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SETS OF EXACT APPROXIMATION ORDER BY RATIONAL NUMBERS II

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ABSTRACT. For a non-increasing function Ψ, let $\mathrm{Exact}(\Psi)$ be the set of real numbers that are approximable by rational numbers to order Ψ, but to no order $c\Psi$ with $0 < c < 1$. In a previous paper, we determined the Hausdorff dimension of the set $\mathrm{Exact}(\Psi)$ when $x \mapsto x^2\Psi(x)$ is non-increasing and the sum $\sum_{x\geq 1} x\Psi(x)$ converges. In the present note we complement this result by establishing that $\mathrm{Exact}(\Psi)$ has full Hausdorff dimension for a large class of functions Ψ that do not decrease too slowly and are such that the sum $\sum_{x\geq 1} x\Psi(x)$ diverges. Furthermore, we discuss the case where $x \mapsto x^2\Psi(x)$ is a constant function.

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1. Introduction

For a function $\Psi : \mathbb{R}_{>0} \to \mathbb{R}_{>0}$, let

$$\mathcal{K}(\Psi) := \left\{ \xi \in \mathbb{R} : \left| \xi - \frac{p}{q} \right| < \Psi(q) \text{ for infinitely many rational numbers } \frac{p}{q} \right\}$$

denote the set of Ψ-approximable real numbers. Jarník [15], Satz 6, used the theory of continued fractions to show explicitly that, if Ψ is non-increasing and satisfies $\Psi(x) = o(x^{-2})$, then there exist real numbers in $\mathcal{K}(\Psi)$ which do not belong to any set $\mathcal{K}(c\Psi)$ with $0 < c < 1$. His result can be restated as follows.

Theorem J. *Let $\Psi : \mathbb{R}_{>0} \to \mathbb{R}_{>0}$ be a non-increasing function satisfying $\Psi(x) = o(x^{-2})$. Then, the set*

$$\mathrm{Exact}(\Psi) := \mathcal{K}(\Psi) \setminus \bigcup_{m\geq 2} \mathcal{K}\big((1 - 1/m)\Psi\big)$$

of real numbers approximable to order Ψ and to no better order is non-empty.

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In other words, $\mathrm{Exact}(\Psi)$ is the set of real numbers ξ such that

$$|\xi - p/q| < \Psi(q) \qquad \text{infinitely often}$$

and

$$|\xi - p/q| \ge c\,\Psi(q) \qquad \text{for any } c < 1 \text{ and any } q \ge q_0(c,\xi),$$

where $q_0(c,\xi)$ denotes a positive real number depending only on c and on ξ. Jarník's method, however, provides no metric statement for that set.

In 1924, Khintchine [18] (see also his book [19]) used the theory of continued fractions to prove that, if $x \mapsto x^2\Psi(x)$ is non-increasing, then $\mathcal{K}(\Psi)$ has Lebesgue measure zero if the sum $\sum_{x\ge 1} x\Psi(x)$ converges and has full Lebesgue measure otherwise. In the convergence case, his result was considerably refined by Jarník who established [15], Satz 5, that, if $\Phi : \mathbb{R}_{>0} \to \mathbb{R}_{>0}$ is a positive continuous function such that $\Phi(x)/x$ tends monotonically to infinity with x, then the sets $\mathcal{K}(\Psi) \setminus \mathcal{K}(\Psi \circ \Phi)$ and $\mathcal{K}(\Psi)$ have the same Hausdorff $\mathcal{H}^f$-measure for a general dimension function f (see [23, 8] for the background on the theory of Hausdorff measure). This is, however, not strong enough to imply that $\mathrm{Exact}(\Psi)$ and $\mathcal{K}(\Psi)$ have the same Hausdorff dimension, a problem raised by Beresnevich, Dickinson and Velani at the end of [1] and solved in [2], where Theorem B below is established. Recall that the *lower order at infinity* $\lambda(g)$ of a function $g : \mathbb{R}_{>0} \to \mathbb{R}_{>0}$ is defined by

$$\lambda(g) = \liminf_{x\to+\infty} \frac{\log g(x)}{\log x}.$$

This notion arises naturally in estimating the Hausdorff dimension of the sets $\mathcal{K}(\Psi)$, see e.g., Dodson [7] and Dickinson [6].

