. Another question raised and solved in base 2 in [16] is the following:

"Is the star discrepancy $D^*(\mathcal{H}_{b,n}^{\boldsymbol{\sigma}})$ independent of the distribution of id and τ in the sequence $\boldsymbol{\sigma} = (\sigma_0, \ldots, \sigma_{n-1}) \in \{id, \tau\}^n$ and does only depend on the number of id and τ as it is the case with the L_2-discrepancy (see [17] *and* [11]*) ?"*

In arbitrary base, the answer is *no* like in base 2, with the same counter-example as in [16]: the sequence $(id, \tau, id, \tau, \ldots, id, \tau)$. To prove this result (see Theorem 4 below) we will utilize the notion of intrication of permutations introduced in [5, Section 3.4.3]:

Definition 3. The *intrication* of two pairs (b, σ) and (c, ρ) is the pair $(bc, \sigma.\rho)$ defined by $\sigma.\rho(l) = c\sigma(h) + \rho(k)$ with $l = bk + h$, $0 \leq h < b$ and $0 \leq k < c$ (Euclidean division of l by b, $0 \leq l < bc$).

Lemma 5. *(i) The function $\psi_{bc}^{\sigma.\rho,+}$ satisfies the relation*

$$\psi_{bc}^{\sigma.\rho,+}(x) = \psi_b^{\sigma,+}(cx) + \psi_c^{\rho,+}(x) \text{ for all } x \in \mathbb{R}$$

and the same relation is also valid for $\psi_{bc}^{\sigma.\rho,-}$ and $\psi_{bc}^{\sigma.\rho}$.

(ii) Let $\Sigma = (\sigma_j)_{j\geq 0}$ and $\mathrm{T} = (\tau_j)_{j\geq 0}$ be two arbitrary sequences of permutations of $\{0, 1, \ldots, b-1\}$. If S_b^{P} is the sequence defined by $\mathrm{P} = (\rho_j)_{j\geq 0}$ with $\rho_{2j} = \sigma_j$ and $\rho_{2j+1} = \tau_j$, then $S_b^{\mathrm{P}} = S_{b^2}^{\Sigma.\mathrm{T}}$ where $\Sigma.\mathrm{T} := (\sigma_j.\tau_j)_{j\geq 0}$.

P r o o f. Part (i) of this lemma is part (i) of [5, Proposition 3.4.3], proved by starting from the definition of $\psi_{bc}^{\sigma.\rho,+}$ and enumerating the different cases. The proof of (ii) follows the same lines as [5, Proposition 3.4.3 (ii)], but with a fixed base b instead of different bases and with two arbitrary sequences of permutations instead of two constant sequences of permutations: starting from the definition of $S_{b^2}^{\Sigma.\mathrm{T}}$, it suffices to write the relations between the expansions of $N-1$ in bases b and b^2 to recover S_b^{P} with Definition 3. $\square$

Theorem 4. *For any even integer $n \geq 2$, let $\widetilde{\boldsymbol{i\tau}} = (id, \tau, id, \tau, \ldots, id, \tau) \in \{id, \tau\}^n$. Then, with some $c_n \in [0, 3]$, we have*

$$D^*(\mathcal{H}_{b,n}^{\widetilde{\boldsymbol{i\tau}}}) = \begin{cases} \dfrac{(b-1)(b+2)}{8(b+1)} n + c_n & \text{if } b \text{ is odd,} \\ \dfrac{b^3}{8(b^2+1)} n + c_n & \text{if } b \text{ is even.} \end{cases}$$

These constants are greater than $\dfrac{b-1}{8}$ *and* $\dfrac{b^2}{8(b+1)}$*, hence the answer to the question above is no.*

