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uniform distribution theory

BERNOULLI POLYNOMIALS AND $(n\alpha)$-SEQUENCES

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ABSTRACT. Let $\alpha \in (0,1)$ be an irrational with continued fraction expansion $\alpha = [0; a_1, ...]$ and convergents $\frac{p_n}{q_n}, n = 0, 1, \ldots$. Given a positive integer N there exists a unique digit expansion, $N = \sum_{i=0}^{m} b_i q_i$, where the digits b_i are non-negative integers satisfying the conditions $b_0 < a_1, b_i \leq a_{i+1}$ and such that $b_i = a_{i+1}$ implies $b_{i-1} = 0$. It is called the Ostrowski expansion of N to base α. In this text we present an explicit formula for $\sum_{n=1}^{N} B_u(\{n\alpha\})$ entirely in terms of the digits $b_0, \ldots, b_m$ if $u = 2$ and an asymptotic formula for $u > 2$. The formula for $u = 2$ allows us to compute $\sum_{n=1}^{N} B_2(n\alpha)$ in $O((\log N)^3)$ steps. Finally we determine all of this $\alpha's$ for which this sum is bounded.

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1. Notations and statement of the result.

Let Ω be the set of all irrational numbers in the interval $[0, 1]$. Then every $\alpha \in \Omega$ has a unique continued fraction expansion $\alpha = [0; a_1, ...]$ and convergents $\frac{p_n}{q_n}$. Given a positive integer N there exists a unique digit expansion,

$$N = \sum_{i=0}^{m} b_i q_i,$$

where the digits b_i are non-negative integers satisfying the conditions $b_0 < a_1$, $b_i \leq a_{i+1}$ and such that $b_i = a_{i+1}$ implies $b_{i-1} = 0$. Is is called the Ostrowski expansion of N to base α.

We define the Bernoulli polynomials $B_n(x)$ by the generating function

$$\frac{te^{tx}}{e^t - 1} = \sum_{n=0}^{\infty} B_n(x) \frac{t^n}{n!}.$$

We use B_n to denote the n-th Bernoulli polynomial and the n-th Bernoulli number as it will always be clear from the context what is meant.

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It is well known that by Weyl's criterion if α is an irrational number and f is a Riemann-integrable function f, periodic of period 1, one has

$$\frac{1}{N}\sum_{n=1}^{N} f(n\alpha) \to \int\limits_0^1 f(x)dx.$$

Especially if $\int_0^1 f(x)dx = 0$ we get $\sum\limits_{n=1}^{N} f(n\alpha) = o(N)$. In this text we are concerned to provide a more precise bound in the case where f is the u-th Bernoulli polynomial. More exactly, our main result (Theorem 2) is:

Let $u > 1$ be an integer and $\alpha = [0; a_1, ...]$ be irrational with convergents $\frac{p_n}{q_n}$ and let $N := \sum\limits_{n=0}^{m} b_n q_n$ be the Ostrowski expansion of N to the base α. Then

$$\sum_{n=1}^{N} B_u(\{n\alpha\}) = \frac{1}{u+1}\sum_{k=0}^{m}(-1)^{ku}\left(B_{u+1}\left(\frac{b_k}{a_{k+1}}\right) - B_{u+1}\right) a_{k+1} q_k^{1-u} + O(1).$$

The O-constant depends only on u.

The asymptotic formula for $\sum_{n=1}^{N} B_2(\{n\alpha\})$ we present, was already announced, without proof, in [9]. Formulas for this sum and its asymptotic value, in a different form of the one presented here, can be found in [10], also without proof.

2. The case $u = 2$.

For the proof we start by giving an explicit formula for $\sum\limits_{n=1}^{N} B_2(\{n\alpha\})$ in terms of the b_i's of the Ostrowski expansion of N to base α.

Let h, k be integers, $(h, k) = 1$ and $k \geq 1$. Dedekind sums are defined by

$$s(h,k) = \sum_{j=1}^{k-1}\left(\left\{\frac{hj}{k}\right\} - \frac{1}{2}\right)\left(\left\{\frac{j}{k}\right\} - \frac{1}{2}\right). \qquad (1)$$

For $\alpha = [0; a_1, \dots] \in \Omega$ with convergents $\frac{p_n}{q_n}$ and $i, j \geq 0$, define $L_j := \sum\limits_{i=0}^{j-1}(-1)^i a_{i+1}$ and $s_{i,j} = q_{\min(i,j)}\left(q_{\max(i,j)}\alpha - p_{\max(i,j)}\right)$, following [8].

There is a close connection between L_j and the Dedekind sums $s(h, k)$:

Proposition 1. *For $m \geq 0$ and $\alpha \in \Omega$,*

$$12q_m s(p_m, q_m) = (-1)^{m-1} q_{m-1} + p_m + q_m L_m - \frac{3q_m}{2}(1-(-1)^m). \tag{2}$$

P r o o f. This relation was seemingly independently proved in [2], [3] and [4]. □

Lemma 1. *Let $\alpha \in \Omega$ and $N = \sum_{i=0}^{m} b_i q_i$ the Ostrowski expansion of $N \geq 1$. Then*

(i) $2\sum_{n=1}^{N} B_1(\{n\alpha\}) = \sum_{j=0}^{m}\sum_{i=0}^{m} s_{i,j} b_i b_j + \sum_{i=0}^{m} b_i (s_{0,i} - (-1)^i).$

(ii) $$6\sum_{n=0}^{q_m-1} B_2(\{n\alpha\}) = \left(\alpha + L_m + 2s_{m,m} - 3s_{0,m} - \frac{3}{2}(1-(-1)^m)\right) s_{0,m} \\ -(-1)^m (q_{m-1}\alpha - p_{m-1}).$$

P r o o f. (i) For a proof see [9].

(ii) Note that for $1 \leq n < q_m$, $\{n\alpha\} = \left\{n\frac{p_m}{q_m}\right\} + n\left(\alpha - \frac{p_m}{q_m}\right)$. We obtain

$$\begin{aligned}\sum_{n=0}^{q_m-1} B_2(\{n\alpha\}) = &\sum_{n=0}^{q_m-1}\left\{n\frac{p_m}{q_m}\right\}^2 + \frac{q_m}{6} - \sum_{n=0}^{q_m-1}\left\{n\frac{p_m}{q_m}\right\} \\ &+ 2\left(\alpha - \frac{p_m}{q_m}\right)\sum_{n=0}^{q_m-1} n\left\{n\frac{p_m}{q_m}\right\} \\ &+ \left(\alpha - \frac{p_m}{q_m}\right)^2 \sum_{n=0}^{q_m-1} n^2 - \left(\alpha - \frac{p_m}{q_m}\right)\sum_{n=0}^{q_m-1} n.\end{aligned}$$

Note that $(p_m, q_m) = 1$ and hence for $k = 1, 2$

$$\sum_{n=0}^{q_m-1}\left\{n\frac{p_m}{q_m}\right\}^k = \sum_{n=0}^{q_m-1}\left\{\frac{n}{q_m}\right\}^k = \sum_{n=0}^{q_m-1}\left(\frac{n}{q_m}\right)^k, \tag{3}$$

hence one can easily compute the first two sums.

