Uniform Distribution Theory **3** (2008), no.2, 147–155

uniform distribution theory

LÉVY CONSTANTS OF QUADRATIC IRRATIONALITIES

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ABSTRACT. An irrational number α is said to have Lévy constant $\beta(\alpha)$ if the limit

$$\lim_{m\to\infty} \frac{1}{m} \log q_m(\alpha) =: \beta(\alpha)$$

exists where $q_m(\alpha)$ denotes the denominator of the mth convergent of α. We give a new proof of the fact that the Lévy constants of quadratic irrationalities are dense in the interval $\left[\log \frac{1+\sqrt{5}}{2}, +\infty\right)$.

Communicated by Cor Kraaikamp

1. Introduction and history of the problem

Let $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ with regular continued fraction expansion

$$\alpha = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\ddots}}} = [a_0, a_1, a_2, \ldots]$$

(i.e., $a_0 \in \mathbb{Z}$ and $a_1, a_2, \ldots \in \mathbb{N}$) and convergents $p_m/q_m = [a_0, a_1, \ldots, a_m]$ (where we set $q_{-1} = 0$ and $p_{-1} = 1$). A celebrated theorem by P. Lévy [7] (after a preliminary result by A.Ya. Khintchine [6]) says that

$$\lim_{m\to\infty} \frac{1}{m} \log q_m(\alpha) = \frac{\pi^2}{12 \log 2} \quad \text{for almost all } \alpha.$$

This result has been thoroughly studied and is well understood. A good starting point for anyone interested in refinements and further developments is W. Philipp's monograph [9].

2000 Mathematics Subject Classification: Primary 11K50; Secondary 11J70.
Keywords: Continued fraction, convergent, quadratic irrational.

Definition. If for $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ the limit

$$\beta(\alpha) := \lim_{m\to\infty} \frac{1}{m} \log q_m(\alpha)$$

exists, then $\beta(\alpha)$ is called the Lévy constant of α.

Remarks. 1) By Lévy's theorem $\beta(\alpha)$ exists and $\beta(\alpha) = \pi^2/(12\log 2)$ for almost all α.
2) It is easy to see that

$$\varliminf_{m\to\infty} \frac{1}{m} \log q_m(\alpha) \geq \log \frac{1+\sqrt{5}}{2} \quad \text{for all } \alpha \in \mathbb{R}\setminus\mathbb{Q}. \tag{1}$$

As $[0, \underbrace{1,\dots,1}_{m \text{ times}}] = F_{m-1}/F_m$ (where F_m denotes the m^{th} Fibonacci number) and q_m is increasing as a function of each of its arguments $a_1,\dots,a_m$ it follows that

$$q_m(\alpha) \geq F_m \sim \frac{1}{\sqrt{5}} \Big(\frac{1+\sqrt{5}}{2}\Big)^{m+1} \qquad \text{as } m \to \infty$$

which immediately implies (1).

H. Jager and P. Liardet [5] proved that every quadratic irrationality α has a Lévy constant $\beta(\alpha)$. C. Faivre [2] showed that for all $\beta \geq \log(1+\sqrt{5})/2$ there exists $\alpha \in \mathbb{R}\setminus\mathbb{Q}$ such that

$$\beta(\alpha) = \lim_{m\to\infty} \frac{1}{m} \log q_m(\alpha) = \beta.$$

C. Baxa [1] proved an analogous result for upper and lower limits: for all

$$\beta^* \geq \beta_* \geq \log \frac{1+\sqrt{5}}{2}$$

there exist non-denumerably many, pairwise not equivalent α such that

$$\varliminf_{m\to\infty} \frac{1}{m} \log q_m(\alpha) = \beta_* \quad \text{and} \quad \varlimsup_{m\to\infty} \frac{1}{m} \log q_m(\alpha) = \beta^*.$$