P r o o f. Let $n = 2m$. First, applying Lemma 5 (ii) with $\sigma_j = id$ for all $j \geq 0$, $\tau_j = \tau$ if $0 \leq j \leq m-1$ and $\tau_j = id$ if $j \geq m$, so that $\mathrm{P} = (id, \tau, id, \tau, \ldots, id, \tau, id, id, \ldots)$, we get $S_b^{\mathrm{P}} = S_{b^2}^{\Sigma.\mathrm{T}}$ with

$$\Sigma.\mathrm{T} = (id.\tau, id.\tau, \ldots, id.\tau, id.id, id.id, \ldots)$$

(m times $id.\tau$ and then $id.id$ only). Therefore, from the definition of generalized Hammersley point sets, we have

$$\begin{aligned}\widetilde{\mathcal{H}_{b,n}^{i\tau}} &= \left\{\left(S_b^{\mathrm{P}}(N), \frac{N-1}{b^n}\right); 1 \leq N \leq b^n\right\} \\ &= \left\{\left(S_{b^2}^{\Sigma.\mathrm{T}}(N), \frac{N-1}{(b^2)^m}\right); 1 \leq N \leq (b^2)^m\right\},\end{aligned}$$

so that $\widetilde{\mathcal{H}_{b,n}^{i\tau}} = \mathcal{H}_{b^2,m}^{\boldsymbol{i}.\boldsymbol{\tau}}$ with $\boldsymbol{i}.\boldsymbol{\tau} := (\underbrace{id.\tau, id.\tau, \ldots, id.\tau}_{m})$.

Next, we apply Theorem 1 to $\mathcal{H}_{b^2,m}^{\boldsymbol{i}.\boldsymbol{\tau}}$ and get, with some $c_m' \in [0,2]$,

$$\begin{aligned}D^*(\mathcal{H}_{b^2,m}^{\boldsymbol{i}.\boldsymbol{\tau}}) = \max &\left(\max_{1\leq N\leq (b^2)^m} \sum_{j=1}^{m} \psi_{b^2}^{id.\tau,+}\left(\frac{N}{b^{2j}}\right), \max_{1\leq N\leq (b^2)^m} \sum_{j=1}^{m} \psi_{b^2}^{id.\tau,-}\left(\frac{N}{b^{2j}}\right)\right) \\ &+ c_m'.\end{aligned}$$

But from Lemma 5 (i), we deduce that $\psi_{b^2}^{id.\tau,+}(x) = \psi_b^{id}(bx)$ and $\psi_{b^2}^{id.\tau,-}(x) = \psi_b^{id}(x)$ for all $x \in \mathbb{R}$. Hence it remains to compute the two maxima:

$$\max_{1\leq N\leq b^{2m}} \sum_{j=1}^{m} \psi_b^{id}\left(\frac{bN}{b^{2j}}\right) \text{ and } \max_{1\leq N\leq b^{2m}} \sum_{j=1}^{m} \psi_b^{id}\left(\frac{N}{b^{2j}}\right).$$

Finally, thanks to the periodicity of ψ functions, in fact, the two sums in these maxima have the same behavior and we will study the second one only. For the same reasons as in the proof of Theorem 2, we have (see Lemma 1 for notations)

$$\max_{1\leq N\leq b^{2m}} \sum_{j=1}^{m} \psi_b^{id}\left(\frac{N}{b^{2j}}\right) = \max_{x\in\mathbb{R}} \sum_{j=1}^{m} \psi_b^{id}\left(\frac{x}{b^{2j}}\right) = \max_{x\in[0,1]} F_m(x) = d_m = \alpha m + \beta_m.$$

The calculation of α and β_m is parallel to that for sequences S_b^{id} performed in [5, 5.5] and since it comes under specific computations according to the parity of b, we defer it to the following Proposition 1 which will complete the proof of Theorem 4. □

