Uniform Distribution Theory **3** (2008), no.1, 73–83

uniform distribution theory

A THEOREM OF KHINTCHINE TYPE

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ABSTRACT. Let $\psi : \mathbb{N} \to [0, \infty)$ be an approximation function with $\sum_{q=1}^{\infty} q\psi(q) = \infty$ and the property that there exists $\delta > 0$ such that $\psi(q) \geq \delta\psi(s)$ for all $q \in \mathbb{N}$ and all $s \in \{q, q+1, \ldots, 2q\}$. Then the set

$$\left\{x \in (0,1) : \left|x - \frac{p}{q}\right| < \psi(q) \text{ for infinitely many reduced rationals } \frac{p}{q}\right\}$$

has Lebesgue measure one.

Communicated by Oto Strauch

1. Introduction

Let $\psi : \mathbb{N} \to [0, \infty)$ be an arbitrary sequence – it shall be called "approximation function", in the sequel. A central question in Diophantine approximation theory consists in determining the size of the Khintchine set

$$K(\psi) := \left\{x \in (0,1) : \left|x - \frac{p}{q}\right| < \psi(q) \text{ for infinitely many rationals } \frac{p}{q}\right\}$$

in terms of its Lebesgue measure $\lambda(K(\psi))$. A fundamental theorem by Khintchine [7] establishes sufficient conditions on ψ for $K(\psi)$ to have Lebesgue measure zero or one (see, e.g., Bugeaud's book [2, Section 1.3]): *If* $\sum_{q=1}^{\infty} q\psi(q) < \infty$*, then* $\lambda(K(\psi)) = 0$*. If* $\sum_{q=1}^{\infty} q\psi(q) = \infty$ *and the map* $q \mapsto q^2\psi(q)$ *is decreasing, then* $\lambda(K(\psi)) = 1$. He also raised the question whether the decay hypothesis in the second half of his theorem could be dropped – in other words, whether it would be possible to characterize the Khintchine sets $K(\psi)$ of full Lebesgue measure as those for which $\sum_{q=1}^{\infty} q\psi(q) = \infty$. An example by Duffin and Schaeffer states that this is not the case (see [5] or [6, Theorem 2.8]). So, if not dropped, *can the decay rate assumption on* ψ *be weakened?*

2000 Mathematics Subject Classification: 11J83, 11K60, 28A12.
Keywords: Khintchine theorem, Borel–Cantelli lemma, approximation function, Euler's ϕ-function, Dawson–Sankoff inequality, Lebesgue measure, Duffin–Schaeffer conjecture.

In their seminal paper [5], Duffin and Schaeffer studied, instead of $K(\psi)$, the modified Khintchine set

$$K^*(\psi) := \left\{ x \in (0,1) : \left| x - \frac{p}{q} \right| < \psi(q) \text{ for infinitely many reduced rationals } \frac{p}{q} \right\}$$

under the assumption $\sum_{q=1}^{\infty} \phi(q)\psi(q) = \infty$[1] – as usual, ϕ stands for the Euler totient function. There are at least two reasons for approving Duffin and Schaeffer's viewpoint: the first is that "the most natural formulation of this problem is in terms of reduced fractions" [5, p.244]; the second is that the task of estimating $\lambda(K^*(\psi))$ is easier than that of estimating $\lambda(K(\psi))$. Indeed, in [5, Theorem III] (see, in alternative, Corollary 1 to Theorem 2.5 in [6]) Duffin and Schaeffer succeeded in improving the Khintchine theorem by imposing a less restrictive decay rate assumption on ψ. Here is their result, not stated in its full generality: *Let $\sum_{q=1}^{\infty} q\psi(q) = \infty$ and $q\psi(q) \leq 1/2$ for all q.*[2] *If for some $c \in \mathbb{R}$ the map $q \mapsto q^c\psi(q)$ is decreasing, then $\lambda(K^*(\psi)) = 1$.*

The way opened by Duffin and Schaeffer has led to remarkable results, such as those of Erdős, Vaaler, Vilchinskii, and Strauch. In this context, Harman's contribution (see [6, Theorem 6.2]) is of particular interest: *Let $\sum_{q=1}^{\infty} q\psi(q) = \infty$. If if there exist two positive constants δ, M such that $M\psi(s) \geq \psi(q) \geq \delta\psi(s)$ for all $q \in \mathbb{N}$ and all $s \in \{q, q+1, \ldots, 2q-1\}$, then $\lambda(K^*(\psi)) = 1$.* In justice to its author we pause to note that the statement above is only a particular instance of Harman's theorem, for he additionally considers various restrictions to the numerator p and the denominator q of the approximating rational p/q – a problem not touched, here.

