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DISCREPANCY ESTIMATE OF NORMAL VECTORS

Mordechay B. Levin — Irina L. Volinsky

ABSTRACT. Let A be an $s \times s$ invertible matrix with integer entries and with eigenvalues $|\lambda_i| > 1, \quad i = 1, \dots, s$. In this paper we prove explicitly that there exists a vector α, such that the discrepancy of the sequence $\{\alpha A^n\}_{n=1}^N$ is equal to $O(N^{-1}(\log N)^{2s+3})$ for $N \longrightarrow \infty$. This estimate can be improved no more than on the logarithmic factor.

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1. Introduction

Let $(x_n)_{n\geq 0}$ be an infinite sequence of points in an $s-$dimensional unit cube $[0,1)^s$; $v = [0,\gamma_1) \times \dots \times [0,\gamma_s)$ a box in $[0,1)^s$; and $J_v(N)$ a number of indexes $n \in [1,N]$ such that x_n lies in v. The sequence $(x_n)_{n\geq 0}$ is said to be *uniformly distributed* in $[0,1)^s$ if for every box v, $J_v(N)/N \to \gamma_1 \dots \gamma_s$. The quantity

$$D((x_n)_{n=1}^N) = \sup_{v\in(0,1]^s} \left| \frac{1}{N} J_v(N) - \gamma_1 \dots \gamma_s \right| \tag{1}$$

is called the *discrepancy* of $(x_n)_{n=1}^N$.

In 1954 Roth (see [DrTi], [KN]) proved that for any sequence in $[0,1)^s$

$$\overline{\lim}_{N\to\infty} ND(N)/\log^{s/2} N > 0. \tag{2}$$

Let A be an $s \times s$ invertible matrix with integer entries. A matrix A is said to be *ergodic* if for almost all $\alpha \in \mathbb{R}^s$ the sequence $\{\alpha A^n\}_{n\geq 1}$ is uniformly distributed.

A vector $\alpha \in \mathbb{R}^s$ is said to be *normal (A normal)* if the sequence $\{\alpha A^n\}_{n\geq 1}$ is uniformly distributed.

Let λ_i $(1 \leq i \leq s)$ denote the eigenvalues of a matrix A. For the case of $|\lambda_i| > 1$, $i = 1, \dots, s$ normal vectors were constructed by Postnikov ($s = 2$) and by Polosuev ($s \geq 2$) (see [Po]). Normal vectors were constructed for the general

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case of an ergodic matrix in [Le1]. The author [Le1] obtained also the following discrepancy estimate

$$D\left(\{\alpha A^n\}_{n=1}^N\right) = O\left(N^{-\frac{1}{2}}(\log N)^{s+3}\right).$$

In [Ko1], Korobov posed the problem of finding a function $\psi(N)$ with maximum decay, such that there exists α with

$$D\left(\{\alpha A^n\}_{n=1}^N\right) = O\left(\psi(N)\right), \quad \text{for } N \longrightarrow \infty.$$

The author [Le2] proved that $\psi(N) = N^{-1}(\log N)^{2s+3}$ for the case of a diagonal ergodic matrix. In this paper we extend this result to the general case of an integer matrix with $|\lambda_i| > 1, \quad i = 1, \dots, s$. By (2) this result can be improved no more than on the logarithmic factor.

2. Construction and Auxiliary results

Let $s \geq 2$, $p \geq 3$ be a prime number, A an $s \times s$ invertible matrix with integer entries, $\lambda_1, ..., \lambda_s$ eigenvalues of the matrix A, where $|\lambda_i| > 1$, $i = 1, \dots, s$, $q = |\det A|$. Let $F_m \subset \mathbb{Z}^s$ be any complete set of coset representatives for the group $\mathbb{Z}^s / A^{2k_0 m}\mathbb{Z}^s$, $\quad m = 1, 2, \dots$. It is easy to see that $\# F_m = q^{2k_0 m}$. Let us take k_0 such that

$$\min_{1 \leq i \leq s} \left(\frac{1 + |\lambda_i|}{2}\right)^{k_0} > p^{2s}. \tag{3}$$

Now let

$$n_1 = 0, \qquad n_m = n_{m-1} + 3k_0(m-1)p^{m-1}, \qquad m = 2, 3, \dots, \tag{4}$$

$$\alpha = \sum_{m=1}^{\infty} \sum_{n=0}^{p^m - 1} \sum_{\nu=0}^{2} \left\{ n b_{\nu,m} A^{-2k_0 m} \right\} A^{-(n_m + k_0 m(3n+\nu))}, \tag{5}$$

where $b_{\nu,m} \in F_m$.

Theorem. *There exists $b_{\nu,m} \in F_m$ $(m = 1, 2, \dots \ \nu = 0, 1, 2)$ such that*

$$D\left(\{\alpha A^n\}_{n=1}^N\right) = O\left(\frac{\log^{2s+3} N}{N}\right), \quad for \quad N \to \infty.$$

We prove this result in Section 3.

