ADDITIVE BASES OF POSITIVE INTEGERS AND RELATED PROBLEMS

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ABSTRACT. Let A be an infinite set of nonnegative integers and let $k \geqslant 2$ be an integer. We investigate the relation between the number of representations of an integer n by sums of the form $a_1 + \cdots + a_k$, where $a_1, \ldots, a_k \in A$, and the size of A. Some related problems are also considered.

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1. Introduction

Let A be an infinite set of nonnegative integers and let $k \geqslant 2$ be an integer. We will denote by $R_k(A, n)$ the number of representations of n in the form $n = a_1 + a_2 + \cdots + a_k$, where $a_1, a_2, \ldots, a_k \in A$. Similarly, let $r_k(A, n)$ denote the number of such representations of n by ordered k-tuples, namely, $n = a_1 + a_2 + \cdots + a_k$ with $a_1 \leqslant a_2 \leqslant \ldots \leqslant a_k$. Setting

$$f(z) = \sum_{j \in A} z^j, \tag{1}$$

we have

$$f(z)^k = \sum_{n=0}^{\infty} R_k(A, n) z^n. \tag{2}$$

Evidently,

$$r_k(A, n) \leqslant R_k(A, n) \leqslant k! r_k(A, n). \tag{3}$$

A set A is called a *base* of $\mathbb{N}$ of order k if $r_k(A, n) \geqslant 1$ for each $n \in \mathbb{N}$. By (3), the condition $r_k(A, n) \geqslant 1$ is equivalent to $R_k(A, n) \geqslant 1$. An old conjecture of Erdős and Turán [4] asserts that if $R_2(A, n) \geqslant 1$ for each sufficiently large n then $\limsup_{n \to \infty} R_2(A, n) = \infty$. In other words, it says that for no positive integer v we have $R_2(A, n) \in [1, v]$ for all sufficiently large n.

Although this Erdős-Turán conjecture (also known as one of USD 500 problems in [3]) remains open, there are several nontrivial results concerning it. Firstly, Grekos, Haddad, Helou and Pihko [5] proved that the numbers $R_2(A, n)$, $n \geqslant 0$, cannot all lie in the interval $[1, 5]$. This was later extended to the interval $[1, 7]$ by Borwein, Choi and Chu [1]. Recently, Sándor [10] showed that if $v = \limsup_{n \to \infty} R_2(A, n)$ then $\liminf_{n \to \infty} R_2(A, n) \leqslant (\sqrt{v} - 1)^2$.

It seems likely that for every integer $k \geqslant 2$ we must have

$$\text{either } \limsup_{n \to \infty} R_k(A, n) = \infty \text{ or } \liminf_{n \to \infty} R_k(A, n) = 0.$$

Of course, such a result, if proved, would imply the Erdős-Turán conjecture (which is the corresponding statement for $k = 2$). In other words, we ask whether there is an interval $[1, v]$, and an integer n_0 such that $R_k(A, n) \in [1, v]$ for each $n \geqslant n_0$. In view of (3) this problem is equivalent to the question on whether there are some integers $n_0, b > 0$ such that $r_k(A, n) \in [1, b]$ for every $n \geqslant n_0$.

The following result describes the size of the set A provided that $r_k(A, n)$ is bounded from below or from above. Here and subsequently, $A(n)$ stands for the number of elements of the set $A \cap [0, n]$.

Theorem 1. Let $k \geqslant 2$ be an integer. If $r_k(A, n) \geqslant a$ for each $n \geqslant n_0$ then $A(n) \geqslant (ak!n)^{1/k} - k + 1$ for every sufficiently large n. If $r_k(A, n) \leqslant b$ for each $n \geqslant n_1$ then $A(n) \leqslant (bk^2(k-1)!n)^{1/k} + 1$ for every sufficiently large n. Finally, if $r_k(A, n) \leqslant 1$ for each $n \geqslant 1$ then

$$A(n) \leqslant \begin{cases} ((k/2)((k/2)!)^2 n)^{1/k} + O(n^{1/(2k)}) & \text{for } k \text{ even,} \\ (((k+1)/2)!)^2 n)^{1/k} + O(n^{1/(2k)}) & \text{for } k \text{ odd.} \end{cases}$$