By only appealing to the rudiments of measure theory and the basic facts of arithmetic, in this paper we shall prove the following:

Theorem 1. *If $\psi : \mathbb{N} \to [0, \infty)$ is an approximation function such that*

$$\sum_{q=1}^{\infty} q\psi(q) = \infty \tag{h_1}$$

and

there is $\delta > 0$ so that $\psi(q) \geq \delta\psi(s)$ for all $q \in \mathbb{N}$ and $s \in \{q, q+1, \ldots, 2q\}$, $\quad (h_2)$

then $\lambda(K^*(\psi)) = 1$ (a fortiori, $\lambda(K(\psi)) = 1$).

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[1] By the convergence half of the Borel–Cantelli lemma, this condition is necessary for $\lambda(K^*(\psi)) = 1$. Whether also sufficient, it is "to date one of the most important unsolved problems in metric number theory" [6, p.27], known as "Duffin–Schaeffer conjecture".

[2] As noticed in [6, p.37], the latter assumption on ψ, omitted by Duffin and Schaeffer, turns out to be necessary in their proof.

It is easy to verify that our hypothesis (h_2) on ψ embraces those of Duffin and Schaeffer (let $\delta := \min\{1, 2^c\}$) and Harman (as (h_2) is even equivalent to: for every $\alpha \in [0,1)$ there exists $\delta > 0$ such that $\psi(q) \geq \delta\psi(s)$ for all $q \in \mathbb{N}$ and $s \in \{q, q+1, \dots, 2q - \lceil \alpha q \rceil\}$). So, in this limited sense, Theorem 1 is a synthesis of the two afore-mentioned results.

The author would like to thank the referee for helpful comments.

2. Another form of the second Borel–Cantelli lemma

The need of some suitable form of the divergence half of the Borel–Cantelli lemma very often emerges in metric number theory: the most frequently used is that dating back to Erdős and Rényi (it can be found in [2, Lemma 6.2] and [6, Lemma 2.3]). As a matter of fact, the following inequality for probabilities is sharper than, and straightforwardly implies, Erdős and Rényi's one [3, Corollary 2].

Lemma 2 (Dawson and Sankoff [3]). *Let $E_1, E_2, \dots, E_n$ be events in a probability space $(\Omega, \mathcal{B}, P)$. Then*

$$P\left(\bigcup_{i=1}^{n} E_i\right) \sum_{i=1}^{n}\sum_{j=1}^{n} P(E_i \cap E_j) \geq \left(\sum_{i=1}^{n} P(E_i)\right)^2 .$$

The easiest way to obtain Lemma 2 (without any appeal to integration theory) is to prove de Caen's inequality for probabilities (see [4]) and then, as indicated there, to immediately deduce Lemma 2 via the Cauchy–Schwarz inequality.

Let Ω be a set and $(E_q)_{q\in\mathbb{N}}$ a sequence of subsets of Ω. By $\limsup E_q$ we denote the subset of Ω consisting of all points in Ω belonging to infinitely many E_q. Namely:

$$\limsup E_q = \bigcap_{q=1}^{\infty} \bigcup_{s=q}^{\infty} E_s .$$

Let us now establish a lower bound for the probability of the "limsup" of a sequence of events in a probability space.