Remark. We will prove a similar result for the case of a hyperbolic matrix ($|\lambda_i| \neq 1$, $i = 1, \dots, s$) in a forthcoming paper.

Let $f(x) = x^s - b_1 x^{s-1} - \ldots - b_{s-1}x - b_s$ be a polynomial with roots λ_i, $\lambda_i| > 1$, $1 \leq i \leq s$. Consider the following recurrence sequence

$$\psi(n) = b_1\psi(n-1) + b_2\psi(n-2) + \ldots + b_s\psi(n-s),$$

where $\psi(i) = \alpha_i$ $(i = 1, \ldots, s)$ and $b_1, \ldots, b_s$ are integers.

Corollary. *There exists* $\alpha = (\alpha_1, \ldots, \alpha_s)$ *such that*

$$D\left((\psi(n), \ldots, \psi(n+s-1))_{n=1}^{N}\right) = O\left(N^{-1}(\log N)^{2s+3}\right).$$

P r o o f. Using the Theorem it follows instantly, if we will denote

$$(\psi(n), \ldots, \psi(n+s-1)) = \alpha A^{n-1},$$

where A is the companion matrix of $f(x)$. □

We will need the following inequalities:

The Erdős-Turán-Koksma inequality(see [DrTi], p. 15):

$$D\left(\{x_n\}_{n=0}^{N-1}\right) \leq \left(\frac{3}{2}\right)^s \left(\frac{2}{M+1} + \frac{1}{N} \sum_{0<\max|m_i|\leq M} \frac{\left|\sum_{n=0}^{N-1} e(\langle m, x_n\rangle)\right|}{\overline{m_1}\ldots\overline{m_s}}\right), \quad (6)$$

where $e(y) = \exp(2\pi i y)$, $x_n = (x_{n,1}, \ldots, x_{n,s})$, $m = (m_1, \ldots, m_s)$, $\overline{m_i} = \max(1, |m_i|)$, and $< (a_1, \ldots, a_s), (b_1, \ldots, b_s) >= a_1 b_1 + \ldots + a_s b_s$.

Lemma A (see [Ko2, p. 1]). *Let β be a real number, M and N natural, then*

$$\left|\sum_{n=M}^{M+N-1} e(n\beta)\right| \leq \min\left(N, \frac{1}{2\parallel\beta\parallel}\right),$$

where $\|\beta\| = \min(\{\beta\}, 1 - \{\beta\})$ and $\{\beta\}$ is the fractional part of β.

Lemma B (see [Ko2, p. 72]). *Let $P \geq 2, (a, P) = 1$, then for any real φ*

$$\sum_{n=1}^{P} \min\left(P, \frac{1}{\parallel an/P + \varphi\parallel}\right) \leq 8P(1 + \log P).$$

Lemma C (see [Ko2, p. 2]). *Let*

$$\delta_q(a) = \begin{cases} 1, & if \quad a \equiv 0 \pmod q \\ 0, & else \end{cases}$$

where $q \geq 1, a \in \mathbb{Z}$. Then

$$\delta_q(a) = \frac{1}{q}\sum_{x=1}^{q} e\left(\frac{ax}{q}\right).$$

3. Proof of the Theorem

Lemma 1. *Let* $0 < |m| \le p^j, \quad 0 \le l < k_0 j$. *Then there exists* $j_0 > 0$, *such that for all* $j \ge j_0$, *we have*

$$\left| < m, b_0 A^{l-2k_0 j} > \right| < \max_{1\le i\le s} \frac{|b_{0i}|}{p^j},$$

where $b_0 = (b_{01}, \dots, b_{0s}), \quad m = (m_1, \dots, m_s)$.

P r o o f. Let us denote

$$\lambda_{\min} = \min_{1\le i\le s} |\lambda_i|, \qquad \lambda_0 = \frac{\lambda_{\min}+1}{2}, \quad a_{ik} = a_{ik}(t) = \left(A^{-t}\right)_{ik}, \quad 1 \le i,k \le s.$$

Using Jordan's form of the matrix A, we obtain

$$a_{ik}(t) = O\left(\lambda_{\min}^{-t} t^s\right) = o\left(\lambda_0^{-t}\right). \tag{7}$$

Bearing in mind (3), we obtain

$$\lambda_0^{-jk_0} < p^{-2j}, \qquad \text{and} \;\; a_{ik}(-l+2k_0 j) = o(p^{-2j}), \quad i,k = 1,\dots,s. \tag{8}$$

Hence

$$\left|\langle m, b_0 A^{l-2k_0 j}\rangle\right| = \left| \sum_{i,k=1}^{s} m_i b_{0k} (A^{l-2k_0 j})_{ki} \right| = o\left(p^{-j} \max_{1\le k\le s} |b_{0k}| \right).$$

Lemma 1 is proved. □

Lemma 2. *Let* $0 < |m| \le p^j$, $m = (m_1, \dots, m_s)$, $j \ge j_0$, $0 \le l < k_0 j$,

$$G(m,j) = \left\{ \{< m, bA^{l-2k_0 j} >\} \;\mid\; b \in F_j \right\} \;\; \text{and} \;\; v_0 = \#G(m,j).$$

Then

$$G(m,j) = \left\{ \frac{\mu}{v_0}, \quad 0 \le \mu < v_0 \right\}$$

with

$$\#G(m,j) > p^j,$$

where $\#G(m,j)$ *is the number of elements of* $G(m,j)$.