In particular, Theorem 1 combined with (3) implies that

$$a^{1/k} \leqslant \liminf_{n \to \infty} A(n) n^{-1/k} \leqslant \limsup_{n \to \infty} A(n) n^{-1/k} \leqslant (bk^2(k-1)!)^{1/k} \qquad (4)$$

provided that $R_k(A, n) \in [a, b]$ for each sufficiently large n.

In this context, Sándor's result [10] can be easily (and by the same method as in [10]) generalized as follows:

Theorem 2. Let $k \geqslant 2$ be an integer. If $\limsup_{n \to \infty} R_k(A, n) < \infty$ then

$$\limsup_{n \to \infty} R_k(A, n)^{1/2} - \liminf_{n \to \infty} R_k(A, n)^{1/2} \geqslant 1.$$

By (1) and (2), Theorem 2 deals with coefficients $b_{n,k}$ of the series

$$\Big(\sum_{n=0}^{\infty} b_n z^n \Big)^k = \sum_{n=0}^{\infty} b_{n,k} z^n,$$

where $b_n \in \{0, 1\}$. In Section 4, we will give an example of such series with $b_n > 0$ for each $n \geqslant 0$ such that $\lim_{n \to \infty} b_{n,k} = \gamma > 0$, so no analogue of Theorem 2 holds if the condition $b_n \in \{0, 1\}$ is replaced by $b_n > 0$. We also ask for the smallest value of the quotient of the largest and the smallest coefficients that appear in the kth power of a polynomial $\left(\sum_{n=0}^{d} b_n z^n \right)^k = \sum_{n=0}^{dk} b_{n,k} z^n$, where $b_0, \ldots, b_d \geqslant 0$ and $b_0, b_d > 0$. It seems that such a question has not been considered earlier, although it seems quite natural. Moreover, its version with *Newman polynomials* (those with coefficients in $\{0, 1\}$) seems to be naturally related to the Erdős-Turán conjecture.

2. Proof of Theorem 1

Put $m = A(n)$. The number of ordered k-tuples $(a_1, \ldots, a_k) \in A^k$, where $0 \leqslant a_1 \leqslant \ldots \leqslant a_k \leqslant n$, is equal to $m(m+1) \ldots (m+k-1)/k!$. Since $r_k(A, N) \geqslant a$ for $N \geqslant n_0$ and $a_{m+1} > n$, every integer $N \in [n_0, n]$ is expressible by the sum of at least a ordered k-tuples as above. Hence

$$m(m+1) \ldots (m+k-1)/k! \geqslant a(n - n_0 + 1).$$

Note that $m(m+1) \ldots (m+k-1) < (m+k-3/2)^k$. Consequently, $(m+k-3/2)^k > ak!(n-n_0+1)$ giving $m+k-3/2 > (ak!(n-n_0+1))^{1/k}$. This yields the required inequality $m > (ak!n)^{1/k} - k + 1$ for n large enough.

We remark that the above proof follows that of Nathanson [9] for $k = 2$ and $a = 1$. It seems, however, that his proof of Theorem 4 gives the inequality $A(0, x) \geqslant \sqrt{2x} - 1$ only instead of $A(0, x) \geqslant 2\sqrt{x} - 1$ as claimed in [9]. (In the notation of [9], $A(0, x)$ is the number of elements of the set A in $[0, x]$. One can only claim that $(k^2 + k)/2 \geqslant x - n_0$, because $n > x$ can be represented as $a + a'$ with $a' > x$.) A corrected inequality $A(0, x) \geqslant \sqrt{2x} - 1$ of [9] is exactly the first part of our Theorem 1 with $k = 2$ and $a = 1$. Of course, this implies that $\liminf_{n \to \infty} A(n)/\sqrt{n} \geqslant \sqrt{2}$ for each set A which is an asymptotic basis of $\mathbb{N}$ of order 2. We stress that a slightly better constant for liminf replaced by limsup, namely, $\limsup_{n \to \infty} A(n)/\sqrt{n} \geqslant 2\sqrt{2/\pi}$ for each A which is an asymptotic basis of $\mathbb{N}$ of order 2 follows from Theorem 3.3 in [6].