Lemma 3. *Let $(\Omega, \mathcal{B}, P)$ be a probability space and $(E_q)_{q\in\mathbb{N}}$ a sequence of events in Ω. Let M be a constant and I an event in Ω with $P(I) > 0$. If there exists a strictly increasing sequence $(q_i)_{i\in\mathbb{N}}$ of natural numbers such that*

$$\sum_{i=1}^{\infty} P(E_{q_i} \cap I) = \infty \tag{1}$$

and

$$P(E_{q_i} \cap E_{q_j} \cap I) \le M \frac{P(E_{q_i} \cap I)P(E_{q_j} \cap I)}{P(I)} \text{ for all } i \in \mathbb{N} \text{ and } j > i, \tag{2}$$

then

$$P(\limsup E_q \cap I) \ge \frac{P(I)}{2(4M+1)}.$$

P r o o f. Fix arbitrarily $n \in \mathbb{N}$. By (1), there exists a natural number $m > n$ such that

$$P(I) \le \sum_{i=n}^{m} P(E_{q_i} \cap I) \le 2P(I). \tag{3}$$

By (2), (3) and Lemma 2, we have

$$\begin{aligned} P\left(\bigcup_{q=n}^{\infty}(E_q \cap I)\right) &\ge P\left(\bigcup_{i=n}^{m}(E_{q_i} \cap I)\right) \ge \frac{(\sum_{i=n}^{m} P(E_{q_i} \cap I))^2}{\sum_{i=n}^{m}\sum_{j=n}^{m} P(E_{q_i} \cap E_{q_j} \cap I)} \\ &= \frac{(\sum_{i=n}^{m} P(E_{q_i} \cap I))^2}{2(\sum_{i=n}^{m-1}\sum_{j=i+1}^{m} P(E_{q_i} \cap E_{q_j} \cap I)) + \sum_{i=n}^{m} P(E_{q_i} \cap I)} \\ &\ge \frac{P^2(I)}{8MP^2(I)/P(I) + 2P(I)} = \frac{P(I)}{2(4M+1)}. \end{aligned}$$

By the arbitrariness of $n \in \mathbb{N}$, this gives

$$P(\limsup E_q \cap I) = \lim_{k\to\infty} P\left(\bigcap_{n=1}^{k}\bigcup_{q=n}^{\infty}(E_q \cap I)\right) \ge \frac{P(I)}{2(4M+1)}$$

and ends the proof. □

Lemma 4. *Let $(\Omega, \mathcal{B}, P)$ be a probability space and let $\mathcal{E}$ be a subfamily of $\mathcal{B}$ such that for every $B \in \mathcal{B}$*

$$P(B) = \inf\left\{\sum_{n=1}^{\infty} P(I_n) : B \subseteq \bigcup_{n=1}^{\infty} I_n \text{ and } I_n \in \mathcal{E} \text{ for all } n \in \mathbb{N}\right\}. \tag{4}$$

Let E be an event in Ω. If there exists a constant $\epsilon > 0$ such that

$$P(E \cap I) \ge \epsilon P(I) \text{ for all } I \in \mathcal{E},$$

then $P(E) = 1$.

Proof. Put $A := \Omega \setminus E$. By subadditivity, for every sequence $(I_n)_{n\in\mathbb{N}}$ in $\mathcal{E}$ covering A we have

$$P(A) \le \sum_{n=1}^{\infty} P(A \cap I_n) \le (1-\epsilon) \sum_{n=1}^{\infty} P(I_n).$$

By (4) and the arbitrariness of the covering $(I_n)_{n\in\mathbb{N}}$, we have $P(A) \le (1-\epsilon)P(A)$ and therefore $P(A) = 0$, i.e., $P(E) = 1$. □

The next proposition is the measure-theoretic ingredient of our proof of Theorem 1:

Proposition 5. *Let $(\Omega, \mathcal{B}, P)$ be a probability space, $(E_q)_{q\in\mathbb{N}}$ a sequence of events in Ω, and $\mathcal{E}$ a subfamily of $\mathcal{B}$ as in Lemma 4. If there exist a positive constant M (depending only on the sequence $(E_q)_{q\in\mathbb{N}}$) and, for each $I \in \mathcal{E}$ with $P(I) > 0$, a strictly increasing sequence $(q_i)_{i\in\mathbb{N}}$ of natural numbers (possibly depending on I) such that*

$$\sum_{i=1}^{\infty} P(E_{q_i} \cap I) = \infty \tag{5}$$

and

$$P(E_{q_i} \cap E_{q_j} \cap I) \le M \frac{P(E_{q_i} \cap I)P(E_{q_j} \cap I)}{P(I)} \quad \text{for all } i \in \mathbb{N} \text{ and } j > i, \tag{6}$$

then $P(\limsup E_q) = 1$.