J. Wu [10] studied the Hausdorff dimension $\dim_{\mathcal{H}} E(\beta_*, \beta^*)$ of the set

$$E(\beta_*,\beta^*) = \Big\{\alpha \in [0,1) \Big| \varliminf_{m\to\infty} \frac{1}{m} \log q_m(\alpha) = \beta_* \text{ and } \varlimsup_{m\to\infty} \frac{1}{m} \log q_m(\alpha) = \beta^* \Big\}.$$

He proved that

$$\dim_{\mathcal{H}} E(\beta_*,\beta^*) \geq \frac{\beta_* - \log \frac{1+\sqrt{5}}{2}}{2\beta^*}.$$

E.P. Golubeva [3, 4] studied Lévy constants of quadratic irrationalities and their connections with real quadratic fields and binary quadratic forms.

Recently J. Wu [11] proved that the Lévy constants of quadratic irrationalities are dense in the interval $\big[\log \frac{1+\sqrt{5}}{2}, +\infty\big)$. It is the purpose of the present note to prove a result that is slightly stronger than Wu's by a different method. Along the way we give short proofs of the results of H. Jager and P. Liardet, and C. Faivre which will be put to good use later on.

2. A proof of the theorem of Jager and Liardet

Lemma 1. *For all $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ we have*

$$\varliminf_{m\to\infty} \frac{1}{m} \log q_m(\alpha) = \varliminf_{m\to\infty} \frac{1}{m} \sum_{i=1}^{m} \log[a_i, a_{i+1}, \ldots]$$

and

$$\varlimsup_{m\to\infty} \frac{1}{m} \log q_m(\alpha) = \varlimsup_{m\to\infty} \frac{1}{m} \sum_{i=1}^{m} \log[a_i, a_{i+1}, \ldots].$$

P r o o f. Our starting point is the well known identity

$$\prod_{i=1}^{m+1} [a_i, a_{i+1}, \ldots] = |q_m\alpha - p_m|^{-1} = q_m \cdot [a_{m+1}, a_{m+2}, \ldots] + q_{m-1}$$

which follows, e.g., from Equations (4) and (7) in § 13 of [8]. It yields

$$\begin{aligned}\sum_{i=1}^{m+1} \log[a_i, a_{i+1}, \ldots] &= \log q_m + \log\Big([a_{m+1}, a_{m+2}, \ldots] + \frac{q_{m-1}}{q_m}\Big) \\ &= \log q_m + \log\Big([a_{m+1}, a_{m+2}, \ldots] + [0, a_m, \ldots, a_1]\Big) \\ &= \log q_m + \log[a_{m+1}, a_{m+2}, \ldots] + O(1)\end{aligned}$$

and therefore

$$\frac{1}{m} \sum_{i=1}^{m} \log[a_i, a_{i+1}, \ldots] = \frac{1}{m} \log q_m + O\Big(\frac{1}{m}\Big) \tag{2}$$

which implies the assertion. □

Definition. We remind the reader that two irrational numbers β and γ are called equivalent if they are of shape

$$\beta = [b_0, b_1, \ldots, b_k, a_1, a_2, a_3, \ldots] \quad \text{and} \quad \gamma = [c_0, c_1, \ldots, c_\ell, a_1, a_2, a_3, \ldots].$$

Theorem 2. *If the irrational numbers β and γ are equivalent then*

$$\underline{\lim}_{m\to\infty} \frac{1}{m} \log q_m(\beta) = \underline{\lim}_{m\to\infty} \frac{1}{m} \log q_m(\gamma)$$

and

$$\overline{\lim}_{m\to\infty} \frac{1}{m} \log q_m(\beta) = \overline{\lim}_{m\to\infty} \frac{1}{m} \log q_m(\gamma).$$