For the third one we note that

$$s(p_m, q_m) = \frac{1}{q_m}\sum_{j=1}^{q_m-1} j\left\{j\frac{p_m}{q_m}\right\} - \frac{1}{4}(q_m - 1).$$

So, using Proposition (1) we obtain

$$\sum_{j=1}^{q_m-1} j\left\{j\frac{p_m}{q_m}\right\} = \frac{1}{12}\left((-1)^{m-1}q_{m-1} + p_m + q_m L_m - \frac{3q_m}{2}(3-(-1)^m - 2q_m)\right). \tag{4}$$

Summing up we obtain the formula above. □

We prove the formula for $\sum\limits_{n=1}^{N} B_2(\{n\alpha\})$ by a nested threefold proof by induction. As this proof does not give the slightest idea how the formula was found we give some hints how we have proceeded.

For short, it was done in the following manner: for every n with Ostrowski expansion $\sum\limits_{i=0}^{m} c_i(n)q_i$ we have the simple formula (see [9])

$$\{n\alpha\} = \sum_{i=0}^{m} c_i(n)s_{i,0} + \frac{1}{2}(1-(-1)^{i_n}), \tag{5}$$

where i_n is the first index j with $c_j(n) \neq 0$. In order to compute $\sum\limits_{n=1}^{N}(\{n\alpha\}^2 - \{n\alpha\}+1/6)$ it is enough - by taking into account Lemma 1(i) above - to compute $\sum\limits_{n=1}^{N}\{n\alpha\}^2$. Using relation (5) we obtain

$$\begin{aligned}\sum_{n=1}^{N}\{n\alpha\}^2 &= \sum_{i=0}^{m}\sum_{j=0}^{m} s_{i,0}s_{j,0}\sum_{n=1}^{N} c_i(n)c_j(n)\\ &\quad + \sum_{i=0}^{m} s_{i,0}\sum_{n=1}^{N}(1-(-1)^{i_n})c_i(n) + \frac{1}{4}\sum_{n=1}^{N}(1-(-1)^{i_n})^2.\end{aligned} \tag{6}$$

The most difficult part is (for $i \leq j$) the sum $\sum\limits_{n=1}^{N-1} c_i(n)c_j(n)$. Instead of summing over $n < N$, we sum c_ic_j over all $m+1$-tuples $(c_0, ..., c_m) \in \mathbb{Z}_+^{m+1}$ with the side conditions $c_0 < a_1,\ c_i \leq a_{i+1},\ c_i = a_{i+1} \implies c_{i-1} = 0$ and $\sum\limits_{i=0}^{m} c_iq_i < \sum\limits_{i=0}^{m} b_iq_i$. This last side condition is equivalent to the existence of a $t,\ 0 \leq t \leq m$ such that $c_j = b_j$ for $j > t$ and $c_t < b_t$. Hence if

$$\begin{aligned}V_t := \{(c_0, ..., c_t) \in \mathbb{Z}_+^{t+1} | c_0 < a_1, c_i \leq a_{i+1}, c_i = a_{i+1} &\implies c_{i-1} = 0,\\ &\text{for } i < t, c_t < b_t\}\end{aligned}$$

one has to compute for $t \geq j$, $\sum_{c \in V_t} c_i c_j$, for $i \leq t < j$, $b_j \sum_{c \in V_t} c_i$ and for $t < i$, $b_i b_j \sum_{c \in V_t} 1$. Finally one has to sum up over t, $0 \leq t \leq m$. This results, after rather tedious calculations, into the following formula which we can prove now by induction.

Theorem 1. *Let $\alpha \in \Omega$ and $N = \sum_{i=0}^{m} b_i q_i$ the Ostrowski expansion of $N \geq 1$. Then*

$$\begin{aligned}\sum_{n=1}^{N} B_2(\{n\alpha\}) &= \frac{1}{3}\sum_{k=0}^{m} s_{k,k} s_{0,k} b_k^3 + \sum_{t=0}^{m}\sum_{r=0}^{t-1} s_{r,t} s_{0,t} b_r b_t^2 \\ &+ \sum_{t=0}^{m}\sum_{r=0}^{t-1} s_{r,t} s_{0,r} b_r^2 b_t + 2\sum_{t=0}^{m}\sum_{k=0}^{t-1}\sum_{r=0}^{k-1} s_{k,r} s_{0,t} b_k b_r b_t \\ &+ \frac{1}{2}\sum_{k=0}^{m} s_{0,k}(s_{0,k} - (-1)^k) b_k^2 + \sum_{t=0}^{m}\sum_{r=0}^{t-1} s_{0,t}(s_{0,r} - (-1)^r) b_r b_t \\ &+ \frac{1}{6}\sum_{k=0}^{m}\left(\left(\alpha + L_k - \frac{3}{2}(1 + (-1)^k)\right) s_{0,k} - (-1)^k (q_{k-1}\alpha - p_{k-1})\right) b_k.\end{aligned}$$

P r o o f. Let $N_i := \sum_{j=0}^{i} b_j q_j$, for $0 \leq i \leq m$. Now observe that

$$\sum_{n=1}^{N} B_2(\{n\alpha\}) = \sum_{i=0}^{m} \sum_{n=N_{i-1}+1}^{N_i} B_2(\{n\alpha\}) = \sum_{i=0}^{m}\sum_{n=1}^{b_i q_i} B_2\left(\{n\alpha + N_{i-1}\alpha\}\right).$$

We proceed by induction on m. If $m = 0$ and $n \leq N$ then n has only one digit c_0, and $i_n = 0$. Then, using (5) and (6)

$$\begin{aligned}&\sum_{i=0}^{m}\sum_{n=1}^{b_i q_i} B_2\left(\{n\alpha + N_{i-1}\alpha\}\right) \\ &\qquad = \sum_{n=1}^{b_0} B_2(\{n\alpha\}) = s_{0,0} s_{0,0} \sum_{n=1}^{b_0} c_0^2 - s_{0,0}\sum_{n=1}^{b_0} c_0 + \sum_{n=1}^{b_0}\frac{1}{6} \\ &\qquad = \frac{1}{3} s_{0,0} s_{0,0} b_0^3 + \frac{1}{2} s_{0,0}(s_{0,0} - 1) b_0^2 + \frac{1}{6}((s_{0,0} - 3)s_{0,0} + 1) b_0.\end{aligned}$$

The same result is obtained using the formula in the theorem, noting that $p_0 = 0$, $q_0 = 1$, $p_{-1} = 1$, and $q_{-1} = 0$. The induction step is equivalent to

$$\sum_{n=1}^{b_m q_m} B_2\left(\{n\alpha + N_{m-1}\alpha\}\right) = \frac{1}{3} s_{m,m} s_{0,m} b_m^3 + s_{0,m} b_m^2 \sum_{r=0}^{m-1} s_{r,m} b_r$$
$$+ b_m \sum_{r=0}^{m-1} s_{r,m} s_{0,r} b_r^2 + 2 s_{0,m} b_m \sum_{k=0}^{m-1} \sum_{r=0}^{k-1} s_{k,r} b_r b_k$$
$$+ \frac{1}{2} s_{0,m}(s_{0,m} - (-1)^m) b_m^2 + s_{0,m} b_m \sum_{r=0}^{m-1} (s_{0,r} - (-1)^r) b_r$$
$$+ \frac{1}{6}\left((\alpha + L_m - \frac{3}{2}(1 + (-1)^m)) s_{0,m} - (-1)^m (q_{m-1}\alpha - p_{m-1})\right) b_m.$$