Proof. It is just a combination of Lemma 3, yielding $P(\limsup E_q \cap I) \ge P(I)/(2(4M+1))$ for all $I \in \mathcal{E}$, and Lemma 4 (applied for $\epsilon = 1/(2(4M+1))$). □

3. Auxiliary results

Let us start with a slight modification of Lemma 4 from [1] (see, in alternative, [2, Lemma 6.3]).

Lemma 6. *If θ is an approximation function that satisfies (h_1) and (h_2), then there exists an approximation function ψ satisfying (h_1), (h_2),*

$$q\psi(q) \le \frac{1}{2} \quad \text{for all } q > 1, \tag{h_3}$$

$$\sum_{q=1}^{\infty} \psi(q) < \infty, \tag{h_4}$$

and such that $K^*(\psi) \subseteq K^*(\theta)$.

P r o o f. Let $\psi(q) := \min\{\theta(q), 1/q^2\}$ for all $q \in \mathbb{N}$. As $\psi(q) \le \theta(q)$ for all $q \in \mathbb{N}$, we have $K^*(\psi) \subseteq K^*(\theta)$. That (h_3) and (h_4) hold for ψ is trivial, and (h_2) follows immediately from the monotonicity of the map $q \mapsto 1/q^2$. So, we only have to check (h_1) for ψ. Assume, towards a contradiction, that $\sum_{q=1}^{\infty} q\psi(q) < \infty$. On the one hand, since $q \le 2\lceil q/2 \rceil$,

$$q^2\psi(q) \le 2\sum_{i=1}^{q} i\psi(q) \le 4\sum_{i=\lceil q/2\rceil}^{q} i\psi(q) \le \frac{4}{\delta}\sum_{i=\lceil q/2\rceil}^{q} i\psi(i) \text{ for every } q \in \mathbb{N},$$

which yields $\limsup q^2\psi(q) = 0$. On the other hand, as $\sum_{q=1}^{\infty} q\theta(q) = \infty$, for infinitely many q we have $\psi(q) = 1/q^2$, i.e., $q^2\psi(q) = 1$. Absurd. □

In view of Lemma 6, it is not limitative for us to assume (h_3) and (h_4) for any approximation function ψ that fulfils (h_1) and (h_2). So:

From now on the approximation function ψ shall be always assumed to satisfy (h_1)–(h_4).

In the sequel we shall adopt the standard Vinogradov symbol $\ll$ to mean "$\le$ up to a positive constant multiplier". Recall that the lower density of a strictly increasing sequence $(q_i)_{i\in\mathbb{N}}$ of natural numbers is defined to be the number $d_L := \liminf_{n\to\infty} \max\{i \in \mathbb{N} : q_i \le n\}/n$. It is intuitive and easy to prove that $d_L > 0$ if and only if $q_i \ll i$ (the reader interested in a proof is referred to [9, Theorem 11.1], for instance). We record this fact for future reference:

Lemma 7. *A strictly increasing sequence $(q_i)_{i\in\mathbb{N}}$ of natural numbers has positive lower density if and only if $q_i \ll i$.*

Lemma 8. *If $(q_i)_{i\in\mathbb{N}}$ is a strictly increasing sequence of natural numbers with positive lower density, then $\sum_{i=1}^{\infty} q_i\psi(q_i) = \infty$.*

P r o o f. Let us firstly prove that

$$\sum_{i=1}^{\infty} i\psi(ki) = \infty \text{ for every } k \in \mathbb{N}. \tag{7}$$