Theorem B. *Let $\Psi : \mathbb{R}_{>0} \to \mathbb{R}_{>0}$ be such that $x \mapsto x^2\Psi(x)$ is non-increasing. Assume that the sum $\sum_{x\ge 1} x\Psi(x)$ converges. If λ denotes the lower order at infinity of the function $1/\Psi$, then*

$$\dim \mathrm{Exact}(\Psi) = \dim \mathcal{K}(\Psi) = \frac{2}{\lambda}.$$

Up to the extra assumption on Ψ, namely the fact that $x \mapsto x^2\Psi(x)$ is non-increasing (which implies that $x \mapsto \Psi(x)$ is decreasing), Theorem B provides a very satisfactory strengthening of Theorem J when the sum $\sum_{x\ge 1} x\Psi(x)$ converges. In view of this result, there remain two natural questions to investigate.

Problem 1. *Let $\Psi : \mathbb{R}_{>0} \to \mathbb{R}_{>0}$ be a non-increasing function satisfying $\Psi(x) = o(x^{-2})$ and such that the sum $\sum_{x\ge 1} x\Psi(x)$ diverges. To find the Hausdorff dimension of the set* $\mathrm{Exact}(\Psi)$.

Problem 2. *Let c be a positive real number and denote by* Exact(c) *the set of real numbers ξ such that*

$$|\xi - p/q| < cq^{-2} \qquad \textit{infinitely often}$$

and

$$|\xi - p/q| \geq (c-\varepsilon)\, q^{-2} \qquad \textit{for any } \varepsilon > 0 \textit{ and any } q \geq q_0(\varepsilon, \xi).$$

To study the set Exact(c).

The main aim of the present paper is to present some new contribution towards Problem 1. We establish in Section 2 that Exact(Ψ) has full Hausdorff dimension if the function Ψ satisfies the assumptions of Problem 1 and does not decrease too slowly. Furthermore, we give in Section 3 a brief survey of known results on Problem 2, together with several open questions.

2. On Problem 1

Apparently, there is in the literature a single paper on Problem 1. It was written in 1952 by Kurzweil [20], a student of Jarník. It contains, among others, the following result.

Theorem K. *Let $\Psi : \mathbb{R}_{>0} \to \mathbb{R}_{>0}$ be a continuous function such that $x \mapsto x^2\Psi(x)$ is non-increasing and tends to 0 as x tends to infinity. Assume that the sum $\sum_{x\geq 1} x\Psi(x)$ diverges and that there exists an integer k such that*

$$\Psi(x) > \frac{1}{x^2(\log x)^k}, \quad \textit{for every large } x.$$

Assume moreover that $(\log x)^2\Psi(x\log x)/\Psi(x)$ tends to 1 as x tends to infinity. Then, the Hausdorff dimension of the set $\mathcal{K}(3\Psi) \setminus \mathcal{K}(\Psi)$ is equal to one.

It is easily seen that the assumptions of Theorem K are satisfied for any function Ψ of the form

$$x \mapsto x^{-2}\,(\log x)^{\alpha_1}\,(\log_2 x)^{\alpha_2}\,\ldots\,(\log_\ell x)^{\alpha_\ell}, \tag{1}$$

where $\log_i$ denotes the i-th iterated logarithm and $\alpha_1, \ldots, \alpha_\ell$ are real numbers, not all zero, with $-1 \leq \alpha_1 \leq 0$ and $\alpha_j < 0$ (resp. $\alpha_j > -1$) if j is the smallest index h for which $\alpha_h \neq 0$ (resp. $\alpha_h \neq -1$). Theorem K follows from the discussion in Section 4 of [20], whose main statement is more general but quite complicated to state. For its proof, Kurzweil used the theory of continued fractions and constructed a dimension function f such that $\mathcal{H}^f(\mathcal{K}(3\Psi))$ is infinite, whereas $\mathcal{H}^f(\mathcal{K}(\Psi))$ is zero. Although the constant 3 occurring in Theorem K can be slightly lowered, we do not believe that Kurzweil's method could give any dimension result for the set $\mathrm{Exact}(\Psi)$.

However, and this is the purpose of the present note, it is possible to refine the construction given in [2] in order to give a partial positive answer to Problem 1.