As the left hand side is equal to

$$\sum_{t=0}^{b_m - 1} \sum_{n=tq_m+1}^{(t+1)q_m} B_2\left(\{n\alpha + N_{m-1}\alpha\}\right) = \sum_{t=0}^{b_m-1} \sum_{n=1}^{q_m} B_2\left(\{n\alpha + tq_m\alpha + N_{m-1}\alpha\}\right),$$

we use again induction to prove this relation, this time on b_m. The case $b_m = 0$ is trivial. Noting that $x^3 - (x-1)^3 = 3x^2 - 3x + 1$, $x^2 - (x-1)^2 = 2x - 1$, and using Lemma 1, the induction step is equivalent to prove for $N < q_{m+1}$ that

$$\sum_{n=1}^{q_m} B_2(\{(n - q_m)\alpha + N\alpha\}) = s_{m,m} s_{0,m}\left(b_m^2 - b_m + \frac{1}{3}\right)$$
$$+ s_{0,m}(2b_m - 1) \sum_{r=0}^{m-1} s_{r,m} b_r + \sum_{r=0}^{m-1} s_{r,m} s_{0,r} b_r^2 + 2 s_{0,m} \sum_{k=0}^{m-1} \sum_{r=0}^{k-1} s_{k,r} b_r b_k$$
$$+ s_{0,m}(s_{0,m} - (-1)^m)\left(b_m - \frac{1}{2}\right) + s_{0,m} \sum_{r=0}^{m-1} (s_{0,r} - (-1)^r) b_r$$
$$+ \frac{1}{6}\left(\left(\alpha + L_m - \frac{3}{2}(1 + (-1)^m)\right) s_{0,m} - (-1)^m (q_{m-1}\alpha - p_{m-1})\right) =$$

$$=\sum_{r=0}^{m} s_{r,m}s_{0,r}b_r^2+2s_{0,m}\sum_{k=0}^{m}\sum_{r=0}^{k-1}s_{k,r}b_rb_k+s_{0,m}\sum_{r=0}^{m}(s_{0,r}-(-1)^r-s_{r,m})b_r$$
$$+\frac{1}{6}\Big(\Big(\alpha+L_m+2s_{m,m}-3s_{0,m}-\frac{3}{2}(1-(-1)^m)\Big)s_{0,m}-(-1)^m(q_{m-1}\alpha-p_{m-1})\Big)$$
$$=2s_{0,m}\sum_{n=1}^{N}B_1(\{n\alpha\})-s_{0,m}^2N+\sum_{n=0}^{q_m-1}B_2(\{n\alpha\}).$$

Observe that the left hand side is equal to $\sum\limits_{n=N+1}^{N+q_m}B_2(\{(n-q_m)\alpha\})$. We prove this formula again by induction, this time on N, for $N<q_{m+1}$. The case $N=0$ is trivial. The induction step is equivalent to

$$B_2(\{N\alpha\})-B_2(\{(N-q_m)\alpha\})=2s_{0,m}B_1(\{N\alpha\})-s_{0,m}^2.$$

Now, for $N<q_{m+1}$, the law of best approximation for continued fraction expansions gives $c_{[0,\{q_m\alpha\})}(\{N\alpha\})=\frac{1}{2}(1-(-1)^m)$, where c_M is the characteristic function of the set M. If m is even we have $\{q_m\alpha\}=q_m\alpha-p_m$, $N\alpha=N\frac{p_m}{q_m}+N(\alpha-\frac{p_m}{q_m})$ and $0<N(\alpha-\frac{p_m}{q_m})<\frac{1}{q_m}$; If $q_m\nmid N$, $\{N\alpha\}\geq\frac{1}{q_m}>q_m\alpha-p_m$; if $q_m|N$, we have $N=b_mq_m$ and $\{N\alpha\}=b_m(q_m\alpha-p_m)\geq q_m\alpha-p_m$ again. Now $\{(N-q_m)\alpha\}=\{N\alpha\}-\{q_m\alpha\}+c_{[0,\{q_m\alpha\})}(\{N\alpha\})=\{N\alpha\}-s_{0,m}$. This implies

$$\begin{aligned}B_2(\{N\alpha\})&-B_2(\{(N-q_m)\alpha\})\\&=\{N\alpha\}^2-\{N\alpha\}-(\{N\alpha\}-s_{0,m})^2+\{N\alpha\}-s_{0,m}\\&=2s_{0,m}\{N\alpha\}-s_{0,m}^2-s_{0,m}=2s_{0,m}B_1(\{N\alpha\})-s_{0,m}^2.\end{aligned}$$

The proof is similar if m is odd. □

Example. Let $\alpha=\pi-3=[0;7,15,1,292,1,1,1,2,1,3,\dots]$ and $N=10^6$. Then $m=9$ and $b_0=5$, $b_1=6$, $b_2=0$, $b_3=42$, $b_4=b_5=b_6=b_7=0$, $b_8=1$, $b_9=2$. So,

$$\begin{aligned}\sum_{n=1}^{10^6}B_2(\{n\pi\})=&\frac{9869619205613893094}{3}-2094398243998885744\pi\\&+333333833333500000\pi^2=-0.1377605692....\end{aligned}$$

To calculate this sum, software Mathematica took 130.469 seconds using the definition and 0.031 seconds using the formula in previous theorem.

We are now in the position to prove an asymptotic expansion for $\sum\limits_{n=1}^{N}B_2(\{n\alpha\})$. We start with an auxiliary result.

Lemma 2. *Let $\alpha \in \Omega$ and $r \geq 0$ an integer. Then*

(i) $\sum\limits_{r=0}^{\infty} \frac{r}{q_r} = O(1)$;

(ii) $\left| \sum\limits_{t=r+1}^{m} b_t(q_t\alpha - p_t) \right| \leq |q_r\alpha - p_r|$.

P r o o f. (i) This follows from the well known fact that, on denoting the r-th Fibonacci number by F_r, $q_r \geq F_r$.
(ii) We have

$$\left| \sum_{t>r}^{m} b_t(q_t\alpha - p_t) \right| \leq \max \left\{ \left| \sum_{t>r,\ 2|t}^{m} b_t(q_t\alpha - p_t) \right|, \left| \sum_{t>r,\ 2\nmid t}^{m} b_t(q_t\alpha - p_t) \right| \right\}.$$

For even r,

$$\begin{aligned} \sum_{t>r,\ 2|t} b_t(q_t\alpha - p_t) &\leq \sum_{t>r,\ 2|t} a_{t+1}(q_t\alpha - p_t) \\ &\leq \sum_{t>r,\ 2|t} ((q_{t+1}\alpha - p_{t+1}) - (q_{t-1}\alpha - p_{t-1})) = q_{r+1}\alpha - p_{r+1}. \end{aligned}$$

Analogously, for odd r

$$\left| \sum_{t>r,\ 2|t} b_t(q_t\alpha - p_t) \right| \leq |q_r\alpha - p_r|.$$

The case $\left| \sum\limits_{t>r, 2\nmid t} b_t(q_t\alpha - p_t) \right| \leq |q_r\alpha - p_r|$ is proved similarly. □

Corollary 1 ((See also [8])). *Let $\alpha = [0; a_1, ...]$ be irrational with convergents $\frac{p_n}{q_n}$ and let $N = \sum\limits_{n=1}^{m} b_n q_n$ be the Ostrowski-expansion of N to base α. Then,*

$$\sum_{n=1}^{N} B_2(\{n\alpha\}) = \frac{1}{3} \sum_{k=0}^{m} B_3\left(\frac{b_k}{a_{k+1}}\right) \frac{a_{k+1}}{q_k} + O(1),$$

where the O-constant neither depends on α nor on N.