P r o o f. Bearing in mind that A is an integer matrix, we get

$$\{\langle m, bA^{l-2k_0 j}\rangle\} = \frac{\mu}{v_1(b)},$$

where $\mu = \mu(b) \ge 0, \;\; v_1(b)$ are integer numbers. Let $v_2 = \max_{b\in F_j} \; v_1(b)$ and let

$$\{\langle m, b_0 A^{l-2k_0 j}\rangle\} = \frac{\mu}{v_2}, \quad (\mu, v_2) = 1$$

for some $b_0 \in F_j$. Taking into account that $G(m,j)$ is the group, we obtain

$$\{\langle nm, b_0 A^{l-2k_0 j}\rangle\} = \left\{\frac{n\mu}{v_2}\right\} \in G(m,j), \quad 0 \le n \le v_2 - 1.$$

Hence, there exists an integer n_0 with $\{\langle n_0 m, b_0 A^{l-2k_0 j}\rangle\} = 1/v_2$ and also $\{n/v_2\} \in G(m,j)$ for all $n \in [0, v_2)$. Suppose that there exists $b \in F_j$ with $\mu(b)/v_1(b) \notin \{0, 1/v_2, \dots, (v_2-1)/v_2\}$. This means that $v_1(b) \nmid v_2$ and $v_1(b) < v_2$. Therefore there exists $d \ge 1$ such that $d | v_1(b)$ and $(d, v_2) = 1$. Bearing in mind that $\{h\mu/v_1(b)\} \in G(m,j)$ $(0 \le h < v_1(b))$, we have $\{h/d\} \in G(m,j)$ for $h \in [0, v_1(b))$. Hence

$$\left\{\frac{n}{v_2} + \frac{h}{d}\right\} \in G(m,j) \ \ (0 \le n < v_2,\ 0 \le h < d) \quad \text{and} \quad \left\{\frac{l}{v_2 d}\right\} \in G(m,j)$$

for $l \in [0, v_2 d - 1]$. But $v_2 = \max_{b \in F_j} v_1(b)$. We have the contradiction. Then, $v_2 = v_0$.

Now let us prove that $v_2 > p^j$. Let $b_{0,i} = (0, 0, \dots, 1_i, \dots, 0)$, $i = 1, \dots, s$, $\gamma = (\gamma_1, \dots, \gamma_s) = mA^{l-2k_0 j}$. We have that there exist integers $c_1, \dots, c_s \ge 0$ with

$$|\langle m, b_{0,i} A^{l-2k_0 j}\rangle| = |\langle \gamma, b_{0,i}\rangle| = |\gamma_i| = \frac{c_i}{v_2}.$$

According to Lemma 1, $c_i/v_2 < 1/p^j$, $i = 1, \dots, s$. Taking into account that $|m| > 0$, we obtain $\gamma \ne 0$. Therefore there exists i_0 with $c_{i_0} > 0$. Thus $v_2 > p^j$. Lemma 2 is proved. □

Lemma 3. *Let φ be a real number, $0 < |m| \le p^j$, $m = (m_1, \dots, m_s)$, $j \ge j_0$, $0 \le l < k_0 j$. Then*

$$\sigma_1(j) := \frac{1}{q^{2k_0 j}} \sum_{b \in F_j} \min\left(p^j, \frac{1}{2\|\langle m, bA^{l-2k_0 j}\rangle + \varphi\|}\right) = O(j),$$

where the O-constant does not depend on φ, m, and l.

P r o o f. Bearing in mind that $b \longrightarrow \{\langle m, bA^{l-2k_0 j}\rangle\}$, where $b \in F_j$, is a group homomorphism, we get

$$\#\{b \in F_j \ \ | \ \ \{\langle m, bA^{l-2k_0 j}\rangle\} = g\} = \#\{b \in F_j \ \ | \ \ \{\langle m, bA^{l-2k_0 j}\rangle\} = 0\}$$

for all $g \in G(m,j)$. Hence, using Lemma 2, we obtain

$$\frac{1}{q^{2k_0 j}} \sum_{b \in F_j} \min\left(p^j, \frac{1}{2\|\langle m, bA^{l-2k_0 j}\rangle + \varphi\|}\right) = \frac{1}{v_0} \sum_{\mu=1}^{v_0} \min\left(p^j, \frac{1}{2\|\mu/v_0 + \varphi\|}\right).$$

Applying Lemma B, we have

$$\sigma_1(j) \leq \frac{1}{v_0} \sum_{\mu=1}^{v_0} \min\left\{v_0, \frac{1}{2\|\mu/v_0 + \varphi\|}\right\} \leq 8(1 + \log v_0).$$