Theorem 1. *Let $\Psi : \mathbb{R}_{>0} \to \mathbb{R}_{>0}$ be such that $x \mapsto x^2\Psi(x)$ is non-increasing. Assume that the sum $\sum_{x\geq 1} x\Psi(x)$ diverges and that*

$$\frac{1}{x^{2+\varepsilon}} \leq \Psi(x) \leq \frac{1}{100x^2 \log x}, \tag{2}$$

for any positive real number ε and any sufficiently large x. Then we have

$$\dim \mathrm{Exact}(\Psi) = \dim \mathcal{K}(\Psi) = 1.$$

No particular importance has to be attached to the numerical constant 100 in the statement of Theorem 1, which can be slightly reduced. Theorem 1 covers the case of functions of the form (1) with $\alpha_1 = -1$ and $-1 < \alpha_2 < 0$, among others.

The proof of Theorem 1 follows the same lines as that of Theorem 1 of [2], except that we have to suitably modify Lemma 6 from [2], whose proof heavily uses the assumption that the sum $\sum_{x\geq 1} x\Psi(x)$ converges.

The idea is to construct a Cantor set as large as possible contained in the set $\mathrm{Exact}(\Psi)$. To do this, we construct trees of continued fraction expansions, and use the fact that the quality of rational approximation to a real number can be easily read on its continued fraction expansion. We construct real numbers $\xi = [0; a_1, a_2, \ldots]$ and an increasing sequence of integers $(n_k)_{k\geq 1}$ such that the order of approximation to ξ by rational numbers is given by the quality of the approximations of ξ by the rationals $[0; a_1, a_2, \ldots, a_{n_k}]$, for $k \geq 1$. In particular, the 'intermediate' partial quotients, that is, the a_n's with n not in $(n_k + 1)_{k\geq 1}$, cannot be too large. We stress that, when the sum $\sum_{x\geq 1} x\Psi(x)$ converges, then we have considerably more choice for these intermediate partial quotients than when this sum diverges. This explains why the 'divergence case' is more delicate than the 'convergence case'.

To keep the present note short, we choose not to give a full proof of Theorem 1, but merely to constantly refer the reader to the proof of Theorem 1 from [2] and to explain which part of it should be modified.

Proof of Theorem 1. We point out how the proof of Theorem 1 from [2] should be modified. We keep the notation from that paper. The main difference is that, because of the divergence of the sum $\sum_{x\geq 1} x\Psi(x)$, no integer M

satisfies (5) from [2]. Since the choice of M does not affect the construction of a suitable integer Q_1 in [2], we have only to adapt the inductive step of the Cantor construction, namely Lemma 6 from [2]. In the sequel, we assume that Q_1 is sufficiently large, and we set

$$Q_k = Q_1^{3^k}, \quad \text{for } k \geq 1. \tag{3}$$

Let k be a positive integer. Following the notation of Lemma 6 from [2], let $I := U_k(p/q)$ be a connected component of E_k with $Q_k/10 \leq q \leq Q_k$. Write $p/q = [0; a_1, \ldots, a_n]$ with n even. Denote by $(p_\ell/q_\ell)_{\ell \geq 1}$ the sequence of the convergents to a number in I. It follows from the definition of $U_k(p/q)$ that

$$a_{n+1} \geq (q^2 \Psi(q))^{-1}/2 \geq 50\, Q_k^{-2} \Psi(Q_k/10)^{-1}.$$

Since $q_n = q \geq Q_k/10$, an easy induction shows that

$$q_{n+j} \geq 2^{j/2}\, Q_k/100, \quad \text{for } j \geq 1. \tag{4}$$

Set

$$M' = Q_k/100 \quad \text{and} \quad h = \lfloor 6 \log Q_k \rfloor,$$

where $\lfloor x \rfloor$ denotes the integer part of the real number x. Instead of the set J occurring in [2], we consider the subset J' of I composed of the real numbers whose partial quotients satisfy

$$a_{n+j+1} \leq \frac{1}{3 \cdot 2^j\, M'^2 \Psi(2^{j/2} M')}, \qquad \text{for } 1 \leq j \leq h. \tag{5}$$