Proof. By Theorem 1, $\sum_{n=1}^{N} B_2(\{n\alpha\})$ is the sum of 7 polynomials in $b_0,\dots b_m$, say $\sum_{u=1}^{7} S_u$. We have

$$S_1 = \frac{1}{3}\sum_{k=0}^{m}\left(\frac{(-1)^k}{a_{k+1}q_k}+O\left(\frac{1}{a_{k+1}^2 q_k}\right)\right)^2 q_k b_k^3 = \frac{1}{3}\sum_{k=0}^{m}\frac{q_k b_k^3}{a_{k+1}^2 q_k^2}+O\left(\sum_{k=0}^{m}\frac{q_k b_k^3}{a_{k+1}^3 q_k^2}\right)$$
$$= \frac{1}{3}\sum_{k=0}^{m}\left(\frac{b_k}{a_{k+1}}\right)^3\frac{a_{k+1}}{q_k}+O(1),$$
$$S_5 = O\left(\sum_{k=0}^{m}|q_k\alpha-p_k|^2 b_k^2\right)-\frac{1}{2}\sum_{k=0}^{m}(-1)^k\frac{(-1)^k b_k^2}{a_{k+1}q_k}+O\left(\sum_{k=0}^{m}\frac{b_k^2}{a_{k+1}^2 q_k}\right)$$
$$= -\frac{1}{2}\sum_{k=0}^{m}\left(\frac{b_k}{a_{k+1}}\right)^2 a_{k+1}\frac{1}{q_k}+O\left(1\right),$$
$$S_7 = O\left(\sum_{i=0}^{m-1}a_{i+1}\left|\sum_{k=i+1}^{m}b_k(q_k\alpha-p_k)\right|\right)+\frac{1}{6}\sum_{k=0}^{m}\frac{b_k}{q_k}+O\left(\left|\sum_{k=0}^{m}(q_k\alpha-p_k)b_k\right|\right)$$
$$= \frac{1}{6}\sum_{k=0}^{m}\left(\frac{b_k}{a_{k+1}}\right)\frac{a_{k+1}}{q_k}+O(1).$$

Also, $S_2 = S_4 = O(1)$, by Lemma 2 (i); interchanging the order of summation and applying Lemma 2 (ii) and (i) we obtain $S_3 = O(1)$. Finally,

$$S_6 = O\left(\sum_{t=0}^{m}|q_t\alpha-p_t|b_t\sum_{r=0}^{t-1}|q_r\alpha-p_r|b_r+\left|\sum_{t=0}^{m}(q_t\alpha-p_t)b_t\sum_{r=0}^{t-1}(-1)^r b_r\right|\right)$$
$$= O\left(\sum_{t=0}^{m}|q_t\alpha-p_t|b_t\right)+O\left(\sum_{r=0}^{m-1}b_r\left|\sum_{t=r+1}^{m}(q_t\alpha-p_t)b_t\right|\right)$$
$$= O(1)+O\left(\sum_{r=0}^{m-1}b_r|q_r\alpha-p_r|\right)=O(1).$$

□

Corollary 2 (See also [8]). *Let α be an irrational number in the unit interval and $m \geq 0$. Then:*

(i) $\max\limits_{1\leq N<q_{m+1}} \sum\limits_{n\leq N} B_2(\{n\alpha\}) = \frac{1}{36\sqrt{3}} \sum\limits_{t=0}^{m} \frac{a_{t+1}}{q_t} + O(1)$,

(ii) $\min\limits_{1\leq N<q_{m+1}} \sum\limits_{n\leq N} B_2(\{n\alpha\}) = -\frac{1}{36\sqrt{3}} \sum\limits_{t=0}^{m} \frac{a_{t+1}}{q_t} + O(1)$.

The O-constant does not depend on α.

P r o o f. It is an immediate consequence of Corollary 1 and the fact that the function $f(x) = x^3 - \frac{3}{2}x^2 + \frac{1}{2}x$ has the maximum and minimum values $\frac{1}{12\sqrt{3}}$ and $-\frac{1}{12\sqrt{3}}$, respectively, in the interval $[0,1[$. □

3. The general case

Lemma 3. *For an integer $q > 0$ and $x, y \in \mathbb{R}$,*

$$\sum_{k=0}^{q-1} B_n\left(\frac{k}{q}\right) = B_n q^{1-n}; \tag{7}$$

$$B_n(x) - B_n(y) = \sum_{j=1}^{n} \binom{n}{j} B_{n-j}(y)\,(x-y)^j. \tag{8}$$

The *proofs* of these elementary relations are omitted and can be found in textbooks.

If n, h, k are integers, $n \geq 0$, $k > 0$ and h, k coprime we define the higher Dedekind sums as

$$s_n(h,k) = \sum_{m=0}^{k-1} \frac{m}{k} B_n\left(\left\{\frac{hm}{k}\right\}\right). \tag{9}$$

In particular one has $s_1(h,k) = s(h,k)$.

Proposition 2. *For n, h, k integers with $n \geq 0$, $k > 0$ and h, k coprime,*

$$\begin{cases} (n+1)\left(hk^n s_n(h,k) + kh^n s_n(k,h)\right) = \\ \qquad = \sum\limits_{t=0}^{n+1} \binom{n+1}{t} (-1)^t B_t B_{n+1-t} h^t k^{n+1-t} + nB_{n+1}, & \text{if } n \text{ is odd.} \\ s_n(h,k) = -\frac{B_n}{2}\left(1-k^{1-n}\right), & \text{if } n \text{ is even.} \end{cases} \tag{10}$$

P r o o f. These formulas were first proved by T. M. Apostol in [1]. $\square$

We need some auxiliary and technical results for the proof of the main theorem.