Taking into account that $v_0 = \#G(m,j) \leq \#F_j = q^{2k_0 j}$, we obtain the assertion of the lemma. □

Let us denote

$$\begin{aligned} S_0(m,l,R,b_{0,j},b_{1,j},b_{2,j}) \\ = \sum_{n=0}^{R-1} e\left(\langle m, b_{2,j}(n-1)A^{l-k_0 j} + b_{0,j} n A^{l-2k_0 j} + \{b_{1,j} n A^{-2k_0 j}\} A^{l-k_0 j}\rangle\right), \end{aligned} \tag{9}$$

$$S_1(m,l,R,b_{0,j},b_{1,j},b_{2,j}) = \sum_{n=0}^{R-1} e\left(\langle m, b_{0,j} n A^{l-k_0 j} + b_{1,j} n A^{l-2k_0 j} + \{b_{2,j} n A^{-2k_0 j}\} A^{l-k_0 j}\rangle\right),$$

$$S_2(m,l,R,b_{0,j},b_{1,j},b_{2,j}) = \sum_{n=0}^{R-1} e\left(\langle m, b_{1,j} n A^{l-k_0 j} + b_{2,j} n A^{l-2k_0 j} + \{b_{0,j} (n+1) A^{-2k_0 j}\} A^{l-k_0 j}\rangle\right),$$

$$\beta_0 = \langle m, b_{2,j} A^{l-k_0 j}\rangle + \langle m, b_{0,j} A^{l-2k_0 j}\rangle - \frac{1}{q^{2k_0 j}} \langle z, b_{1,j} q^{2k_0 j} A^{-2k_0 j}\rangle, \tag{10}$$

$$\beta_1 = \langle m, b_{0,j} A^{l-k_0 j}\rangle + \langle m, b_{1,j} A^{l-2k_0 j}\rangle - \frac{1}{q^{2k_0 j}} \langle z, b_{2,j} q^{2k_0 j} A^{-2k_0 j}\rangle,$$

$$\beta_2 = \langle m, b_{1,j} A^{l-k_0 j}\rangle + \langle m, b_{2,j} A^{l-2k_0 j}\rangle - \frac{1}{q^{2k_0 j}} \langle z, b_{0,j} q^{2k_0 j} A^{-2k_0 j}\rangle.$$

LEMMA 4. *For $\nu = 0,1,2$, we have*

$$|S_\nu(m,l,p^j - 1, b_{0,j}, b_{1,j}, b_{2,j})|$$

$$\leq \sum_{z_1,\dots,z_s=0}^{q^{2k_0 j}-1} \min\left(p^j, \frac{1}{2\|\beta_\nu\|}\right) \prod_{i=1}^{s} \min\left(1, \frac{1}{2q^{2k_0 j}\| \frac{m_1 a_{i1} + \dots + m_s a_{is} + z_i}{q^{2k_0 j}} \|}\right), \tag{11}$$

where $(a_{ij})_{i,j=1}^s = A^{l-2k_0 j}$.

P r o o f. It is easy to see that $A^{-1} = B_0/\det A$, where B_0 is an integer matrix, so $A^{-2k_0 j} = B_1/q^{2k_0 j}$, where B_1 is an integer matrix. Let $b_1 \in F_j$, $n \in \mathbb{Z}$ and

$k \equiv nq^{2k_0 j} b_1 A^{-2k_0 j} \left(\bmod q^{2k_0 j}\mathbb{Z}^s\right)$ with $k = (k_1, \dots, k_s) \in \mathbb{Z}^s \cap [0, q^{2k_0 j})^s$. Hence

$$\left\{nb_1 A^{-2k_0 j}\right\} = k/q^{2k_0 j}.$$

Let $\nu = 0$. Removing a fractional part, we get

$$S_0(m, l, p^j - 1, b_{0,j}, b_{1,j}, b_{2,j})$$

$$= \sum_{k_1,\dots,k_s=0}^{q^{2k_0 j}-1} \sum_{n=0}^{p^j-1} e\left(\left\langle m, b_{2,j}(n-1)A^{l-k_0 j} + b_{0,j} n A^{l-2k_0 j} + \frac{k}{q^{2k_0 j}} A^{l-k_0 j}\right\rangle\right)$$

$$\times \prod_{i=1}^{s} \delta_{q^{2k_0 j}}(k_i - (b_{1,j} q^{2k_0 j} A^{-2k_0 j})_i n).$$