In other words, we do not assume that all the partial quotients are small, like in the definition of J, but that only some of them are. Lemma 3 from [2] yields that

$$\mathrm{Leb}(J') \geq \mathrm{Leb}(I) \prod_{j=1}^{h} \bigl(1 - 9 \cdot 2^j\, M'^2\, \Psi(2^{j/2} M')\bigr),$$

where Leb denotes the Lebesgue measure. It follows from the right-hand side of (2) that

$$9 \cdot 2^j\, M'^2\, \Psi(2^{j/2} M') \leq \frac{9}{100 \log(2^{j/2} M')}, \quad \text{for } 1 \leq j \leq h.$$

Consequently,

$$\begin{aligned}\log \prod_{j=1}^{h} \bigl(1 - 9 \cdot 2^j \, M'^2 \, \Psi(2^{j/2} M')\bigr) &\ge -\frac{10}{9} \sum_{j=1}^{h} 9 \cdot 2^j \, M'^2 \, \Psi(2^{j/2} M') \\ &\ge - \sum_{j=1}^{h} \frac{1}{10 \log (2^{j/2} M')} \\ &\ge - \frac{3 \log Q_k}{5 \log M'} \ge - \frac{31}{50},\end{aligned}$$

and we get

$$\mathrm{Leb}(J') \ge e^{-31/50} \, \mathrm{Leb}(I) \ge \mathrm{Leb}(I)/2.$$

As in [2], we check that if ξ is in J' then (4) and (5) yield that

$$a_{n+j+1} \le \frac{1}{3 \, q_{n+j}^2 \, \Psi(q_{n+j})}, \quad \text{for } 1 \le j \le h, \tag{6}$$

since $x \mapsto x^2 \Psi(x)$ is non-increasing. We infer from (6) and Lemma 2 from [2] that

$$\left| \xi - \frac{p_{n+j}}{q_{n+j}} \right| > \Psi(q_{n+j}), \qquad \text{for } j = 1, \ldots, h \text{ such that } \xi \ne p_{n+j}/q_{n+j}.$$

Consequently, since

$$q_{n+h+1} \ge 2^{(h+1)/2} Q_k / 100 \ge Q_k^{3.05},$$

we have, for ξ in J',

$$\left| \xi - \frac{u}{v} \right| \ge \Psi(v) \qquad \text{for any } u/v \text{ with } q_{n+1} \le v \le Q_k^3 \text{ and } u/v \ne \xi.$$

Furthermore, since ξ belongs to the interval $[p_n/q_n + c_k \Psi(q_n), p_n/q_n + \Psi(q_n)]$ with $c_k = 1 - 2^{-k}$, and since $x \mapsto x\Psi(x)$ is non-increasing, we infer from the inequalities

$$|q_n \xi - p_n| \ge c_k q_n \Psi(q_n) \ge c_k v \Psi(v)$$

that

$$\left| \xi - \frac{u}{v} \right| \ge c_k \Psi(v)$$

holds for any u/v with $q_n < v < q_{n+1}$.

Let Q be a real number with

$$1000 (\mathrm{Leb}(I))^{-1} \log (1/\mathrm{Leb}(I)) \le Q_k^{2.6} \le Q \le Q_k^{3.05}. \tag{7}$$

Observe that, by Lemma 2 of [2], our choice for Q_k implies that $\log q \ge 2^k$, which in turns yields that I is non-empty (namely, the integers A_{n+1} and B_{n+1}

defined on page 178 of [2] satisfy $B_{n+1} < A_{n+1}$) and the lower bound $\mathrm{Leb}(I) \geq 2^{-k-1}\Psi(q)$. A number Q as in (7) does exist since

$$\mathrm{Leb}(I) \geq 2^{-k-1}\Psi(q) \geq 2^{-k-1}\,\Psi(Q_k) \geq 2^{-k-1}Q_k^{-5/2},$$

by (3) and the left-hand side of (2) with $\varepsilon = 1/2$ (recall that Q_1 is assumed to be large enough). The Lebesgue measure of the set A of $\xi \in I$ for which

$$\left|\xi - \frac{p}{q}\right| < \frac{1}{qQ}$$

is satisfied by a rational number p/q with $q \leq Q/10$ is less than or equal to $\mathrm{Leb}(I)/4$. Consequently,

$$\mathrm{Leb}\big((I \setminus A) \cap J'\big) \geq \mathrm{Leb}(I)/4,$$

and we can then argue exactly as in [2] to prove that there are at least

$$\lfloor Q^2 \mathrm{Leb}(I)/88 \rfloor$$

evenly spaced rational numbers in $I \cap J'$ whose denominator lies between $Q/10$ and Q. Since, by (3) and (7), Q_{k+1} is a suitable choice for Q, the proof of the inductive step is completed.