We may assume $k > 1$ (the case $k = 1$ coincides with our hypothesis (h_1)). By (h_2)

$$\begin{aligned}\sum_{q=1}^{\infty} q\psi(q) &= \sum_{q=1}^{k-1} q\psi(q) + \sum_{i=1}^{\infty}\sum_{m=0}^{k-1}(ki+m)\psi(ki+m)\\ &\leq \sum_{q=1}^{k-1} q\psi(q) + \sum_{i=1}^{\infty}\sum_{m=0}^{k-1}\frac{(ki+m)\psi(ki)}{\delta}\\ &= \sum_{q=1}^{k-1} q\psi(q) + \frac{1}{\delta}\sum_{i=1}^{\infty}\left(k^2 i + \frac{k(k-1)}{2}\right)\psi(ki)\\ &= \sum_{q=1}^{k-1} q\psi(q) + \frac{k(k-1)}{2\delta}\sum_{i=1}^{\infty}\psi(ki) + \frac{k^2}{\delta}\sum_{i=1}^{\infty} i\psi(ki). \end{aligned} \tag{8}$$

Since by (h_4)

$$\sum_{q=1}^{k-1} q\psi(q) + \frac{k(k-1)}{2\delta}\sum_{i=1}^{\infty}\psi(ki) < \infty,$$

(7) follows from (8) and (h_1).

Now, in view of Lemma 7 we have

$$i \leq q_i \leq ki \leq kq_i \text{ for some } k \in \mathbb{N} \text{ and all } i \in \mathbb{N};$$

together with (h_2), this gives

$$\psi(q_i) \geq \delta^{\lceil \log_2 k \rceil}\psi(ki) \text{ for all } i \in \mathbb{N}$$

and consequently

$$\sum_{i=1}^{\infty} q_i\psi(q_i) \geq \sum_{i=1}^{\infty} i\psi(q_i) \geq \delta^{\lceil \log_2 k \rceil}\sum_{i=1}^{\infty} i\psi(ki).$$

The proof is finally completed with the aid of (7). □

The asymptotic behavior on average of Euler's ϕ-function is vital for the following proposition, the number-theoretic ingredient of our proof of Theorem 1.

Proposition 9. *There exists a strictly increasing sequence $(q_i)_{i\in\mathbb{N}}$ of natural numbers with lower density $\geq 1/4$ and with $\phi(q_i)/q_i \geq 1/4$ for all $i \in \mathbb{N}$.*

Proof. It is known that for all large n we have

$$\sum_{q=1}^{n}\frac{\phi(q)}{q} \geq \frac{n}{2};$$

LeVeque presents this fact as an exercise [8, p.171] based on the asymptotic formula for the mean value of Euler's ϕ-function [8, Theorem 6.32] and on Abel's partial summation formula [8, p.148]. This inequality (together with $\phi(q)/q \le 1$ for all q) plainly ensures that for all large n we have $\phi(q)/q \ge 1/4$ for at least one fourth of those q belonging to $\{1, 2, \dots, n\}$. □

It immediately follows from Lemma 8 and Proposition 9 that under (h_2) the two conditions $\sum_{q=1}^{\infty} q\psi(q) = \infty$ and $\sum_{q=1}^{\infty} \phi(q)\psi(q) = \infty$ on ψ are equivalent.

4. Proof of Theorem 1

For every $q > 1$ put

$$E_q := \bigcup_{\substack{1\le i\le q-1\\ \gcd(i,q)=1}} \left(\frac{i}{q} - \psi(q), \frac{i}{q} + \psi(q)\right). \tag{9}$$

Note that the set E_q consists of the disjoint (by (h_3)) union of $\phi(q)$ intervals, each of length $2\psi(q)$ and included in $(0, 1)$, centered in reduced rationals with denominator q; by definition, $K^*(\psi) = \limsup E_q$.

The plan of the proof is to apply Proposition 5 to the case where: the probability space $(\Omega, \mathcal{B}, P)$ is the interval $(0, 1)$ equipped with the Lebesgue measure λ restricted to the Borel σ-algebra $\mathcal{B}$ of $(0, 1)$; the sets E_q are those defined in (9); the family $\mathcal{E}$ contains all intervals of the particular form $(m/p, (m+1)/p)$, being p a prime number and $m \in \{0, 1, \dots, p-1\}$ (by a routine density argument, such family $\mathcal{E}$ is easily seen to fulfil (4)). As shall be seen in a moment, the advantage in limiting our study to such intervals is that the necessary task of estimating the size of the sets E_{p^2q} and their overlaps is very easy, there.