By Lemma C, we have

$$\left|S_0(m, l, p^j - 1, b_{0,j}, b_{1,j}, b_{2,j})\right|$$

$$= \frac{1}{q^{2k_0 js}} \Bigg| \sum_{k_1,\dots,k_s=0}^{q^{2k_0 j}-1} \sum_{n=0}^{p^j-1} \sum_{z_1,\dots,z_s=0}^{q^{2k_0 j}-1} e(\langle m, b_{2,j}(n-1)A^{l-k_0 j}$$

$$+ b_{0,j} n A^{l-2k_0 j} + \frac{k}{q^{2k_0 j}} A^{l-k_0 j}\rangle) e(\langle z, \frac{k - b_{1,j} q^{2k_0 j} A^{-2k_0 j} n}{q^{2k_0 j}}\rangle)\Bigg|$$

$$= \frac{1}{q^{2k_0 js}} \Bigg| \sum_{k_1,\dots,k_s=0}^{q^{2k_0 j}-1} \sum_{z_1,\dots,z_s=0}^{q^{2k_0 j}-1} e\left(\langle m, kA^{l-k_0 j}\rangle \frac{1}{q^{2k_0 j}} - \langle m, b_{2,j} A^{l-k_0 j}\rangle + \frac{1}{q^{2k_0 j}}\langle z, k\rangle\right)$$

$$\times \sum_{n=0}^{p^j-1} e\left(n\left(\langle m, b_{2,j} A^{l-k_0 j}\rangle + \langle m, b_{0,j} A^{l-2k_0 j}\rangle - \frac{1}{q^{2k_0 j}}\langle z, b_{1,j} q^{2k_0 j} A^{-2k_0 j}\rangle\right)\right)\Bigg|$$

$$= \frac{1}{q^{2k_0 js}} \left| \sum_{k_1,\dots,k_s=0}^{q^{2k_0 j}-1} \sum_{z_1,\dots,z_s=0}^{q^{2k_0 j}-1} e(\varphi_0) \sum_{n=0}^{p^j-1} e(n\beta_0)\right|$$

$$\leq \sum_{z_1,\dots,z_s=0}^{q^{2k_0 j}-1} \left|\sum_{n=0}^{p^j-1} e(n\beta_0)\right| \left|\frac{1}{q^{2k_0 js}} \sum_{k_1,\dots,k_s=0}^{q^{2k_0 j}-1} e(\varphi_0)\right|, \tag{12}$$

where

$$\varphi_0 = \langle m, kA^{l-k_0 j}\rangle \frac{1}{q^{2k_0 j}} - \langle m, b_{2,j} A^{l-k_0 j}\rangle + \frac{1}{q^{2k_0 j}}\langle z, k\rangle \tag{13}$$

and β_0 defined in (10).
Let us approximate the following

$$\sigma := \left| \frac{1}{q^{2k_0sj}} \sum_{k_1,\ldots,k_s=0}^{q^{2k_0j}-1} e(\varphi_0) \right|$$
$$= \left| \frac{1}{q^{2k_0sj}} \sum_{k_1,\ldots,k_s=0}^{q^{2k_0j}-1} e\left(\frac{1}{q^{2k_0j}} \left(\langle m, kA^{l-k_0j} \rangle + \langle z, k \rangle \right) \right) \right|.$$

We have for $(a_{ij})_{1\le i,j\le s} = A^{l-k_0j}$ that

$$\sigma = \left| \frac{1}{q^{2k_0sj}} \sum_{k_1,\ldots,k_s=0}^{q^{2k_0j}-1} e\left(\frac{1}{q^{2k_0j}} \sum_{i=1}^{s} (k_i(m_1a_{i1} + \ldots + m_sa_{is} + z_i) \right) \right|.$$

Using Lemma A, we get

$$\sigma \le \prod_{i=1}^{s} \min\left(1, \frac{1}{2q^{2k_0j} \| \frac{m_1a_{i1}+\ldots+m_sa_{is}+z_i}{q^{2k_0j}} \|}\right) \tag{14}$$

and

$$\left| \sum_{n=0}^{p^j-1} e(n\beta_0) \right| \le \min\left(p^j, \frac{1}{2\|\beta_0\|}\right).$$

By (12) we obtain

$$|S_0(m, l, p^j - 1, b_{0,j}, b_{1,j}, b_{2,j})|$$
$$\le \sum_{z_1,\ldots,z_s=0}^{q^{2k_0j}-1} \min\left(p^j, \frac{1}{2\|\beta_0\|}\right) \prod_{i=1}^{s} \min\left(1, \frac{1}{2q^{2k_0j} \| \frac{m_1a_{i1}+\ldots+m_sa_{is}+z_i}{q^{2k_0j}} \|}\right).$$

So, (11) is proved. In the same way we will get the inequalities for $\nu = 1$ and $\nu = 2$. Lemma 4 is proved. □

Lemma 5. *For $\nu = 0, 1, 2$ and $R \in [0, p^j)$, we have*

$$|S_\nu(m, l, R, b_{0,j}, b_{1,j}, b_{2,j})| \le \sum_{m_{s+1}=-[p^j/2]}^{[p^j/2]} \frac{1}{m_{s+1}}$$
$$\times \sum_{z_1,\ldots,z_s=0}^{q^{2k_0j}-1} \min\left(p^j, \frac{1}{2 \parallel \beta_\nu + \frac{m_{s+1}}{p^j} \parallel}\right) \prod_{i=1}^{s} \min\left(1, \frac{1}{2q^{2k_0j} \parallel \frac{m_1a_{i1}+\ldots+m_sa_{is}+z_i}{q^{2k_0j}} \parallel}\right). \tag{15}$$