Lemma 4. *For $n \geq 2$ we have*

$$\sum_{k=0}^{q_m-1} B_n\left(\{k\alpha\}\right) = B_n q_m^{1-n} + n s_{0,m} s_{n-1}\left(p_m, q_m\right) + O\left(\frac{1}{q_{m+1}}\right).$$

P r o o f. For $0 \leq k < q_m$ we have $\left|\left(\alpha - \frac{p_m}{q_m}\right)k\right| < \frac{1}{q_m}$ and hence

$$\{\alpha k\} - \left\{\frac{p_m}{q_m}k\right\} = \left\{\left(\alpha - \frac{p_m}{q_m}\right)k + \frac{p_m}{q_m}k\right\} - \left\{\frac{p_m}{q_m}k\right\} = \left(\alpha - \frac{p_m}{q_m}\right)k.$$

Relation (7) implies

$$\begin{aligned}
&B_n\left(\{k\alpha\}\right) - B_n\left(\left\{\frac{p_m}{q_m}k\right\}\right) \\
&= \sum_{j=1}^{n}\binom{n}{j}B_{n-j}\left(\left\{\frac{p_m}{q_m}k\right\}\right)\left(\{\alpha k\} - \left\{\frac{p_m}{q_m}k\right\}\right)^j \\
&= nk\left(\alpha - \frac{p_m}{q_m}\right)B_{n-1}\left(\left\{\frac{p_m}{q_m}k\right\}\right) + \sum_{j=2}^{n}\binom{n}{j}B_{n-j}\left(\left\{\frac{p_m}{q_m}k\right\}\right)k^j\left(\alpha - \frac{p_m}{q_m}\right)^j \\
&= nk\left(\alpha - \frac{p_m}{q_m}\right)B_{n-1}\left(\left\{\frac{p_m}{q_m}k\right\}\right) + O\left(\sum_{j=2}^{n}\frac{k^j}{\left(q_m q_{m+1}\right)^j}\right) \\
&= nk\left(\alpha - \frac{p_m}{q_m}\right)B_{n-1}\left(\left\{\frac{p_m}{q_m}k\right\}\right) + O\left(\sum_{j=2}^{n}\frac{1}{q_{m+1}^j}\right) \\
&= nk\left(\alpha - \frac{p_m}{q_m}\right)B_{n-1}\left(\left\{\frac{p_m}{q_m}k\right\}\right) + O\left(\frac{1}{q_{m+1}^2}\right).
\end{aligned}$$

Hence by the definition of $s_n(h,k)$ and relation (7) we get

$$\begin{aligned}
&\sum_{k=0}^{q_m-1} B_n(\{k\alpha\}) \\
&= \sum_{k=0}^{q_m-1} B_n\left(\left\{\frac{p_m}{q_m}k\right\}\right) + n\left(q_m\alpha - p_m\right)\sum_{k=0}^{q_m-1}\frac{k}{q_m}B_{n-1}\left(\left\{\frac{p_m}{q_m}k\right\}\right) + O\left(\frac{1}{q_{m+1}}\right) \\
&= B_n q_m^{1-n} + n\left(q_m\alpha - p_m\right)s_{n-1}\left(p_m, q_m\right) + O\left(\frac{1}{q_{m+1}}\right).
\end{aligned}$$

□

The following result is a generalization of Proposition 1.

Proposition 3. *Consider an odd integer $n \geq 1$ and define*

$$u_{m,i} = (-1)^i\left(p_m q_i - p_i q_m\right),$$

for $i \geq -1$. Then

$$s_n\left(p_m, q_m\right) = \frac{1}{n+1}\sum_{t=0}^{n+1}\binom{n+1}{t}(-1)^t B_t B_{n+1-t} q_m^{1-n}\sum_{i=0}^{m-1}(-1)^i u_{m,i}^{t-1}u_{m,i-1}^{n-t} + (-1)^{m+1}\frac{nB_{n+1}q_{m-1}}{(n+1)\,q_m^n}.$$

P r o o f. Evidently for $h \equiv h' \pmod{k}$ we have $s_n(h,k) = s_n(h',k)$. We put

$$F_n(h,k) = \frac{1}{n+1}\sum_{t=0}^{n+1}\binom{n+1}{t}(-1)^t B_t B_{n+1-t}\left(\frac{h}{k}\right)^{t-1} + \frac{nB_{n+1}}{(n+1)\,hk^n}.$$

Relation (10) implies $s_n(h,k) = -\left(\frac{h}{k}\right)^{n-1}s_n(k,h) + F_n(h,k)$. In particular,

$$s_n\left(u_{m,i+1}, u_{m,i}\right) = -\left(\frac{u_{m,i+1}}{u_{m,i}}\right)^{n-1}s_n\left(u_{m,i}, u_{m,i+1}\right) + F_n\left(u_{m,i+1}, u_{m,i}\right).$$

From $u_{m,i} = u_{m,i+2} + a_{i+2}u_{m,i+1} \equiv u_{m,i+2} \pmod{u_{m,i+1}}$ we get

$$s_n\left(u_{m,i+1}, u_{m,i}\right) = -\left(\frac{u_{m,i+1}}{u_{m,i}}\right)^{n-1}s_n\left(u_{m,i+2}, u_{m,i+1}\right) + F_n\left(u_{m,i+1}, u_{m,i}\right).$$

With $t_i := (-1)^i \left(\frac{u_{m,i}}{q_m}\right)^{n-1} s_n\left(u_{m,i+1}, u_{m,i}\right)$ this results in

$$(-1)^i\, t_i \left(\frac{q_m}{u_{m,i}}\right)^{n-1} = -\,(-1)^{i+1}\left(\frac{q_m}{u_{m,i+1}}\right)^{n-1}\left(\frac{u_{m,i+1}}{u_{m,i}}\right)^{n-1} t_{i+1} + F_n\left(u_{m,i+1}, u_{m,i}\right),$$

that is $t_i = t_{i+1} + (-1)^i \left(\frac{u_{m,i}}{q_m}\right)^{n-1} F_n\left(u_{m,i+1}, u_{m,i}\right)$. As a corollary

$$t_{-1} - t_{m-1} = \sum_{i=-1}^{m-2} (-1)^i \left(\frac{u_{m,i}}{q_m}\right)^{n-1} F_n\left(u_{m,i+1}, u_{m,i}\right).$$

As $u_{m,-1} = q_m,\ u_{m,0} = p_m,\ u_{m,m} = 0,\ u_{m,m-1} = 1$ and $s_n(0,1) = 0$ we get $t_{m-1} = 0$, $t_{-1} = -s_n(p_m, q_m)$ and hence

$$s_n(p_m, q_m) = \sum_{i=0}^{m-1} (-1)^i \left(\frac{u_{m,i-1}}{q_m}\right)^{n-1} F_n\left(u_{m,i}, u_{m,i-1}\right).$$

The formula

$$\begin{aligned}
&(-1)^m\Big(\left(q_{m-1}p_{i+1} - p_{m-1}q_{i+1}\right)\left(p_m q_i - p_i q_m\right) \\
&- \left(q_{m-1}p_i - p_{m-1}q_i\right)\left(p_m q_{i+1} - p_{i+1}q_m\right)\Big) \\
&= (-1)^m\big(q_{m-1}p_{i+1}p_m q_i + p_{m-1}q_m p_i q_{i+1} - p_m q_{m-1}q_{i+1}p_i - p_{m-1}p_{i+1}q_m q_i\big) \\
&= (-1)^m\left(q_{m-1}p_m\left(p_{i+1}q_i - p_i q_{i+1}\right) - p_{m-1}q_m\left(p_{i+1}q_i - p_i q_{i+1}\right)\right) \\
&= (-1)^{m+i}\left(p_m q_{m-1} - p_{m-1}q_m\right) = (-1)^{i+1}
\end{aligned}$$

implies

$$\frac{(-1)^m\left(q_{m-1}p_{i+1} - p_{m-1}q_{i+1}\right)}{p_m q_{i+1} - p_{i+1}q_m} - \frac{(-1)^m\left(q_{m-1}p_i - p_{m-1}q_i\right)}{p_m q_i - p_i q_m} = \frac{(-1)^i}{u_{m,i}u_{m,i+1}}.$$