For the second part, suppose that $m = A(n)$ and $r_k(A, N) \leqslant b$ for $N \geqslant n_1$. Put $B = \max_{0 \leqslant n \leqslant n_1 - 1} r_k(A, n)$. As above, the number of ordered k-tuples $(a_1, \ldots, a_k) \in A^k$, where $0 \leqslant a_1 \leqslant \ldots \leqslant a_k \leqslant n$, is equal to $m(m+1) \ldots (m+k-1)/k!$ and $a_{m+1} > n$. Note that the sum $a_1 + \cdots + a_k$ lies in the interval $[0, kn]$. Hence

$$m^k/k! < m(m+1) \ldots (m+k-1)/k! \leqslant Bn_1 + b(kn - n_1 + 1).$$

This yields $m = A(n) < (bk^2(k-1)!n)^{1/k} + 1$ for n sufficiently large.

Recall that a set of integers E is called a $B_k[1]$ set if all possible sums of k (not necessarily distinct) elements of E are distinct. Since $r_k(A, j) \leqslant 1$ for all $j \geqslant 0$, the set $A \cap [0, n]$ is a $B_k[1]$ set. However, the largest such set which is a subset of $\{0, 1, \ldots, n\}$ contains at most $((k/2)((k/2)!)^2 n)^{1/k} + O(n^{1/(2k)})$ elements for k even (see [7], [8]) and at most $(((k+1)/2)!)^2 n)^{1/k} + O(n^{1/(2k)})$ elements for k odd (see [2]). This gives the required bound for $A(n)$.

3. Proof of Theorem 2

The result is obvious if $\liminf_{n \to \infty} R_k(A, n)^{1/2} = 0$, because A is infinite and so $R_k(A, n) \geqslant 1$ for infinitely many n's.

Assume that $\liminf_{n \to \infty} R_k(A, n)^{1/2} > 0$, but the required inequality does not hold. Then there exist a positive integer n_0 and two positive numbers u, v such that $0 < u < v < \infty$, $v > 1$, $v - u < 1$ and $u \leqslant R_k(A, n)^{1/2} \leqslant v$ for every $n \geqslant n_0$.

The inequality $v - u < 1$ implies that $v^2 - v < u^2 + u$. Take $w = (v^2 - v + u^2 + u)/2$. Then $0 < w - u^2 < u$ and $0 < v^2 - w < v$, hence $(u^2 - w)^2 < u^2$ and $(v^2 - w)^2 < v^2$. It follows that there is a positive number ε_0 such that for each $y \in [u^2, v^2]$ we have $(y - w)^2 < (1 - \varepsilon_0)y$. Hence, as $u^2 \leqslant R_k(A, n) \leqslant v^2$ for $n \geqslant n_0$, we obtain

$$(R_k(A, n) - w)^2 < (1 - \varepsilon_0)R_k(A, n). \tag{5}$$

Let r be a fixed number satisfying $1/2 < r < 1$. Consider the integral

$$I = \int_0^1 |f(re^{2\pi it})^k - w \sum_{n=0}^{\infty} r^n e^{2\pi itn}| \mathrm{d}t = \int_0^1 |f(re^{2\pi it})^k - \frac{w}{1 - re^{2\pi it}}| \mathrm{d}t.$$

Subtracting $w/(1 - z)$ from both sides of (2) and substituting $z = re^{2\pi it}$ with $t \in [0, 1]$, we obtain

$$f(re^{2\pi it})^k - w/(1 - re^{2\pi it}) = \sum_{n=0}^{\infty} (R_k(A, n) - w) r^n e^{2\pi itn}.$$