Remark 3. Before starting with Proposition 1 and its proof, we wish to make clear two things: first note that we deal with the function $\psi_{b^2} := \psi_b^{id}$ in base b^2, i.e., the 1-periodic function which is equal to the restriction of ψ_b^{id} on intervals $[\frac{k-1}{b^2}, \frac{k}{b^2}]$ $(1 \leq k \leq b^2)$. Secondly, we observe that it should be easy to obtain the leading constants as lower bounds for α with the help of Lemma 1 (ii): if b is odd, taking $\nu = 1$ and $a = \dfrac{b(b-1)}{2}$ would lead to $\alpha \geq \dfrac{(b-1)(b+2)}{4(b+1)}$ and if b is even, taking $\nu = 2$ and $a = \dfrac{(b^2+2)(b^2-1)}{2}$ would lead to $\alpha \geq \dfrac{b^3}{4(b^2+1)}$ (we omit the proof since we will get the equality below). Therefore, the answer to the question above would be *no*, without any more computations. But, like Kritzer, Larcher and Pillichshammer did in the case of base 2 (see [16, Section 4]), we think it is interesting in itself to obtain the exact value of α together with the remaining term β_m.

Proposition 1. *For any integer $m \geq 1$ we have*

$$\max_{1\leq N\leq b^{2m}} \sum_{j=1}^{m} \psi_b^{id}\left(\frac{N}{b^{2j}}\right) = \begin{cases} \dfrac{(b-1)(b+2)}{4(b+1)}m + \dfrac{b}{4(b+1)^2}\left(1 - \dfrac{1}{b^{2m}}\right) & \text{if } b \text{ is odd,} \\ \dfrac{b^3}{4(b^2+1)}m + \dfrac{b^3}{4(b^2+1)^2}\left(1 - \dfrac{(-1)^m}{b^{2m}}\right) & \text{if } b \text{ is even.} \end{cases}$$

P r o o f. According to Lemma 1, we have to compute

$$d_m = \alpha m + \beta_m = \max_{x\in[0,1]} F_m(x) \text{ with } F_m(x) = \sum_{k=0}^{m-1} \psi_{b^2}(x(b^2)^k).$$

We proceed by induction on $m \geq 1$. We have found the induction hypothesis by plotting the first graphs of the functions $\psi_{b^2} = \psi_b^{id}$ for small bases ($b \leq 7$) and looking at the dominating intervals, i.e., intervals $I_{k_0} = [\frac{k_0-1}{b^2}, \frac{k_0}{b^2}]$ such that ψ_{b^2} is dominated by its restriction to I_{k_0} within a translation or a symmetry. Then verifying the induction hypothesis reduces to add $\psi_{b^2}(xb^{2m})$ to $F_m(x)$ and check the formulas for $m+1$. In the following, we give in detail the induction hypothesis but we leave out the tedious verifications which are long but straightforward. For the sake of completeness, recall the formulas for ψ_b^{id} [5, 5.5.1]: $\psi_b^{id}(x) = \max(k(1-x), (b-k-1)x)$ for all $x \in [k/b, (k+1)/b]$ and integers $k \in [0, b-1]$.

The case of an odd base b.

There is one dominating half interval

$$\left[u_m, u_m + \frac{1}{2b^{2m}}\right] \text{ with } u_m = \frac{b}{2(b+1)}\left(1 - \frac{1}{b^{2m}}\right).$$

On this interval, F_m is the linear function $F_m(x) = q_m x + F_m(u_m)$ with

$$q_m = -\frac{b^{2m}-1}{2(b+1)},\ F_m(u_m) = \frac{(b-1)(b+2)}{4(b+1)}m + \frac{b}{4(b+1)^2}\left(1-\frac{1}{b^{2m}}\right) \text{ and}$$
$$d_m = \max_{x\in[0,1]} F_m(x) = F_m(u_m).$$

The case of an even base b.