Proof. Let $\alpha = [0, a_1, a_2, a_3, \ldots]$. It suffices to prove the assertions for α and $\beta = [b_0, b_1, \ldots, b_k, a_1, a_2, a_3, \ldots]$. Using Equation (2) we get

$$\begin{aligned}
&\frac{1}{n+k} \log q_{n+k}(\beta)\\
=&\frac{1}{n+k}\Big(\sum_{i=1}^{k} \log[b_i, b_{i+1}, \ldots, b_k, a_1, a_2, \ldots] + \sum_{i=1}^{n} \log[a_i, a_{i+1}, \ldots]\Big) + O\Big(\frac{1}{n+k}\Big)\\
=&\frac{n}{n+k}\frac{1}{n} \log q_n(\alpha) + O\Big(\frac{1}{n}\Big)
\end{aligned}$$

which implies the assertions. □

Theorem 3 (Jager, Liardet). *If $\beta = [b_0, \ldots, b_k, \overline{a_1, \ldots, a_n}]$ is a quadratic irrationality then*

$$\lim_{m\to\infty} \frac{1}{m} \log q_m(\beta) = \frac{1}{n} \sum_{i=1}^{n} \log[\overline{a_i, \ldots, a_{i+n-1}}].$$

Proof. Let $\alpha = [0, \overline{a_1, \ldots, a_n}]$. Due to Theorem 2 it suffices to prove the assertion for α. Let m be a positive integer and $m = qn + r$ (with $q \geq 0$ and $0 \leq r < n$). By using equation (2) we get

$$\begin{aligned}
\frac{1}{m} \log q_m(\alpha) &= \frac{1}{qn+r} \sum_{i=1}^{qn+r} \log[\overline{a_i, \ldots, a_{i+n-1}}] + O\Big(\frac{1}{qn+r}\Big)\\
&= \frac{1}{qn+r}\Big(q \sum_{i=1}^{n} \log[\overline{a_i, \ldots, a_{i+n-1}}] + \sum_{i=1}^{r} \log[\overline{a_i, \ldots, a_{i+n-1}}]\Big) + O\Big(\frac{1}{q}\Big)\\
&= \frac{q}{qn+r} \sum_{i=1}^{n} \log[\overline{a_i, \ldots, a_{i+n-1}}] + O\Big(\frac{1}{q}\Big)\\
&\to \frac{1}{n} \sum_{i=1}^{n} \log[\overline{a_i, \ldots, a_{i+n-1}}]
\end{aligned}$$

as $m \to \infty$ (and therefore $q \to \infty$). □

3. Proofs of the theorems of Faivre and Wu

Definition. Let $a_1, \dots, a_n$ be positive integers. The continuant $K_n(a_1, \dots, a_n)$ is defined as the determinant

$$K_n(a_1, \dots, a_n) = \begin{vmatrix} a_1 & 1 & & & \\ -1 & a_2 & 1 & & \\ & -1 & a_3 & 1 & \\ & & \ddots & \ddots & \ddots \\ & & & -1 & a_n \end{vmatrix}$$

In addition we set $K_0 := 1$, $K_{-1} := 0$.

Remarks. 1) Continuants are closely connected to continued fractions by the following fact: if $\alpha = [a_0, a_1, a_2, \dots]$ then $q_m(\alpha) = K_m(a_1, \dots, a_m)$.
2) Obviously continuants satisfy the relation $K_m(a_1, \dots, a_m) = K_m(a_m, \dots, a_1)$.
3) For $0 \le m \le n$ we have

$$\begin{aligned} K_n(a_1, \dots, a_n) =& K_m(a_1, \dots, a_m) K_{n-m}(a_{m+1}, \dots, a_n) \\ &+ K_{m-1}(a_1, \dots, a_{m-1}) K_{n-m-1}(a_{m+2}, \dots, a_n). \end{aligned} \tag{3}$$

This relation contains the recursion relation $q_{m+1} = a_{m+1} q_m + q_{m-1}$ as the special case $n = m + 1$ and is proved under the name "Fundamentalformeln" in O. Perron's classic textbook [8]. From now on we will drop the index and write $K(a_1, \dots, a_n)$.