Hence $I = \int_0^1 |\sum_{n=0}^{\infty} (R_k(A, n) - w) r^n e^{2\pi itn}| \mathrm{d}t$. Applying the inequality of Cauchy-Schwarz $\int_0^1 |g(t)| \mathrm{d}t \leqslant \left(\int_0^1 |g(t)|^2 \mathrm{d}t \right)^{1/2}$ and the Parseval identity

$$\int_0^1 |\sum_{n=0}^{\infty} b_n e^{2\pi itn}|^2 \mathrm{d}t = \sum_{n=0}^{\infty} |b_n|^2, \tag{6}$$

we find that $I^2 \leqslant \sum_{n=0}^{\infty}(R_k(A,n) - w)^2 r^{2n}$. Estimating the sum of all terms in the range $0 \leqslant n \leqslant n_0 - 1$ by the absolute constant

$$c_1 = n_0 \max_{0 \leqslant n \leqslant n_0 - 1}(R_k(A,n) - w)^2$$

and each of the terms $n \geqslant n_0$ by (5), we obtain the inequality

$$I^2 \leqslant c_1 + (1 - \varepsilon_0)\sum_{n=n_0}^{\infty} R_k(A,n)r^{2n}.$$

Using (2) with $z = r^2$ we further have $I^2 \leqslant c_1 + (1 - \varepsilon_0)f(r^2)^k$. This yields

$$I \leqslant c_2 + (1 - \varepsilon_0/2)f(r^2)^{k/2}. \tag{7}$$

(Here and below, $c_1, c_2, \ldots$ are some positive constants that do not depend on r.)

Next, we shall estimate I from below using

$$I \geqslant \int_0^1 |f(re^{2\pi i})|^k \mathrm{d}t - w\int_0^1 \frac{\mathrm{d}t}{|1 - re^{2\pi it}|}.$$

Firstly, by (1) and (6), we have

$$\int_0^1 |f(re^{2\pi it})|^2 \mathrm{d}t = \int_0^1 \left|\sum_{j \in A} r^j e^{2\pi itj}\right|^2 \mathrm{d}t = \sum_{j \in A} r^{2j} = f(r^2).$$

Combining this with the inequality $\int_0^1 |g(t)|^2 \mathrm{d}t \leqslant \left(\int_0^1 |g(t)|^k \mathrm{d}t\right)^{2/k}$, where $k \geqslant 2$ (which follows from Hölder's inequality), we have

$$\int_0^1 |f(re^{2\pi it})|^k \mathrm{d}t \geqslant \left(\int_0^1 |f(re^{2\pi it})|^2 \mathrm{d}t\right)^{k/2} = f(r^2)^{k/2}.$$

Note that $|1 - re^{2\pi it}| = \left((1-r)^2 + 4r\sin^2(\pi t)\right)^{1/2}$. So $|1 - re^{2\pi it}| \geqslant 1 - r$ in the interval $0 \leqslant t \leqslant 1 - r$ and $|1 - re^{2\pi it}| \geqslant 2\sqrt{r}|\sin(\pi t)| \geqslant 4\sqrt{r}t > 2t$ in the interval $1 - r \leqslant t \leqslant 1/2$, because $\sin x \geqslant 2x/\pi$ for each $x \in [0, \pi/2]$. It follows that

$$\int_0^1 \frac{\mathrm{d}t}{|1 - re^{2\pi it}|} = 2\int_0^{1/2} \frac{\mathrm{d}t}{|1 - re^{2\pi it}|} \leqslant 2\int_0^{1-r} \frac{\mathrm{d}t}{1 - r} + 2\int_{1-r}^{1/2} \frac{\mathrm{d}t}{2t}$$

$$= 2 + \log\frac{1}{2 - 2r} < 2 + \log\frac{1}{1 - r}.$$

Consequently,

$$I \geqslant f(r^2)^{k/2} - w\left(2 + \log\frac{1}{1 - r}\right).$$

Combining this inequality with (7), we obtain

$$f(r^2)^{k/2} - w\left(2 + \log\frac{1}{1-r}\right) \leqslant c_2 + (1 - \varepsilon_0/2)f(r^2)^{k/2}.$$