P r o o f. By the same way, as in Lemma 4 we get

$$|S_0(m,l,R,b_{0,j},b_{1,j},b_{2,j})| = \frac{1}{q^{2k_0js}} \left| \sum_{k_1,\dots,k_s=0}^{q^{2k_0j}-1} \sum_{z_1,\dots,z_s=0}^{q^{2k_0j}-1} e(\varphi_0) \sum_{n=0}^{R-1} e(n\beta_0) \right|,$$

where φ_0 defined in (13) and β_0 defined in (10).
Applying Lemma C, we have

$$\sum_{n=0}^{R-1} e(n\beta_0+\varphi_0) = \sum_{n=0}^{R-1}\sum_{n_1=0}^{p^j-1} e\left(n_1\beta_0+\varphi_0\right)\delta_{p^j}(n_1-n)$$

$$= \sum_{n=0}^{R-1}\sum_{n_1=0}^{p^j-1} \frac{1}{p^j} \sum_{m_{s+1}=-[p^j/2]}^{[p^j/2]} e\left(n_1\beta_0+\varphi_0+\frac{m_{s+1}(n_1-n)}{p^j}\right)$$

$$= \frac{1}{p^j} \sum_{m_{s+1}=-[p^j/2]}^{[p^j/2]} \sum_{n=0}^{R-1} e\left(-\frac{m_{s+1}n}{p^j}\right) \sum_{n_1=0}^{p^j-1} e\left(n_1\beta_0+\varphi_0+\frac{m_{s+1}n_1}{p^j}\right).$$

According to [Ni, p. 35]

$$\left|\frac{1}{p^j}\sum_{n=0}^{R-1} e\left(\frac{m_{s+1}n}{p^j}\right)\right| \le \frac{1}{\overline{m_{s+1}}}, \quad 1 \le R \le p^j.$$

Now the proof of the following inequality is the same as that of [Ko2, p. 13].

$$\left| \sum_{k_1,\dots,k_s=0}^{q^{2k_0j}-1} \sum_{z_1,\dots,z_s=0}^{q^{2k_0j}-1} \sum_{n=0}^{R-1} e(n\beta_0+\varphi_0) \right|$$

$$\le \sum_{m_{s+1}=-[p^j/2]}^{[p^j/2]} \frac{1}{\overline{m_{s+1}}} \left| \sum_{k_1,\dots,k_s=0}^{q^{2k_0j}-1} \sum_{z_1,\dots,z_s=0}^{q^{2k_0j}-1} \sum_{n_1=0}^{p^j-1} e\left(n_1\beta_0+\varphi_0+\frac{m_{s+1}n_1}{p^j}\right) \right|.$$

Therefore

$$|S_0(m,l,R,b_{0,j},b_{1,j},b_{2,j})| \le \frac{1}{q^{2k_0js}}$$

$$\times \sum_{m_{s+1}=-[p^j/2]}^{[p^j/2]} \frac{1}{\overline{m_{s+1}}} \left| \sum_{k_1,\dots,k_s=0}^{q^{2k_0j}-1} \sum_{z_1,\dots,z_s=0}^{q^{2k_0j}-1} e(\varphi_0) \right| \left| \sum_{n_1=0}^{p^j-1} e\left(n_1\left(\beta_0+\frac{m_{s+1}}{p^j}\right)\right) \right|.$$

Using Lemma A, (13), and (14), we get

$$|S_0(m,l,R,b_{0,j},b_{1,j},b_{2,j})| \le \sum_{m_{s+1}=-[p^j/2]}^{[p^j/2]} \frac{1}{\overline{m_{s+1}}}$$

$$\times \sum_{z_1,\dots,z_s=0}^{q^{2k_0j}-1} \min\left(p^j, \frac{1}{2 \parallel \beta_0 + \frac{m_{s+1}}{p^j} \parallel}\right) \prod_{i=1}^{s} \min\left(1, \frac{1}{2q^{2k_0j} \parallel \frac{m_1a_{i1}+\ldots+m_sa_{is}+z_i}{q^{2k_0j}} \parallel}\right).$$

Hence (15) is proved for $\nu = 0$. In the same way, we will get (15) for $\nu = 1$ and $\nu = 2$. Lemma 5 is proved. □