Hence

$$\begin{aligned}
&\sum_{i=0}^{m-1}(-1)^i\left(\frac{u_{m,i-1}}{q_m}\right)^{n-1}\frac{n}{n+1}B_{n+1}\frac{1}{u_{m,i}u_{m,i-1}^n}\\
=&\frac{n}{n+1}B_{n+1}q_m^{1-n}\sum_{i=0}^{m-1}\frac{(-1)^i}{u_{m,i}u_{m,i-1}}\\
=&-\frac{n}{n+1}B_{n+1}q_m^{1-n}(-1)^m\sum_{i=0}^{m-1}\left(\frac{q_{m-1}p_i-p_{m-1}q_i}{p_mq_i-p_iq_m}-\frac{q_{m-1}p_{i-1}-p_{m-1}q_{i-1}}{p_mq_{i-1}-p_{i-1}q_m}\right)\\
=&-\frac{n}{n+1}B_{n+1}q_m^{1-n}(-1)^m\frac{q_{m-1}}{q_m}=\frac{n}{n+1}B_{n+1}(-1)^{m+1}\frac{q_{m-1}}{q_m^n}.
\end{aligned}$$

From this we get the formula

$$\begin{aligned}
s_n(p_m,q_m)=&\sum_{i=0}^{m-1}(-1)^i\left(\frac{u_{m,i-1}}{q_m}\right)^{n-1}\sum_{t=0}^{n+1}\binom{n+1}{t}\frac{(-1)^t}{n+1}B_tB_{n+1-t}\left(\frac{u_{m,i}}{u_{m,i-1}}\right)^{t-1}\\
&+(-1)^{m+1}\frac{nB_{n+1}}{n+1}\frac{q_{m-1}}{q_m^n},
\end{aligned}$$

which is the assertion initially made. □

Lemma 5. *Let* $N=\sum_{i=0}^{m}b_iq_i$ *be the Ostrowski-expansion of* N *to base* α. *Then for* $n>1$ *we get*

$$\sum_{i=0}^{m}b_i\sum_{k=0}^{q_i-1}B_n(\{k\alpha\})=B_n\sum_{i=0}^{m}b_iq_i^{1-n}+O(1).$$

P r o o f. By Lemma 4 we have

$$\begin{aligned}
\sum_{i=0}^{m}b_i\sum_{k=0}^{q_i-1}B_n(\{k\alpha\})=&B_n\sum_{i=0}^{m}b_iq_i^{1-n}+n\sum_{i=0}^{m}b_i(q_i\alpha-p_i)s_{n-1}(p_i,q_i)\\
&+O\left(\sum_{i=0}^{m}\frac{b_i}{q_{i+1}}\right).
\end{aligned}$$

The last term is $O(1)$. We note that for $j < i$, $\frac{q_i}{2q_{j+1}} \leq |p_i q_j - p_j q_i| \leq \frac{q_i}{q_{j+1}}$. By formula (10) for n odd the second summand is therefore equal to

$$\begin{aligned}
&\sum_{i=0}^{m} (q_i\alpha - p_i)\, b_i\, (-1)^{i+1} \frac{(n-1)\, B_n q_{i-1}}{q_i^{n-1}} \\
&+ \sum_{i=0}^{m} (q_i\alpha - p_i)\, b_i \sum_{t=0}^{n} \binom{n}{t} (-1)^t\, B_t B_{n-t} q_i^{2-n} \times \\
&\times \sum_{j=0}^{i-1} (-1)^j\, |p_i q_j - p_j q_i|^{t-1} |p_i q_{j-1} - p_{j-1} q_i|^{n-1-t} = \\
&= O\left(\sum_{i=0}^{m} \frac{b_i}{q_{i+1}}\right) + \sum_{t=0}^{n} \binom{n}{t} (-1)^t\, B_t B_{n-t} \sum_{j=0}^{m-1} (-1)^j \times \\
&\times \sum_{i=j+1}^{m} b_i\, (q_i\alpha - p_i)\, q_i^{2-n} |p_i q_j - p_j q_i|^{t-1} |p_i q_{j-1} - p_{j-1} q_i|^{n-1-t} \\
&= O\left(1 + \sum_{t=0}^{n} \sum_{j=0}^{m-1} \sum_{i=j+1}^{m} \frac{b_i}{q_{i+1}} q_i^{2-n} \left(\frac{q_i}{q_{j+1}}\right)^{t-1} \left(\frac{q_i}{q_j}\right)^{n-1-t}\right) \\
&= O\left(1 + \sum_{t=0}^{n} \sum_{j=0}^{m-1} \sum_{i=j+1}^{m} \frac{b_i}{q_{i+1}} q_i^{2-n} q_i^{t-1+n-1-t} q_j^{t+1-n} q_{j+1}^{1-t}\right) \\
&= O\left(1 + \sum_{t=0}^{n} \sum_{j=0}^{m-1} q_j^{t+1-n} q_{j+1}^{1-t} \sum_{i=j+1}^{m} q_i^{-1}\right) \\
&= O\left(1 + \sum_{t=0}^{n} \sum_{j=0}^{m-1} q_j^{t+1-n} q_{j+1}^{1-t} q_{j+1}^{-1}\right) = O\left(1 + \sum_{t=0}^{n} \sum_{j=0}^{m-1} q_j^{1-n}\right) = O(1).
\end{aligned}$$

This is the assertion if n is odd. If n is even, we get likewise

$$\begin{aligned}
n \sum_{i=0}^{m} b_i\, (q_i\alpha - p_i)\, s_{n-1}(p_i, q_i) &= -n \frac{B_{n-1}}{2} \sum_{i=0}^{m} b_i\, (q_i\alpha - p_i) \left(1 - q_i^{2-n}\right) \\
&= O\left(\sum_{i=0}^{m} \frac{b_i}{q_{i+1}}\right) = O(1).
\end{aligned}$$

□

Theorem 2. *Let $u \geq 2,\ \alpha = [0; a_1, ...]$ be irrational with convergents $\frac{p_n}{q_n}$ and let $N = \sum_{n=1}^{m} b_n q_n$ be the Ostrowski-expansion of N to base α. Then*

$$\sum_{n=1}^{N} B_u(\{n\alpha\}) = \frac{1}{u+1} \sum_{k=0}^{m} (-1)^{ku} \left(B_{u+1}\left(\frac{b_k}{a_{k+1}}\right) - B_{u+1} \right) a_{k+1} q_k^{1-u} + O(1).$$

The O-constant depends on u only.

P r o o f. We prove the assertion by induction on u. The case $u = 2$ is Corollary 1 (note that $B_{2n+1} = 0$, for $n \geq 1$).