With our modified version of Lemma 6 from [2], we define a Cantor set $\mathcal{C}$ exactly as in [2]. Using the left-hand side of (2), it then follows from Example 4.6 of Falconer [8] that $\mathcal{C}$ has full Hausdorff dimension.

The left-hand side of (2) is a necessary assumption in the present proof. Indeed, we cannot allow arbitrarily large gaps in the sequence $(Q_k)_{k\geq 1}$, unlike what is done on page 183 of [2]. □

3. The Lagrange spectrum and Problem 2

The aim of the present section is to collect some results on Problem 2 and to point out several open questions.

We restrict our attention to the set of badly approximable numbers, that is, to the set ***Bad*** composed of the irrational numbers ξ for which there exists a positive constant $c(\xi)$ such that

$$|\xi - p/q| \geq c(\xi)\, q^{-2} \qquad \text{for all } p/q \text{ with } q \geq 1.$$

Jarník [14] established that ***Bad*** has full Hausdorff dimension. Recall also that an irrational number ξ is in ***Bad*** if, and only if, the sequence of its partial quotients in the continued fraction expansion

$$\xi = [a_0; a_1, a_2, \ldots]$$

is a bounded sequence.

For ξ in ***Bad***, set

$$\lambda(\xi) = \liminf_{q \to +\infty} q \cdot ||q\xi||.$$

The set $\mathcal{L}$ of values taken by the function λ is called the *Lagrange spectrum*.

For any positive real number c, denote by $\mathrm{Exact}'(c)$ the set of real numbers ξ such that

$$|\xi - p/q| \le (c + \varepsilon) q^{-2} \qquad \text{has infinitely many solutions for every } \varepsilon > 0$$

and

$$|\xi - p/q| \ge c\, q^{-2} \qquad \text{for any } q \ge q_0.$$

Clearly, any ξ in ***Bad*** belongs either to $\mathrm{Exact}(\lambda(\xi))$ or to $\mathrm{Exact}'(\lambda(\xi))$, and the set of positive real numbers c such that $\mathrm{Exact}(c)$ is non-empty is contained in $\mathcal{L}$.

According to Malyshev [22], the set of positive real numbers c such that $\mathrm{Exact}(c)$ is non-empty is equal to the Lagrange spectrum $\mathcal{L}$.

It follows from the theory of continued fractions (see e.g., [3]) that if $\xi = [a_0; a_1, a_2, \ldots]$ is in ***Bad*** then, setting $p_n/q_n = [a_0; a_1, \ldots a_n]$ for $n \ge 1$, we have

$$\frac{1}{(a_{n+1}+2)q_n^2} < \frac{1}{(q_{n+1}+q_n)q_n} < \left|\xi - \frac{p_n}{q_n}\right| < \frac{1}{q_{n+1}q_n} < \frac{1}{a_{n+1}q_n^2}$$

and

$$q_n \cdot ||q_n\xi|| = 1/([a_{n+1}; a_{n+2}, a_{n+3}, \ldots] + [0; a_n, a_{n-1}, \ldots, a_1]),$$

whence

$$1/\lambda(\xi) = \limsup_{n \to +\infty} ([a_{n+1}; a_{n+2}, a_{n+3}, \ldots] + [0; a_n, a_{n-1}, \ldots, a_1]). \tag{8}$$

This expression of $\lambda(\xi)$ is used in the proofs of most of the results on $\mathcal{L}$.

Theorem 2. *For any positive real number c with $c < 1/6$, the set* $\mathrm{Exact}(c)$ *is non-empty.*