Lemma 4.

$$\log K(a_1, \dots, a_n, b_m, \dots, b_1) = \log K(a_1, \dots, a_n) + \log K(b_1, \dots, b_m) + O(1)$$

with an absolute implied constant.

Proof. Using Equation (3) we get

$$\begin{aligned} \log & K(a_1, \dots, a_n, b_m, \dots, b_1) \\ &= \log\Big(K(a_1, \dots, a_n) K(b_1, \dots, b_m) + K(a_1, \dots, a_{n-1}) K(b_1, \dots, b_{m-1}) \Big) \\ &= \log K(a_1, \dots, a_n) + \log K(b_1, \dots, b_m) \\ &\quad + \underbrace{\log\left(1 + \frac{K(a_1, \dots, a_{n-1}) K(b_1, \dots, b_{m-1})}{K(a_1, \dots, a_n) K(b_1, \dots, b_m)} \right)}_{\in (0, \log 2)} \end{aligned}$$

which immediately implies the assertion. □

Lemma 5. *If $\alpha = [0, \overline{a}] = [0, a, a, a, \ldots]$ then*

$$\log q_m(\alpha) = m \log[\overline{a}] + O(1) = m \log \frac{a + \sqrt{a^2+4}}{2} + O(1).$$

P r o o f. This follows from the fact that

$$\begin{aligned} q_m(\alpha) &= \frac{1}{\sqrt{a^2+4}} \Big([\overline{a}]^{m+1} - (-[0,\overline{a}])^{m+1} \Big) \\ &= \frac{1}{\sqrt{a^2+4}} \bigg(\Big(\frac{a+\sqrt{a^2+4}}{2} \Big)^{m+1} - \Big(\frac{a-\sqrt{a^2+4}}{2} \Big)^{m+1} \bigg) \end{aligned}$$

for all $m \geq -1$ which can be easily proved by induction. □

Theorem 6 (Faivre)**.** *For all $\beta \geq \log \frac{1+\sqrt{5}}{2}$ there exists an $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ such that*

$$\lim_{m\to\infty} \frac{1}{m} \log q_m(\alpha) = \beta.$$

P r o o f. If $\beta = \log[\overline{a}]$ for some $a \in \mathbb{N}$ the assertion is proved in Lemma 5.

Let $\log[\overline{a}] < \beta < \log[\overline{b}]$ for positive integers $a < b$. Set

$$x = \frac{\beta - \log[\overline{a}]}{\log[\overline{b}] - \log[\overline{a}]}.$$

Then

$$\beta = (1-x)\log[\overline{a}] + x \log[\overline{b}].$$

Choose a positive integer $n_0 \in \mathbb{N}$ such that

$$\big[(1-x)(n+1)^2\big] - \big[(1-x)n^2\big] \geq 1 \quad \text{and} \quad \big[x(n+1)^2\big] - \big[xn^2\big] \geq 1 \quad \text{for all } n \geq n_0$$

and set

$$\lambda_n = \big[(1-x)(n+1)^2\big] - \big[(1-x)n^2\big] \quad \text{and} \quad \mu_n = \big[x(n+1)^2\big] - \big[xn^2\big].$$

Then

$$\lambda_{n_0} + \cdots + \lambda_n = (1-x)(n+1)^2 + O(1) \quad \text{and} \quad \mu_{n_0} + \cdots + \mu_n = x(n+1)^2 + O(1)$$

for all $n \geq n_0$ which implies

$$\begin{aligned} \lim_{n\to\infty} \frac{\lambda_{n_0} + \cdots + \lambda_n}{\lambda_{n_0} + \cdots + \lambda_n + \mu_{n_0} + \cdots + \mu_n} &= 1 - x, \\ \lim_{n\to\infty} \frac{\mu_{n_0} + \cdots + \mu_n}{\lambda_{n_0} + \cdots + \lambda_n + \mu_{n_0} + \cdots + \mu_n} &= x \\ \text{and} \quad \lim_{n\to\infty} \frac{n}{\lambda_{n_0} + \cdots + \lambda_n + \mu_{n_0} + \cdots + \mu_n} &= 0. \end{aligned} \tag{4}$$