First of all, we prove an auxiliary formula: if $1 \leq N < q_{i+1} - q_i$, then

$$\sum_{n=1}^{q_i} B_u(\{(n+N)\alpha\}) = \sum_{n=1}^{q_i} B_u(\{n\alpha\}) + (q_i\alpha - p_i) \sum_{j=0}^{i} (-1)^{(u-1)j} \left(B_u\left(\frac{b_j}{a_{j+1}}\right) - B_u \right) a_{j+1} q_j^{2-u} + O\left(q_{i+1}^{-1}\right). \tag{11}$$

Let $S_{N,i} := \sum_{n=1}^{q_i} B_u(\{(n+N)\alpha\})$. Then

$$\begin{aligned} S_{N,i} - S_{N-1,i} &= \sum_{n=N+1}^{N+q_i} B_u(\{n\alpha\}) - \sum_{n=N}^{N+q_i-1} B_u(\{n\alpha\}) \\ &= B_u(\{(N+q_i)\alpha\}) - B_u(\{N\alpha\}) \end{aligned}$$

and hence

$$S_{N,i} = \sum_{n=1}^{N} \left(B_u(\{(n+q_i)\alpha\}) - B_u(\{n\alpha\}) \right) + \sum_{n=1}^{q_i} B_u(\{n\alpha\}).$$

As for $n \leq N$, $n + q_i < q_{i+1}$, we get $\{(n+q_i)\alpha\} = \{n\alpha\} + q_i\alpha - p_i$.

Then, formula (8) and the induction hypothesis give

$$\begin{aligned}
&\sum_{n=1}^{q_i}\left(B_u\left(\{(n+N)\,\alpha\}\right)-B_u\left(\{n\alpha\}\right)\right)=\sum_{n=1}^{N}\left(B_u\left(\{(n+q_i)\,\alpha\}\right)-B_u\left(\{n\alpha\}\right)\right)\\
&=\sum_{n=1}^{N}\left(B_u\left(\{n\alpha\}+(q_i\alpha-p_i)\right)-B_u\left(\{n\alpha\}\right)\right)\\
&=\sum_{j=1}^{u}\binom{u}{j}\sum_{n=1}^{N}B_{u-j}\left(\{n\alpha\}\right)(q_i\alpha-p_i)^j\\
&=u\sum_{n=1}^{N}B_{u-1}\left(\{n\alpha\}\right)(q_i\alpha-p_i)+O\left(\sum_{j=2}^{u}\frac{N}{q_{i+1}^j}\right)\\
&=(q_i\alpha-p_i)\left(\sum_{k=0}^{i}(-1)^{(u-1)i}\left(B_u\left(\frac{b_k}{a_{k+1}}\right)-B_u\right)a_{k+1}q_k^{2-u}+O\left(1\right)\right)\\
&\quad+O\left(\sum_{j=2}^{u}q_{i+1}^{1-j}\right)\\
&=(q_i\alpha-p_i)\sum_{k=0}^{i}(-1)^{(u-1)i}\left(B_u\left(\frac{b_k}{a_{k+1}}\right)-B_u\right)a_{k+1}q_k^{2-u}+O\left(\frac{1}{q_{i+1}}\right).
\end{aligned}$$

We have $N_k < q_{k+1}$ and

$$\begin{aligned}
\sum_{n=1}^{N}B_u\left(\{n\alpha\}\right)&=\sum_{i=0}^{m}\sum_{n=N_{i-1}+1}^{N_i}B_u\left(\{n\alpha\}\right)=\sum_{i=0}^{m}\sum_{j=1}^{b_iq_i}B_u\left(\{(j+N_{i-1})\,\alpha\}\right)\\
&=\sum_{i=0}^{m}\sum_{t=0}^{b_i-1}\sum_{j=tq_i+1}^{(t+1)q_i}B_u\left(\{(j+N_{i-1})\,\alpha\}\right)\\
&=\sum_{i=0}^{m}\sum_{t=0}^{b_i-1}\sum_{j=1}^{q_i}B_u\left(\{(j+tq_i+N_{i-1})\,\alpha\}\right).
\end{aligned}$$

Note that if $b_i < a_{i+1}$, $tq_i+N_{i-1} < (b_i-1)\,q_i+q_i = b_iq_i \le a_{i+1}q_i-q_i \le q_{i+1}-q_i$ and if $b_i = a_{i+1}$ then $b_{i-1}=0$, and hence again $tq_i+N_{i-1} < a_{i+1}q_i-q_i+q_{i-1} =$

$q_{i+1} - q_i$. Therefore we get, by (11):

$$\sum_{n=1}^{N} B_u\left(\{n\alpha\}\right) = \sum_{i=0}^{m} b_i \sum_{n=1}^{q_i} B_u\left(\{n\alpha\}\right)$$
$$+ \sum_{i=0}^{m} \left(q_i\alpha - p_i\right) b_i \sum_{j=0}^{i-1} (-1)^{(u-1)j} \left(B_u\left(\frac{b_j}{a_{j+1}}\right) - B_u \right) a_{j+1} q_j^{2-u}$$
$$+ \sum_{i=0}^{m} \left(q_i\alpha - p_i\right) (-1)^{(u-1)i} q_i^{2-u} a_{i+1} \sum_{t=0}^{b_i-1} \left(B_u\left(\frac{t}{a_{i+1}}\right) - B_u \right) + O\left(\sum_{i=0}^{m} \frac{b_i}{q_{i+1}} \right).$$

The second and fourth sum yield

$$\sum_{j=0}^{m-1} (-1)^{(u-1)j} \left(B_u\left(\frac{b_j}{a_{j+1}}\right) - B_u \right) a_{j+1} q_j^{2-u} \sum_{i=j+1}^{m} b_i \left(q_i\alpha - p_i\right)$$
$$= O\left(\sum_{j=0}^{m-1} a_{j+1} q_j^{2-u} |q_j\alpha - p_j| \right) = O\left(\sum_{j=0}^{m-1} \frac{a_{j+1}}{q_{j+1}} \right) = O\left(1\right).$$

Furthermore

$$\sum_{t=0}^{b_i-1} B_u\left(\frac{t}{a_{i+1}}\right) = \sum_{t=0}^{b_i} B_u\left(\frac{t}{a_{i+1}}\right) + O\left(1\right) = \int_0^{b_i} B_u\left(\frac{x}{a_{i+1}}\right) \mathrm{d}x + O\left(1\right)$$
$$= a_{i+1} \int_0^{\frac{b_i}{a_{i+1}}} B_u\left(x\right) \mathrm{d}x + O\left(1\right) = \frac{a_{i+1}}{u+1} \left(B_{u+1}\left(\frac{b_i}{a_{i+1}}\right) - B_{u+1} \right) + O\left(1\right).$$

Note that the O-constant depends only on u, as in equality

$$\frac{1}{a_{i+1}} \sum_{t=1}^{b_i} B_u\left(\frac{t}{a_{i+1}}\right) = \int_0^{\frac{b_i}{a_{i+1}}} B_u(x)\mathrm{d}x + O,$$

by Koksma theorem, one has $O \leq V.D$, where V is the variation of $B_u(x)$ and D the discrepancy of the sequence $\frac{1}{a_{i+1}}, \frac{2}{a_{i+1}}, \ldots, \frac{a_{i+1}}{a_{i+1}}$ which is $\frac{1}{a_{i+1}}$.