Let

$$S^{(1)}(m,j) = \sum_{\nu=0}^{2} \frac{1}{\overline{m_1}\ldots\overline{m_s}} \tag{16}$$

$$\times \sum_{z_1,\dots,z_s=0}^{q^{2k_0j}-1} \min\left(p^j, \frac{1}{2\|\beta_\nu\|}\right) \prod_{i=1}^{s} \min\left(1, \frac{1}{2q^{2k_0j}\| \frac{m_1a_{i1}+\ldots+m_sa_{is}+z_i}{q^{2k_0j}}\|}\right),$$

$$S^{(2)}(m,j) = \sum_{m_{s+1}=-p^j/2}^{p^j/2} \sum_{\nu=0}^{2} \frac{1}{\overline{m_1}\ldots\overline{m_s}\ \overline{m_{s+1}}}$$

$$\times \sum_{z_1,\dots,z_s=0}^{q^{2k_0j}} \min\left(p^j, \frac{1}{2 \parallel \beta_\nu + \frac{m_{s+1}}{p^j} \parallel}\right) \prod_{i=1}^{s} \min\left(1, \frac{1}{2q^{2k_0j} \parallel \frac{m_1a_{i1}+\ldots+m_sa_{is}+z_i}{q^{2k_0j}} \parallel}\right),$$

$$T_i(b_{0,j},b_{1,j},b_{2,j}) = \sum_{0<\max|m_i|\le M} \sum_{l=0}^{k_0j-1} S^{(i)}(m,j), \tag{17}$$

where $M = [p^j/2]$, $i = 1,2$.

Lemma 6. *Let us take $b_{0,j}, b_{1,j}, b_{2,j}$ so that*

$$T_1(b_{0,j},b_{1,j},b_{2,j}) + T_2(b_{0,j},b_{1,j},b_{2,j})/j$$

will be minimal. Then

$$T_1(b_{0,j},b_{1,j},b_{2,j}) = O(j^{2s+2})$$

and

$$T_2(b_{0,j},b_{1,j},b_{2,j}) = O(j^{2s+3}).$$

P r o o f. Consider the mean values

$$\widetilde{T_i} = \frac{1}{q^{6k_0 j}} \sum_{b_0, b_1, b_2 \in F_j} T_i(b_0, b_1, b_2), \quad i = 1, 2. \tag{18}$$

It is easy to see that Lemma 6 goes after from the following assertion

$$\widetilde{T_1} + \widetilde{T_2}/j = O(j^{2s+2}). \tag{19}$$

Let

$$\sigma_1 = \frac{1}{q^{6k_0 j}} \sum_{b_0, b_1, b_2 \in F_j} \min\left(p^j, \frac{1}{2\|\beta_0\|}\right).$$

According to (10) and Lemma 3, we have

$$\sigma_1 = \frac{1}{q^{6k_0 j}} \sum_{b_0, b_1, b_2 \in F_j} \min\left(p^j, \frac{1}{2\| < m, b_0 A^{l-2k_0 j} > + \varphi(z, b_0, b_2) \, \|}\right) = O(j),$$

where the O-constant does not depend on z, l.
By (16) and (17) we obtain

$$\widetilde{T_1} =$$

$$O\left(\sum_{0 < \max|m_i| \le M} \frac{j^2}{\overline{m_1} \ldots \overline{m_s}} \sum_{z_1, \ldots, z_s = 0}^{q^{2k_0 j} - 1} \prod_{i=1}^{s} \min\left(1, \frac{1}{2q^{2k_0 j} \| \; \frac{m_1 a_{i1} + \ldots + m_s a_{is} + z_i}{q^{2k_0 j}} \|}\right)\right).$$

Using Lemma B, we get:

$$\sum_{z_1, \ldots, z_s = 0}^{q^{2k_0 j} - 1} \prod_{i=1}^{s} \min\left(1, \frac{1}{2q^{2k_0 j} \| \; \frac{m_1 a_{i1} + \ldots + m_s a_{is} + z_i}{q^{2k_0 j}} \|}\right)$$

$$\le \left(8(1 + 2\log(q^{2k_0 j})\right)^s = O\left((\log q^{2k_0 j})^s\right) = O(j^s),$$

and we have

$$\sum_{0 < \max|m_i| \le p^j/2} \frac{1}{\overline{m_1} \ldots \overline{m_s}} \le (3 + 2\log p^j)^s = O(j^s).$$

Thus

$$\widetilde{T_1} = O(j^{2s+2}). \tag{20}$$

Approximation for $\widetilde{T_2}$ is the same as for $\widetilde{T_1}$. So we get

$$\widetilde{T_2} = O(j^{2s+3}). \tag{21}$$

Now from (18), (20) and (21), we get (19) and the assertion of the lemma. □

We will use vectors $b_{0,j}, b_{1,j}, b_{2,j}$ $(j = 1, 2, \ldots)$ in (5).