There is one dominating half interval

$$\left[u_m, u_m + \frac{(-1)^m}{2b^{2m}}\right] \text{ with } u_m = \frac{b^2}{2(b^2+1)}\left(1-\frac{(-1)^m}{b^{2m}}\right).$$

On this interval, $F_m(x) = q_m x + F_m(u_m)$ with

$$q_m = (-1)^{m-1}\frac{b}{2(b^2+1)}(b^{2m} + (-1)^{m-1}),$$
$$F_m(u_m) = \frac{b^3}{4(b^2+1)}m + \frac{b^3}{4(b^2+1)^2}\left(1-\frac{(-1)^m}{b^{2m}}\right) \text{ and}$$
$$d_m = \max_{x\in[0,1]} F_m(x) = F_m(u_m).$$

These formulas achieve the proof of Proposition 1 and therefore the proof of Theorem 4 too. □

Corollary 3. *Asymptotically, we get (with even n)*

$$\lim_{n\to\infty}\frac{D^*(\mathcal{H}_{b,n}^{\widetilde{\boldsymbol{i\tau}}})}{\log b^n} = \begin{cases} \dfrac{(b-1)(b+2)}{8(b+1)\log b} & \textit{if } b \textit{ is odd and} \\ \dfrac{b^3}{8(b^2+1)\log b} & \textit{if } b \textit{ is even.} \end{cases}$$

Remark 4. Of course, for $b = 2$ we recover the result of [16, end of Section 4] with the constant $\frac{1}{5}\cdot$ In fact, this result was known for a long time with much more precision since Halton and Zaremba [13, 1969] obtained exact formulas for $D^*(\mathcal{H}_{2,n}^{\boldsymbol{id}})$ and $D^*(\mathcal{H}_{2,n}^{\widetilde{\boldsymbol{i\tau}}})$ after a lot of technical computations (see [18, Sections 1 and 4] for comments on this paper).

5. Swapping with an arbitrary permutation

In this section, we fix an arbitrary permutation σ of $\{0, 1, \ldots, b-1\}$ and consider sequences produced by swapping σ with τ, that is we mix the permutations σ and $\overline{\sigma} := \tau \circ \sigma$ to obtain sequences $\boldsymbol{\sigma} = (\sigma_0, \ldots, \sigma_{n-1}) \in \{\sigma, \overline{\sigma}\}^n$. The situation is not so clear as with the identity and we will only consider sequences $\boldsymbol{\sigma\overline{\sigma}} = (\overbrace{\sigma,\ldots,\sigma}^{\frac{n}{2}},\overbrace{\overline{\sigma},\ldots\overline{\sigma}}^{\frac{n}{2}})$, if n is even and $\boldsymbol{\sigma\overline{\sigma}} = (\overbrace{\sigma,\ldots,\sigma}^{\frac{n-1}{2}},\overbrace{\overline{\sigma},\ldots\overline{\sigma}}^{\frac{n+1}{2}})$, if n is odd. This choice permits to improve upon the discrepancy, but until now we are not able to prove it is the best, like in Section 4 with the identity and Theorem 3.

Let us recall two notations adapted to the present context (see Lemma 1) before to state the result for the sequence $\boldsymbol{\sigma\overline{\sigma}}$. Set

$$\alpha_b^{\sigma,+} := \inf_{n\geq 1}\max_{x\in\mathbb{R}}\left(\frac{1}{n}\sum_{j=1}^{n}\psi_b^{\sigma,+}\left(\frac{x}{b^j}\right)\right) \text{ and } \alpha_b^{\sigma,-} := \inf_{n\geq 1}\max_{x\in\mathbb{R}}\left(\frac{1}{n}\sum_{j=1}^{n}\psi_b^{\sigma,-}\left(\frac{x}{b^j}\right)\right).$$

Then:

Theorem 5. . *For any integer $n \geq 1$ we have, with some $c_n \in [-1,4]$,*

$$D^*(\mathcal{H}_{b,n}^{\boldsymbol{\sigma\overline{\sigma}}}) = \begin{cases} \dfrac{\alpha_b^{\sigma,+}+\alpha_b^{\sigma,-}}{2}n + c_n & \textit{if } n \textit{ is even} \\ \dfrac{\alpha_b^{\sigma,+}+\alpha_b^{\sigma,-}}{2}(n+1) + c_n & \textit{if } n \textit{ is odd.} \end{cases}$$