The following lemma appears in [5, Lemma I] without the proof; Harman provides us with it in [6, p.39].

Lemma 10. *Let $q, s \in \mathbb{N}$ and $A > 0$. Then the number of integers pairs (i, j) which satisfy $0 < |is - jq| \le A$ and $1 \le i \le q$, $1 \le j \le s$, does not exceed $2A$.*

Lemma 11. *Let p be a prime number and $m \in \{0, 1, \dots, p-1\}$. Put $I := (m/p, (m+1)/p)$. Then*

$$\lambda(E_{p^2q} \cap I) = 2\phi(pq)\psi(p^2q) \text{ for every } q \in \mathbb{N} \tag{10}$$

and

$$\lambda(E_{p^2q} \cap E_{p^2s} \cap I) \le \frac{8}{\lambda(I)} pq\psi(p^2q)ps\psi(p^2s) \text{ for every } q \in \mathbb{N} \text{ and } s > q. \tag{11}$$

P r o o f. Basically, the proof is the "scaled" version of the original one by Duffin and Schaeffer (see [5, Lemma II] or [6, p.39]). Let us fix $q \in \mathbb{N}$ and write I as follows (it is maybe useful to visualize I in this way)

$$I = \left(\frac{m}{p}, \frac{m+1}{p}\right) = \left(\frac{mpq}{p^2q}, \frac{mpq+pq}{p^2q}\right).$$

Thanks to the equivalence (of elementary verification)

$$\gcd(i, p^2q) = 1 \text{ if and only if } \gcd(i, pq) = 1 \text{ for any } i \in \mathbb{N},$$

we infer that the interval I intersects – even more than this, properly includes – exactly $\phi(pq)$ intervals of E_{p^2q}, each of length $2\psi(p^2q)$. By disjointness, this proves the first part of the statement.

Now, let $q \in \mathbb{N}$ and $s > q$. Suppose there are two intervals with nonempty overlap, one in $E_{p^2q} \cap I$ and one in $E_{p^2s} \cap I$, and let $(i+mpq)/p^2q$ and $(j+mps)/p^2s$ be their respective centers – here $1 \le i \le pq$ and $1 \le j \le ps$. Then

$$\begin{aligned} 0 < \left|\frac{i}{p^2q} - \frac{j}{p^2s}\right| &= \left|\frac{i+mpq}{p^2q} - \frac{j+mps}{p^2s}\right| \\ &\le \psi(p^2q) + \psi(p^2s) \le 2\max\{\psi(p^2q), \psi(p^2s)\} \end{aligned}$$

(note that the first inequality is strict by coprimeness) and consequently

$$0 < |ips - jpq| \le 2p^3qs \max\{\psi(p^2q), \psi(p^2s)\}.$$

By Lemma 10 there are no more than $4p^3qs\max\{\psi(p^2q), \psi(p^2s)\}$ pairs of intervals having nonempty intersection; moreover, each overlap has measure at most $2\min\{\psi(p^2q), \psi(p^2s)\}$. To sum up:

$$\begin{aligned} \lambda(E_{p^2q} \cap E_{p^2s} \cap I) &\le 8p^3qs\max\{\psi(p^2q), \psi(p^2s)\}\min\{\psi(p^2q), \psi(p^2s)\} \\ &= 8p^3qs\psi(p^2q)\psi(p^2s) = \frac{8}{\lambda(I)}pq\psi(p^2q)ps\psi(p^2s) \text{ for every } q \in \mathbb{N} \text{ and } s > q. \end{aligned}$$

The proof is complete. □

To reach (5) and (6) trough the estimates (10) and (11) it would suffice to find a positive constant, say c, and a strictly increasing sequence $(q_i)_{i\in\mathbb{N}}$ of natural numbers such that, for each prime number p, $\sum_{i=1}^{\infty} pq_i\psi(p^2q_i) = \infty$ and $\phi(pq_i)/pq_i \ge c$ for all $i \in \mathbb{N}$. We are in this position, now.