Set $\alpha = [0, \overline{a}^{\lambda_{n_0}}, \overline{b}^{\mu_{n_0}}, \overline{a}^{\lambda_{n_0+1}}, \overline{b}^{\mu_{n_0+1}}, \ldots]$ where $\overline{a}^n$ denotes a block $a, \ldots, a$ of length n. For a positive integer m let k be such that

$$\sum_{i=n_0}^{k} (\lambda_i + \mu_i) \leq m < \sum_{i=n_0}^{k+1} (\lambda_i + \mu_i),$$

i.e.,

$$m = \sum_{i=n_0}^{k} (\lambda_i + \mu_i) + \widetilde{\lambda}_{k+1} + \widetilde{\mu}_{k+1},$$

where either $0 \leq \widetilde{\lambda}_{k+1} \leq \lambda_{k+1}$ and $\widetilde{\mu}_{k+1} = 0$ or $\widetilde{\lambda}_{k+1} = \lambda_{k+1}$ and $0 < \widetilde{\mu}_{k+1} < \mu_{k+1}$. Using Lemmata 4 and 5 we get

$$\begin{aligned}
\log q_m &= \log K\Big(\overline{a}^{\lambda_{n_0}}, \overline{b}^{\mu_{n_0}}, \ldots, \overline{a}^{\lambda_k}, \overline{b}^{\mu_k}, \overline{a}^{\widetilde{\lambda}_{k+1}}, \overline{b}^{\widetilde{\mu}_{k+1}}\Big) \\
&= \sum_{i=n_0}^{k} \log K(\overline{a}^{\lambda_i}) + \sum_{i=n_0}^{k} \log K(\overline{b}^{\mu_i}) + \log K(\overline{a}^{\widetilde{\lambda}_{k+1}}) + \log K(\overline{b}^{\widetilde{\mu}_{k+1}}) + O(k) \\
&= \sum_{i=n_0}^{k} \Big(\lambda_i \log[\overline{a}]\Big) + \sum_{i=n_0}^{k} \Big(\mu_i \log[\overline{b}]\Big) + \widetilde{\lambda}_{k+1} \log[\overline{a}] + \widetilde{\mu}_{k+1} \log[\overline{b}] + O(k) \\
&= \Big(\sum_{i=n_0}^{k} \lambda_i\Big) \log[\overline{a}] + \Big(\sum_{i=n_0}^{k} \mu_i\Big) \log[\overline{b}] + O(k)
\end{aligned}$$

and

$$m = \sum_{i=n_0}^{k} \lambda_i + \sum_{i=n_0}^{k} \mu_i + \widetilde{\lambda}_{k+1} + \widetilde{\mu}_{k+1} = \sum_{i=n_0}^{k} \lambda_i + \sum_{i=n_0}^{k} \mu_i + O(k).$$

Using Equation (4) we arrive at

$$\begin{aligned}
\lim_{m\to\infty} \frac{1}{m} \log q_m(\alpha) = &\lim_{k\to\infty} \frac{\Big(\sum_{i=n_0}^{k} \lambda_i\Big) \log[\overline{a}] + \Big(\sum_{i=n_0}^{k} \mu_i\Big) \log[\overline{b}] + O(k)}{\sum_{i=n_0}^{k} \lambda_i + \sum_{i=n_0}^{k} \mu_i + O(k)} \\
=&(1-x) \log[\overline{a}] + x \log[\overline{b}] = \beta.
\end{aligned}$$

□

Theorem 7 (Wu).