So

$$\varepsilon_0 f(r^2)^{k/2} \leqslant c_3 + 2w\log\frac{1}{1-r}.$$

Select $r = 1 - 1/T$ with a large integer T to be chosen later. Then

$$\varepsilon_0 f(1 - 2/T)^{k/2} \leqslant \varepsilon_0 f(r^2)^{k/2} \leqslant c_3 + 2w\log\frac{1}{1-r} \leqslant c_4\log T.$$

From $R_k(A, n) \geqslant u^2$ for $n \geqslant n_0$ using (4) we deduce that $A(n) > c_5 n^{1/k}$. Thus $A(T) > c_5 T^{1/k}$ for each sufficiently large integer T. Hence, by (1),

$$f(1 - 2/T) = \sum_{j \in A}(1 - 2/T)^j > \sum_{j \in A,\, j \leqslant T}(1 - 2/T)^j$$

$$\geqslant A(T)(1 - 2/T)^T > c_6 T^{1/k}.$$

It follows that

$$c_4\log T \geqslant \varepsilon_0 f(1 - 2/T)^{k/2} > \varepsilon_0(c_6 T^{1/k})^{k/2} > \varepsilon_0 c_7 T^{1/2},$$

which is a contradiction for T large enough.

4. Some related problems

Let us consider the series

$$f(z) = \sum_{n=0}^{\infty}\frac{z^n}{(n+1)^{1-1/k}}.$$

Then

$$f(z)^k = \sum_{n=0}^{\infty}b_{n,k}z^n,$$

where

$$b_{n,k} = \sum_{j_1,\ldots,j_k \geqslant 0,\, j_1 + \cdots + j_k = n}((j_1 + 1)(j_2 + 1)\ldots(j_k + 1))^{-1+1/k}.$$

Write $b_{n,k}$ in the form

$$b_{n,k} = \frac{1}{n^{k-1}}\sum\left(\frac{j_1 + 1}{n}\ldots\frac{j_{k-1} + 1}{n}\left(1 + \frac{k}{n} - \sum_{l=1}^{k-1}\frac{j_l + 1}{n}\right)\right)^{-1+1/k},$$

where the sum is taken over the indices satisfying the inequalities $j_1, \ldots, j_{k-1}$ $\geqslant 0$, $j_1 + \cdots + j_{k-1} \leqslant n$. Put

$$B_{n,k} = \frac{1}{n^{k-1}} \sum_{i_1, \ldots, i_{k-1} \geqslant 1,\, i_1 + \cdots + i_{k-1} \leqslant n-1} \left(\frac{i_1}{n} \ldots \frac{i_{k-1}}{n} \left(1 - \sum_{l=1}^{k-1} \frac{i_l}{n} \right) \right)^{-1+1/k}.$$

For each $k \geqslant 2$, let T_k be a subset of $\mathbb{R}^{k-1}$ consisting of the points $(\theta_1, \ldots, \theta_{k-1})$ $\in \mathbb{R}^{k-1}$ satisfying $\theta_1, \ldots, \theta_{k-1} \geqslant 0$ and $\theta_1 + \cdots + \theta_{k-1} \leqslant 1$. Then the sum $B_{n,k}$ is a Riemann sum of the integral

$$\mathcal{I}_k = \int_{T_k} (x_1 \ldots x_{k-1}(1 - x_1 - \cdots - x_{k-1}))^{-1+1/k} dx_1 \ldots dx_{k-1}.$$

