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uniform distribution theory

ADDENDUM TO: *A THEOREM OF KHINTCHINE TYPE*

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Throughout, notation and definitions are consistent with those in [4]. By making appeal only to elementary facts of measure theory and arithmetic, we have proved in [4] that the set

$$K^*(\psi) := \Big\{ x \in (0,1) : \Big| x - \frac{p}{q} \Big| < \psi(q) \text{ for infinitely many reduced rationals } \frac{p}{q} \Big\}$$

has Lebesgue measure one, provided $\psi : \mathbb{N} \to [0,\infty)$ is an approximation function with the necessary condition

$$\sum_{q=1}^{\infty} q\psi(q) = \infty \tag{h_1}$$

and with

$$\text{there is } \delta > 0 \text{ so that } \psi(q) \geq \delta\psi(s) \text{ for all } q \in \mathbb{N} \text{ and } s \in \{q, q+1, \dots, 2q\}. \tag{h_2}$$

With the aid of Gallagher's zero-one law, in this note we shall generalize that result as follows:

Theorem 1. *If $\psi : \mathbb{N} \to [0,\infty)$ is an approximation function with (h_1) and (h_2), and if $\mathbf{q} := (q_i)_{i\in\mathbb{N}}$ is a strictly increasing sequence of natural numbers with positive lower density, then the set*

$$K^*_{\mathbf{q}}(\psi) := \Big\{ x \in (0,1) : \Big| x - \frac{p}{q_i} \Big| < \psi(q_i) \textit{ for infinitely many reduced rationals } \frac{p}{q_i} \Big\}$$

has Lebesgue measure one.

Note that Theorem 1 extends Duffin and Schaeffer's Theorem III in [1] (for (h_2) is weaker than their decay rate assumption on ψ, namely: for some real c the map $q \mapsto q^c\psi(q)$ is decreasing); also, note that Theorem 1 is not implied by Harman's Theorem 6.2 [3] (as Harman's hypotheses (6.1.3) and (6.1.4) are stronger than (h_2)).

Let us now proceed with the (brief) proof of the theorem. Let ψ and $\mathbf{q}$ be as in Theorem 1. By [4, Lemma 6] it is not limitative to assume

$$q\psi(q) \leq \frac{1}{2} \quad \text{for all } q > 1 \tag{1}$$

and $\sum_{q=1}^{\infty} \psi(q) < \infty$; by [4, Lemma 8] the latter leads to

$$\sum_{i=1}^{\infty} q_i \psi(q_i) = \infty. \tag{2}$$

Moreover, it follows from the first part of the proof of Corollary 3 in [3, p.42] that we may even assume

$$\frac{\phi(q_i)}{q_i} \geq c \quad \text{for all } i \in \mathbb{N} \text{ and some } c > 0 \tag{3}$$

(incidentally, Duffin and Schaeffer deduce this fact in [1, p.252] by resorting to Schönberg's theorem on the continuity of the distribution function of the sequence $q \mapsto \phi(q)/q$, whereas Harman's argument is based on the simpler [3, Lemma 2.5]). Now, for all $i > 1$ consider the sets

$$E_{q_i} := \bigcup_{\substack{1 \leq m \leq q_i - 1 \\ \gcd(m, q_i) = 1}} \left(\frac{m}{q_i} - \psi(q_i), \frac{m}{q_i} + \psi(q_i) \right);$$

by definition, we have

$$K^*_{\mathbf{q}}(\psi) = \limsup E_{q_i}. \tag{4}$$

Since the intervals forming each E_{q_i} are disjoint (trivially due to (1)), we have

$$\lambda(E_{q_i}) = 2\phi(q_i)\psi(q_i) \quad \text{for all } i > 1; \tag{5}$$

moreover, by [1, Lemma II] (see, in alternative, [3, p.39]) we have

$$\lambda(E_{q_i} \cap E_{q_j}) \leq 8 q_i \psi(q_i) q_j \psi(q_j) \quad \text{for all } i, j \text{ with } j > i > 1. \tag{6}$$

Combining (2), (3) and (5), we get

$$\sum_{i=1}^{\infty} \lambda(E_{q_i}) = \infty \tag{7}$$

and, combining (3), (5) and (6),

$$\lambda(E_{q_i} \cap E_{q_j}) \leq \frac{2}{c^2} \lambda(E_{q_i}) \lambda(E_{q_j}) \quad \text{for all } i, j \text{ with } j > i > 1. \tag{8}$$

By [4, Lemma 3], items (4), (7), and (8) together imply

$$\lambda(K^*_{\mathbf{q}}(\psi)) > 0. \tag{9}$$

Consider now the approximation function $\theta : \mathbb{N} \to [0, \infty)$ defined by letting $\theta(q) := \psi(q_i)$ if $q = q_i$ for some i, or $\theta(q) := 0$ otherwise. Since $K^*_{\mathbf{q}}(\psi) = K^*(\theta)$, by (9) and Gallagher's zero-one law (see [2] or [3, Theorem 2.7 (B)]) we conclude

$$\lambda(K^*_{\mathbf{q}}(\psi)) = \lambda(K^*(\theta)) = 1$$

(note that Gallagher's Lemma 1 [2] is unnecessary here, for $\psi(q) \to 0$ is already ensured by (1)). The proof of Theorem 1 is complete.

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