$$\overline{\Big\{\beta(\alpha) \;\Big|\; \alpha \textit{ is a quadratic irrationality}\Big\}} = \Big[\log \frac{1+\sqrt{5}}{2}, +\infty\Big)$$

P r o o f. Let $\log[\overline{a}] < \beta = (1-x)\log[\overline{a}] + x\log[\overline{b}] < \log[\overline{b}]$ as above. We know that there is $\alpha = [0, \overline{a}^{\lambda_1}, \overline{b}^{\mu_1}, \overline{a}^{\lambda_2}, \overline{b}^{\mu_2}, \ldots]$ such that $\lim_{m\to\infty} \frac{1}{m}\log q_m(\alpha) = \beta$. Let

$$\alpha_p = \left[0, \overline{\overline{a}^{\lambda_1}, \overline{b}^{\mu_1}, \ldots, \overline{a}^{\lambda_p}, \overline{b}^{\mu_p}}\right].$$

The existence of $\beta(\alpha_p) = \lim_{m\to\infty} \frac{1}{m}\log q_m(\alpha_p)$ follows from Theorem 3. We claim that $\lim_{p\to\infty} \beta(\alpha_p) = \beta(\alpha)$. We write

$$\alpha_p = [0, \overline{a}^{\lambda_1'}, \overline{b}^{\mu_1'}, \overline{a}^{\lambda_2'}, \overline{b}^{\mu_2'}, \ldots],$$

i.e., $\lambda_i' = \lambda_i$ and $\mu_i' = \mu_i$ for $1 \le i \le p$ and $\lambda_i' = \lambda_j'$ and $\mu_i' = \mu_j'$ if $i \equiv j \pmod p$. For a positive integer m let k be such that

$$\sum_{i=1}^{k} (\lambda_i' + \mu_i') \le m < \sum_{i=1}^{k+1} (\lambda_i' + \mu_i')$$

and $k = qp + r$ (with $q \ge 0$ and $0 \le r < p$). Employing Lemmata 4 and 5 as in the proof of Theorem 6 we get

$$\log q_m(\alpha_p) = q\Big(\sum_{i=1}^{p} \lambda_i\Big)\log[\overline{a}] + q\Big(\sum_{i=1}^{p} \mu_i\Big)\log[\overline{b}] + \Delta_m$$

where $\Delta_m = O(qp + r)$ and

$$m = q\sum_{i=1}^{p} \lambda_i + q\sum_{i=1}^{p} \mu_i + O(1).$$

Therefore

$$\begin{aligned}\beta(\alpha_p) &= \lim_{m\to\infty} \frac{1}{m}\log q_m(\alpha_p)\\ &= \frac{\sum_{i=1}^{p}\lambda_i}{\sum_{i=1}^{p}\lambda_i + \sum_{i=1}^{p}\mu_i}\log[\overline{a}] + \frac{\sum_{i=1}^{p}\mu_i}{\sum_{i=1}^{p}\lambda_i + \sum_{i=1}^{p}\mu_i}\log[\overline{b}] + \frac{\lim_{q\to\infty}\frac{1}{q}\Delta_m}{\sum_{i=1}^{p}\lambda_i + \sum_{i=1}^{p}\mu_i}\end{aligned}$$

where $\lim_{q\to\infty} \Delta_m/q \ll p$ and thus

$$\lim_{p\to\infty} \beta(\alpha_p) = (1-x)\log[\overline{a}] + x\log[\overline{b}] = \beta(\alpha).$$

□

Remark. In fact we proved the following stronger result: for positive integers $a < b$ let

$$Q_{a,b} = \{\alpha = [0, a_1, a_2, \ldots] \mid \alpha \text{ is a quadratic irrationality and } a_i \in \{a, b\} \text{ for all } i \geq 1\}.$$

Then $\overline{\{\beta(\alpha) \mid \alpha \in Q_{ab}\}} = \big[\log[\overline{a}], \log[\overline{b}]\big]$.

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Received July 28, 2008
Accepted February 27, 2009

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