Since k is fixed and $n \to \infty$, it is easily seen that

$$\lim_{n \to \infty} b_{n,k} = \lim_{n \to \infty} B_{n.k} = \mathcal{I}_k.$$

Hence no analogue of Theorem 2 holds for the series with nonnegative coefficients. In particular, for $k = 2$, the integral is expressible by Euler's beta function. Indeed,

$$\mathcal{I}_2 = \int_{T_2} (x_1(1 - x_1))^{-1/2} dx_1 = \int_0^1 \frac{dx_1}{\sqrt{x_1(1 - x_1)}} = \frac{\Gamma(1/2)^2}{\Gamma(1)} = \pi,$$

so the coefficients $b_{n,2}$ of the series $\sum_{n=0}^{\infty} b_{n,2} z^n = (\sum_{n=0}^{\infty} \frac{z^n}{\sqrt{n+1}})^2$ tend to π as $n \to \infty$.

Suppose that P is a polynomial in one variable with positive coefficients and let $q(P)$ be the quotient of the largest and the smallest coefficients of P. Let $\mathcal{P}_d$ denote the set of polynomials of degree d with nonnegative coefficients, i.e.,

$$\mathcal{P}_d = \{\sum_{n=0}^{d} b_n z^n \; : \; b_0, \ldots b_d \geqslant 0, b_0, b_d > 0\}.$$

Consider the quantity

$$m_k(d) = \inf_{P \in \mathcal{P}_d} q(P^k) = \min_{P \in \mathcal{P}_d} q(P^k).$$

Note that without loss of generality it is sufficient to consider only those polynomials of $\mathcal{P}_d$ which satisfy $b_0 = 1$ and $b_d \geqslant 1$. Indeed, we can multiply P by a constant. In addition, instead of P we can consider the reciprocal polynomial of P, defined by

$$P^*(z) = z^d P(1/z) = b_d + b_{d-1} z + \cdots + b_0 z^d,$$

for which $q(P^{*k}) = q(P^k)$. Also, by compactness, the infimum of $q(P^k)$ is attained.

For example, the quantity $m_2(d)$ evaluates how 'flat' can the square of a polynomial with nonnegative coefficients be in terms of its coefficients. It is easy to see that $m_2(1) = 2$ (the minimum is attained at $P(z) = 1 + z$) and $m_2(2) = 2.25$ (the minimum is attained at $P(z) = 1 + z/2 + z^2$). More generally, let us take

$$P(z) = 1 + \frac{1}{2}(z + z^2 + \cdots + z^{d-1}) + z^d \in \mathcal{P}_d.$$

Then each coefficient of P^2 is greater than or equal to 1 and two extreme coefficients of P^2 are equal to 1. The largest coefficient of P^2 is that of z^d. It is equal to $(d+7)/4$. Consequently,

$$m_2(d) \leqslant q(P^2) = (d+7)/4 \quad \text{for each} \quad d \geqslant 1.$$

This inequality is not optimal for $d \geqslant 3$. Consider, for instance, the sequence whose first two terms are $y_0 = 1$, $y_1 = 1/2$ and whose mth element y_m, $m \geqslant 2$, is defined by the following recurrent formulas depending on the parity of m:

$$2y_{2k}y_0 + 2y_{2k-1}y_1 + \cdots + 2y_{k+1}y_{k-1} + y_k^2 = 1,$$

$$2y_{2k+1}y_0 + 2y_{2k}y_1 + \cdots + 2y_{k+2}y_{k-1} + 2y_{k+1}y_k = 1.$$

Then

$$y_0 = 1, \ y_1 = \frac{1}{2}, \ y_2 = \frac{3}{8}, \ y_3 = \frac{5}{16}, \ y_4 = \frac{35}{128}, \ y_5 = \frac{63}{256}, \ y_6 = \frac{231}{1024}, \ldots.$$

Consider the following reciprocal polynomial

$$P_d(z) = 1 + y_1 z + y_2 z^2 + y_3 z^3 + \cdots + y_3 z^{d-3} + y_2 z^{d-2} + y_1 z^{d-1} + z^d.$$