Lemma 12. *For any prime number p and any $q \in \mathbb{N}$, $\phi(pq) \ge (p-1)\phi(q)$.*

P r o o f. Immediately deduced from the equality $\phi(p^n) = p^{n-1}(p-1)$ valid for every $n \in \mathbb{N}$ and the fact that Euler's ϕ-function is multiplicative. □

Lemma 13. *Let p and I be as in Lemma 11, and $(q_i)_{i\in\mathbb{N}}$ as in Proposition 9. Then*

$$\sum_{i=1}^{\infty} \lambda(E_{p^2q_i} \cap I) = \infty \tag{12}$$

and

$$\lambda(E_{p^2q_i} \cap E_{p^2q_j} \cap I) \leq 128 \frac{\lambda(E_{p^2q_i} \cap I)\lambda(E_{p^2q_j} \cap I)}{\lambda(I)} \text{ for all } i \in \mathbb{N} \text{ and } j > i. \tag{13}$$

P r o o f. First we have, by Lemma 12,

$$\phi(pq_i) \geq (p-1)\phi(q_i) \geq \frac{p\phi(q_i)}{2} \geq \frac{pq_i}{8} \text{ for all } i \in \mathbb{N}, \tag{14}$$

which through the estimates (10) and (11) straightforwardly leads to (13):

$$\begin{aligned}\lambda(E_{p^2q_i} \cap E_{p^2q_j} \cap I) &\leq \frac{8}{\lambda(I)} pq_i\psi(p^2q_i)pq_j\psi(p^2q_j)\\ &\leq \frac{8^3}{\lambda(I)} \phi(pq_i)\psi(p^2q_i)\phi(pq_j)\psi(p^2q_j)\\ &= 128\frac{\lambda(E_{p^2q_i} \cap I)\lambda(E_{p^2q_j} \cap I)}{\lambda(I)} \text{ for all } i \in \mathbb{N} \text{ and } j > i.\end{aligned}$$

It remains to prove (12), by (10) equivalent to $\sum_{i=1}^{\infty} \phi(pq_i)\psi(p^2q_i) = \infty$. By (14) it is enough to check that $\sum_{i=1}^{\infty} pq_i\psi(p^2q_i) = \infty$ or, equivalently,

$$\sum_{i=1}^{\infty} p^2q_i\psi(p^2q_i) = \infty.$$

The latter is a consequence of the inequality chain $p^2q_i \ll q_i \ll i$, Lemma 7 and Lemma 8. $\square$

We are finally in a position to apply Proposition 5: we have the constant to put in (6), namely, $M := 128$; moreover, for any interval I of the form $(m/p, (m+1)/p)$ we have verified that the sequence $(E_{p^2q_i})_{i\in\mathbb{N}}$ fulfils (5) and (6). All this gives $\lambda(K^*(\psi)) = 1$.

REFERENCES

[1] BERESNEVICH, V.: *On approximation of real numbers by real algebraic numbers*, Acta Arith. **90** (1999), 97–112.

[2] BUGEAUD, Y.: *Approximation by Algebraic Numbers*, Cambridge University Press, Cambridge, 2004.

[3] DAWSON, D.A. – SANKOFF, D.: *An inequality for probabilities*, Proc. Amer. Math. Soc. **18** (1967), 504–507.

[4] DE CAEN, D.: *A lower bound on the probability of a union*, Discrete Math. **169** (1997), 217–220.

[5] DUFFIN, R.J. – SCHAEFFER, A.C.: *Khintchine's problem in metric Diophantine approximation*, Duke J. **8** (1941), 243–255.

[6] HARMAN, G.: *Metric Number Theory*, Clarendon Press, Oxford, 1998.

[7] KHINTCHINE, A.: *Einige Sätze über Kettenbrüche mit Anwendungen auf die Theorie der Diophantischen Approximationen*, Math. Ann. **92** (1924), 115–125.

[8] LEVEQUE, W.J.: *Fundamentals of Number Theory*, Addison-Wesley, Reading, 1977.

[9] NIVEN, I. – ZUCKERMAN, H.S.: *An Introduction to the Theory of Numbers*, Wiley, New York, 1972.

Received February 13, 2008
Accepted August 23, 2008

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