P r o o f. It is sufficient to slightly modify the proof of the corresponding statement for the set $\mathcal{L}$. The key ingredient is a deep result of Hall [10], claiming that any real number can be written in the form $a + [0; b_1, b_2, \ldots] + [0; c_1, c_2, \ldots]$, where a is an integer and the partial quotients b_i and c_i, $i \ge 1$, do not exceed 4. Let λ be a real number with $\lambda > 6$ and write it under the above form. Then, defining

$$\xi = [c_1; c_2, \ldots, c_{k_1}, f_1, g_1, b_{k_2}, \ldots, b_1, a, c_1, \ldots, c_{k_2}, f_2, g_2, b_{k_3}, \ldots, b_1, a, \ldots],$$

where $a \ge 5$, the f_j's and the g_j's belong to $\{1, 2, 3, 4\}$ and $(k_j)_{j \ge 1}$ is a rapidly increasing sequence of positive integers, we easily check that $\lambda(\xi) = 1/\lambda$. We would like a more precise result, namely to decide whether ξ belongs to $\mathrm{Exact}(c)$ or to $\mathrm{Exact}'(c)$. Recall that, regardless of the positive integers $d_1, \ldots, d_k, d_{k+1}$, we have

$$[0; d_1, \ldots, d_k, d_{k+1}] < [0; d_1, \ldots, d_k], \quad \text{if } k \text{ is odd},$$

and

$$[0; d_1, \ldots, d_k, d_{k+1}] > [0; d_1, \ldots, d_k], \quad \text{if } k \text{ is even}.$$

Using this and (8), it is possible to choose the f_j's and the g_j's and the parity of the k_j's accordingly to decide whether ξ belongs to $\mathrm{Exact}(\lambda(\xi))$ or to $\mathrm{Exact}'(\lambda(\xi))$, both sets being consequently non-empty. We omit the details. □

For a positive real number c, write $\mathcal{B}_c$ for the set composed of the real numbers ξ with $\lambda(\xi) \geq c$. We propose open questions and briefly survey some of the results on the Lagrange spectrum and on the sets $\mathcal{B}_c$. The interested reader is directed to the survey [22] and to the book of Cusick and Flahive [5] for more information and bibliographical references.

Problem 3. *To study the function*

$$\delta : c \mapsto \dim(\mathcal{B}_c).$$

Is δ continuous ?

It is known that $\mathcal{B}_c$ is countable (and, consequently, $\dim(\mathcal{B}_c) = 0$) for any $c > 1/3$ and that $\dim(\mathcal{B}_c)$ tends to 1 as c tends to 0.

Problem 4. *Does there exist a positive real number c such that* $\dim \mathrm{Exact}(c)$ *is positive ?*

Hightower [13] proved in 1970 that there are countably many disjoint intervals in $(0, 1/3)$ that contain no element from $\mathcal{L}$, see also [9, 21].

Problem 5. *To study the functions*

$$\tau : c \mapsto \mathrm{Leb}(\mathcal{L} \cap (c, 1))$$

and

$$\sigma : c \mapsto \dim(\mathcal{L} \cap (c, 1)).$$

Recalling that $\mathcal{L}$ contains the interval $(0, 1/6)$, we get that $\tau(c)$ is positive and thus $\sigma(c) = 1$ if c is sufficiently small. Furthermore, Bumby [4] established that $\tau(8/\sqrt{689}) = 0$ (see [5], Chapter 6).

We conclude this section by mentionning that many authors have studied sets of real numbers whose partial quotients belong to some fixed set, rather than the sets $\mathcal{B}_c$. Let Λ be a subset of the set of positive integers and set

$$d_\Lambda := \dim\{\xi = [0; a_1, a_2, \ldots] : a_i \in \Lambda \text{ for } i \geq 1\}.$$

No close formula is known for $d_{\{1,2\}}$, whose first decimal digits are $0.5312805\ldots$, see [12, 16].

For simplicity, write d_M for $d_{\{1,\ldots,M\}}$, for every integer $M \geq 1$. The first upper and lower bounds for d_M were obtained by Jarník [14]. We quote below an asymptotic estimate of Hensley [11].

Theorem H. *As M tends to infinity, we have*

$$d_M = 1 - \frac{6}{\pi^2 M} - \frac{72 \log M}{\pi^4 M^2} + O(M^{-2}).$$

Theorem H complements a result of Kurzweil [20] asserting that

$$1 - 0.99c \leq \dim(\mathcal{B}_c) \leq 1 - 0.25c$$

holds for any positive real number $c < 1/1000$.

Finally, we point out a recent, beautiful result of Kesseböhmer and Zhu [17].

Theorem KZ. *For any real number d in $(0,1)$, there exists a subset Λ of the set of positive integers such that $d_\Lambda = d$. The set*

$$\{d_\Lambda : \Lambda \textit{ is a finite subset of the integers}\}$$

is dense in $(0,1)$.

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