Then

$$P_3(z)^2 = (1 + \frac{1}{2}(x + x^2) + x^3)^2 = 1 + x + \frac{5}{4}x^2 + \frac{5}{2}x^3 + \frac{5}{4}x^4 + x^5 + x^6,$$

so $q(P_3^2) = 5/2$ and $m_3(3) \leqslant 5/2$. Similarly,

$$P_4(z)^2 = 1 + x + x^2 + \frac{11}{8}x^3 + \frac{169}{64}x^4 + \frac{11}{8}x^5 + x^6 + x^7 + x^8,$$

which implies that $q(P_4^2) = 169/64$ and $m_2(4) \leqslant 169/64 = 2.640625$. In the same manner, we obtain

$$m_2(5) \leqslant q(P_5^2) = \frac{89}{32} = 2.78125,$$

$$m_2(6) \leqslant q(P_6^2) = \frac{737}{256} = 2.87890625,$$

$$m_2(7) \leqslant q(P_7^2) = \frac{381}{128} = 2.9765625,$$

$$m_2(8) \leqslant q(P_8^2) = \frac{49993}{16384} = 3.05133056\ldots,$$

$$m_2(9) \leqslant q(P_9^2) = \frac{25609}{8192} = 3.12609863\ldots,$$

$$m_2(10) \leqslant q(P_{10}^2) = \frac{208841}{65536} = 3.18666076\ldots,$$

$$m_2(11) \leqslant q(P_{11}^2) = \frac{106405}{32768} = 3.24722290\ldots,$$

$$m_2(12) \leqslant q(P_{12}^2) = \frac{3458321}{1048576} = 3.29811191\ldots.$$

Is it true that $m_2(d) = q(P_d^2)$ for each d? Is $m_2(d)$ bounded or unbounded in terms of d?

A polynomial analogue of the problem raised in Section 1 can be stated as follows. *Is it true that, for every $k \geqslant 2$, there is an absolute constant $b = b(k) \geqslant 1$ such that, for each $d \in \mathbb{N}$, the kth power of some Newman polynomial of degree d has all of its coefficients in $[1, b]$?*

Of course, if the quantity $m_2(d)$ and, more generally, $m_k(d)$ is unbounded, then the answer to this question is negative. Indeed, the set of all Newman polynomials of degree d is just a small subset of $\mathcal{P}_d$. For $k = 2$, this conjecture about squares of Newman polynomials may be viewed as a polynomial analogue of the Erdős-Turán conjecture.

REFERENCES

[1] BORWEIN, P. – CHOI, S. – CHU, F.: *An old conjecture of Erdős-Turán on additive basis*, Math. Comp. **75** (2006), 475–484.

[2] CHEN, S.: *On the size of finite Sidon sequences*, Proc. Amer. Math. Soc. **121** (1994), 353–356.

[3] ERDŐS, P.: *Some old and new problems on additive and combinatorial number theory*, in: Combinatorial Mathematics: Proc. of the Third Intern. Conf. (New York, 1985), New York Acad. Sci., New York, 1989, pp. 181–186.

[4] ERDŐS, P. – TURÁN, P.: *On a problem of Sidon in additive number theory and some related problems*, J. London Math. Soc. **16** (1941), 212–215.

[5] GREKOS, G. – HADDAD, L. – HELOU, C. – PIHKO, J.: *On the Erdős-Turán conjecture*, J. Number Theory **102** (2003), 339–352.

[6] GREKOS, G. – HADDAD, L. – HELOU, C. – PIHKO, J.: *Variations on a theme of Cassels for additive bases*, Int. J. Number Theory **2** (2006), 249–265.

[7] XING DE JIA: *On finite Sidon sequences*, J. Number Theory **44** (1993), 84–92.

[8] KOLOUNTZAKIS, M.N.: *The density of $B_h[g]$ sequences and the minimum of dense cosine sums*, J. Number Theory **56** (1996), 4–11.

[9] NATHANSON, M.B.: *Unique representation bases for the integers*, Acta Arith. **108** (2003), 1–8.

[10] SÁNDOR, C.: *A note on a conjecture of Erdős-Turán*, Integers **8** (2008), #A30, 4 p.

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