P r o o f. We condider only the case of an even integer $n = 2m$ (the case of an odd n is handled in the same way as in Theorem 2). Applying Theorem 1 with $\psi_b^{\overline{\sigma},+} = \psi_b^{\tau\circ\sigma,+} = \psi_b^{\sigma,-}$ and $\psi_b^{\overline{\sigma},-} = \psi_b^{\tau\circ\sigma,-} = \psi_b^{\sigma,+}$, we obtain, with some $c'_n \in [0,2]$,

$$D^*(\mathcal{H}_{b,n}^{\boldsymbol{\sigma\overline{\sigma}}}) = \max\left\{\begin{array}{l} \max\limits_{1\leq N\leq b^n}\left(\sum\limits_{j=1}^{m}\psi_b^{\sigma,+}\left(\dfrac{N}{b^j}\right) + \sum\limits_{j=m+1}^{2m}\psi_b^{\sigma,-}\left(\dfrac{N}{b^j}\right)\right) \\ \max\limits_{1\leq N\leq b^n}\left(\sum\limits_{j=1}^{m}\psi_b^{\sigma,-}\left(\dfrac{N}{b^j}\right) + \sum\limits_{j=m+1}^{2m}\psi_b^{\sigma,+}\left(\dfrac{N}{b^j}\right)\right) \end{array}\right\} + c'_n.$$

Now, of course we have (with obvious A_N and B_N)

$$\max_{1\leq N\leq b^n}(A_N + B_N) \leq \max_{1\leq N\leq b^n} A_N + \max_{1\leq N\leq b^n} B_N.$$

But, due to the properties of the ψ functions (see [5, Lemma 4.2.1], we also have

$$\max_{1\leq N\leq b^n} A_N + \max_{1\leq N\leq b^n} B_N \leq \max_{1\leq N\leq b^n}(A_N + B_N) + 1.$$

Discussing the maxima and the sums like in the proof of Theorem 2 and applying Lemma 1 with functions $\psi_b^{\sigma,+}$ and $\psi_b^{\sigma,-}$, we obtain (with β_m^+ and $\beta_m^- \in [0,1]$)

$$\begin{aligned}(\alpha_b^{\sigma,+}m + \beta_m^+) + (\alpha_b^{\sigma,-}m + \beta_m^-) - 1 &\le \max_{1\le N\le b^n}(A_N + B_N)\\ &\le (\alpha_b^{\sigma,+}m + \beta_m^+) + (\alpha_b^{\sigma,-}m + \beta_m^-).\end{aligned}$$

Finally, bringing back these bounds in $D^*(\mathcal{H}_{b,n}^{\sigma\overline{\sigma}})$, we get the desired result with some $c_n \in [-1,4]$. □

Corollary 4. *Asymptotically, we get*

$$\lim_{n\to\infty}\frac{D^*(\mathcal{H}_{b,n}^{\sigma\overline{\sigma}})}{\log b^n} = \frac{\alpha_b^{\sigma,+} + \alpha_b^{\sigma,-}}{2\log b}.$$

Remark 5. Of course, we recover Corollary 2 when $\sigma = id$. Until now, the best result coming from [5, Théorème 5] is obtained for $b = 12$ and the permutation $\sigma = (15)(29)(47)(6\overline{10})$ (product of cycles) with

$$\lim_{n\to\infty}\frac{D^*(\mathcal{H}_{12,n}^{\sigma\overline{\sigma}})}{\log 12^n} = \frac{\alpha_{12}^{\sigma,+} + \alpha_{12}^{\sigma,-}}{2\log 12} = \frac{1919}{3454\log 12} = .223\ldots,$$

a bit better than $b = 3$ and $\sigma = id$ with $\frac{1}{4\log 3} = 0.227\ldots$. Even if such improvements seem small, they concern the leading constants in discrepancy formulas and we think it is more important to improve upon these constants rather than to search for exact formulas or to reduce the complementary terms c_n in estimations.

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