We will also use $a_{i+1}q_i|q_i\alpha - p_i| = 1 + O\left(\frac{1}{a_{i+1}}\right)$. Then the first and third sum result in - if we use Lemma 5 -

$$\begin{aligned}
&B_u \sum_{i=0}^{m} b_i q_i^{1-u} \\
&+ \frac{1}{u+1}\sum_{i=0}^{m} |q_i\alpha - p_i| \left(-1\right)^{ui} a_{i+1}^2 q_i^{2-u} \left(B_{u+1}\left(\frac{b_i}{a_{i+1}}\right) - B_{u+1}\right) + O\left(1\right) \\
&- B_u \sum_{i=0}^{m} |q_i\alpha - p_i| \left(-1\right)^{ui} q_i^{2-u} a_{i+1} b_i \\
&= B_u \sum_{i=0}^{m} b_i q_i^{1-u} \\
&+ \frac{1}{u+1}\sum_{i=0}^{m} \left(-1\right)^{ui} q_i^{1-u} a_{i+1}\left(1 + O\left(\frac{1}{a_{i+1}}\right)\right)\left(B_{u+1}\left(\frac{b_i}{a_{i+1}}\right) - B_{u+1}\right) \\
&- B_u \sum_{i=0}^{m} \left(-1\right)^{ui} q_i^{1-u} b_i \left(1 + O\left(\frac{1}{a_{i+1}}\right)\right) + O\left(1\right) \\
&= \frac{1}{u+1}\sum_{i=0}^{m} \left(-1\right)^{ui} q_i^{1-u} a_{i+1}\left(B_{u+1}\left(\frac{b_i}{a_{i+1}}\right) - B_{u+1}\right) + O\left(\sum_{i=0}^{\infty}\frac{1}{q_i}\right) \\
&+ B_u \sum_{i=0}^{m} b_i q_i^{1-u}\left(1 - \left(-1\right)^{ui}\right) + O\left(1\right),
\end{aligned}$$

and this is the assertion made, if we take into account that for u odd, $B_u = 0$. □

4. Some consequences

Theorem 3. *Let u be a positive integer and let*

$$K_u = \left\{\alpha \in \Omega : \sum_{n=1}^{N} B_u(\{n\alpha\}) = O(1)\right\}.$$

Then

(i) $K_u = \left\{\alpha \in \Omega : \sum\limits_{k=0}^{\infty} \frac{a_{k+1}}{q_k^{u-1}} \text{ is convergent } \right\}$;

(ii) $K_1 = \emptyset; K_u \subseteq K_{u+1}$;

(iii) $[0,1] \setminus K_2$ *is a set of measure 0.*

P r o o f. (i) Consider $\alpha \in \Omega$ such that $\sum\limits_{k=0}^{\infty} \frac{a_{k+1}}{q_k^{u-1}} < \infty$. Note that $B_{u+1} = B_{u+1}(0)$. Using Theorem 2 and formula (8), one has

$$\sum_{n=1}^{N} B_u(\{n\alpha\}) = \frac{1}{u+1}\sum_{k=0}^{m}(-1)^{ku}\left(B_{u+1}\left(\frac{b_k}{a_{k+1}}\right) - B_{u+1}\right)a_{k+1}q_k^{1-u} + O(1)$$
$$= O\left(\sum_{k=0}^{m} a_{k+1}q_k^{1-u}\right) + O(1) = O(1).$$

Assume now that $\sum\limits_{n=1}^{N} B_u(\{n\alpha\})$ is bounded. Let $x_0 \in (0,1)$ be chosen such that $B_{u+1}(x_0) \neq B_{u+1}$. Let $\epsilon \in \{0,1\}$, $b_k^{(\epsilon)} := \frac{1}{2}[x_0 a_{k+1}](1+(-1)^{ku+\epsilon})$ and $N_m^{(\epsilon)} := \sum_{k=0}^{m} b_k^{(\epsilon)} q_k$. Clearly $0 \le b_k < a_{k+1}$. We have

$$\frac{b_k}{a_{k+1}} = \frac{1}{2}(1+(-1)^{ku+\epsilon})x_0 + O\left(\frac{1}{a_{k+1}}\right)$$

and, as B_{u+1} is Lipschitz-continuous,

$$B_{u+1}\left(\frac{b_k^{(\epsilon)}}{a_{k+1}}\right) = B_{u+1}\left(\frac{1}{2}(1+(-1)^{ku+\epsilon})x_0\right) + O\left(\frac{1}{a_{k+1}}\right)$$
$$= \frac{1}{2}(1+(-1)^{ku+\epsilon})B_{u+1}(x_0) + \frac{1}{2}(1-(-1)^{ku+\epsilon})B_{u+1} + O\left(\frac{1}{a_{k+1}}\right).$$

This implies that

$$(B_{u+1}(x_0) - B_{u+1})\sum_{k=0}^{m}\frac{1}{2}((-1)^{ku} + (-1)^{\epsilon})a_{k+1}q_k^{1-u} =$$
$$= \sum_{k=0}^{m}(-1)^{ku}\frac{1}{2}(1+(-1)^{ku+\epsilon})(B_{u+1}(x_0) - B_{u+1})a_{k+1}q_k^{1-u}$$
$$= \sum_{k=0}^{m}(-1)^{ku}\left(B_{u+1}\left(\frac{b_k^{(\epsilon)}}{a_{k+1}}\right) - B_{u+1} + O\left(\frac{1}{a_{k+1}}a_{k+1}q_k^{1-u}\right)\right) = O(1),$$

for $\epsilon \in \{0,1\}$. Choosing $\epsilon = 0$, we get that $\sum_{2|k} \frac{a_{k+1}}{q_k^{u-1}}$ is convergent. If we choose $\epsilon \equiv u \pmod 2$ we get that $\sum_{2\nmid k} \frac{a_{k+1}}{q_k^{u-1}}$ is convergent. Hence $\sum_{k=0}^{\infty} \frac{a_{k+1}}{q_k^{u-1}} < \infty$.

(ii) The first assertion has been firstly proved by Ostrowski in [6]. The second assertion is an immediate consequence of (i).

(iii) We have

Theorem 4 (Borel-Cantelli Lemma)**.** *Let* Ψ *be a positive function such that* $\sum_q \frac{\Psi(q)}{q} < \infty$*. Then*

$$T_\Psi := \left\{\alpha \in \Omega : \left|\alpha - \frac{p}{q}\right| \leq \frac{\Psi(q)}{q^2} \text{ has infinitely many solutions } (p,q) \in \mathbb{Z} \times \mathbb{N}\right\}$$

has measure 0.

As a consequence we have that for almost all $\alpha \in \Omega$, $a_{k+1}(\alpha) \leq q_k(\alpha)^{1/4}$ for all except a finite number of positive integers k. In fact, if $a_{k+1}(\alpha) > q_k(\alpha)^{1/4}$ for infinitely many k, then

$$\left|\alpha - \frac{p_k}{q_k}\right| < \frac{1}{q_k q_{k+1}} \leq \frac{1}{q_k^2 a_{k+1}} < \frac{1}{q_k^2} \cdot \frac{1}{q_k^{1/4}}.$$

Consider an α such that $a_{k+1}(\alpha) \leq q_k(\alpha)^{1/4}$ for all except a finite number of positive integers k. Then, by Corollary 2,

$$\sum_{n=1}^{N} B_2(\{n\alpha\}) = O\left(\sum_{k=0}^{m} \frac{a_{k+1}}{q_k}\right) + O(1) = O\left(\sum_{k=0}^{m} \frac{q_k^{1/4}}{q_k}\right) + O(1) = O(1).$$

□

Remark. This text is part of my unpublished PhD thesis presented at Universidade de Trs-os-Montes e Alto Douro, Portugal.

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