Completion of the proof of the Theorem

Let us decompose our interval $[1, N]$ into subintervals:

$$[1, n_2), [n_2, n_3), \dots, [n_{r-1}, n_r), [n_r, N],$$

where $n_{r+1} > N \geq n_r$ (see (4)). Hence by (4)

$$4k_0 r p^r > N \geq (r-1)p^{r-1}. \tag{22}$$

Let us take a full interval, i.e., $k \in [n_j, n_{j+1})$, where $j = 2, \dots, r-1$,

$$k = n_j + k_0 j(3n^* + \nu^*) + l^*, \quad 0 \leq l^* < k_0 j, \ 0 \leq n^* \leq p^j - 1, \ 0 \leq \nu^* \leq 2. \tag{23}$$

By (5) we have:

$$\alpha A^k = \sum_{m=1}^{\infty} \sum_{n=0}^{p^m-1} \sum_{\nu=0}^{2} \left\{ n b_{\nu,m} A^{-2k_0 m} \right\} A^{k-(n_m + k_0 m(3n+\nu))}$$

$$= \sum_{m=1}^{\infty} \sum_{n=0}^{p^m-1} \sum_{\nu=0}^{2} \left\{ n b_{\nu,m} A^{-2k_0 m} \right\} A^{n_j + k_0 j(3n^*+\nu^*) + l^* - (n_m + k_0 m(3n+\nu))}.$$

Let

$$R_{\nu^*, n^*} =$$

$$\sum_{m=j}^{\infty} \sum_{n=0}^{p^m-1} \sum_{\substack{n_m + k_0 m(3n+\nu-1) > k \\ \nu \in [0,2]}} \left\{ n b_{\nu,m} A^{-2k_0 m} \right\} A^{n_j + k_0 j(3n^*+\nu^*) + l^* - (n_m + k_0 m(3n+\nu))}.$$

We have, for example, for $\nu^* = 0$

$$\begin{aligned} R_{0,n^*} = & \sum_{n=n^*+1}^{p^j-1} \Big(\left\{ n b_{0,j} A^{-2k_0 j} \right\} A^{3k_0 j n^* + l^* - 3k_0 j n} \\ & + \left\{ n b_{1,j} A^{-2k_0 j} \right\} A^{3k_0 j n^* + l^* - (3k_0 j n + k_0 j)} \Big) \\ & + \sum_{n=n^*}^{p^j-1} \left\{ n b_{2,j} A^{-2k_0 j} \right\} A^{3k_0 j n^* + l^* - (3k_0 j n + 2k_0 j)} \\ & + \sum_{m=j+1}^{\infty} \sum_{n=0}^{p^m-1} \sum_{\nu=0}^{2} \left\{ n b_{\nu,m} A^{-2k_0 m} \right\} A^{n_j + 3k_0 j n^* + l^* - (n_m + k_0 m(3n+\nu))}. \end{aligned}$$

Bearing in mind that

$$\left\{ \left\{ n b_{\nu,m} A^{-2k_0 m} \right\} A^{n_j + k_0 j(3n^*+\nu^*) + l^* - (n_m + k_0 m(3n+\nu))} \right\} = 0$$

for $n_j + k_0 j(3n^* + \nu^*) + l^* - (n_m + k_0 m(3n + \nu)) \geq 2k_0 m$, we obtain

$$\{\alpha A^k\} = \{f_{\nu^*,n^*} + R_{\nu^*,n^*}\}, \tag{24}$$

for $n^* \neq 0$ and $n^* \neq p^j - 1$, where

$$\begin{aligned} f_{0,n^*} =& \{b_2(n^*-1)A^{-2k_0 j}\}A^{l^*+k_0 j} + \{b_0 n^* A^{-2k_0 j}\}A^{l^*} + \{b_1 n^* A^{-2k_0 j}\}A^{l^*-k_0 j}, \\ f_{1,n^*} =& \{b_0 n^* A^{-2k_0 j}\}A^{l^*+k_0 j} + \{b_1 n^* A^{-2k_0 j}\}A^{l^*} + \{b_2 n^* A^{-2k_0 j}\}A^{l^*-k_0 j}, \\ f_{2,n^*} =& \{b_1(n^*-1)A^{-2k_0 j}\}A^{l^*+k_0 j} + \{b_2 n^* A^{-2k_0 j}\}A^{l^*} \\ &+ \{b_0(n^*+1)A^{-2k_0 j}\}A^{l^*-k_0 j}. \end{aligned}$$

Let us approximate R_{0,n^*}. By (7) and (23) we have

$$R_{0,n^*} = O\Bigg(\sum_{n=n^*+1}^{p^j-1} \lambda_0^{3k_0 jn^* + l^* - 3k_0 jn}$$

$$+ \sum_{n=n^*}^{p^j-1} \lambda_0^{3k_0 jn^* + l^* - (3k_0 jn + 2k_0 j)} + \sum_{m=j+1}^{\infty} \sum_{n=0}^{p^m-1} \sum_{\nu=0}^{2} \lambda_0^{n_j + 3k_0 jn^* + l^* - (n_m + k_0 m(3n+\nu))} \Bigg).$$

According to (4) and (8), we obtain

$$R_{\nu^*,n^*} = O(\lambda_0^{-k_0 j} + \sum_{n \geq 0} \lambda_0^{-2k_0 j - n}) = O(\lambda_0^{-k_0 j}) = O(p^{-2j})$$

for $\nu^* = 0$. We have the same estimate for $\nu^* = 1, 2$.

Using the inequality

$$|e(x) - 1| = |2\sin(\pi x)| \leq 2\pi |x|,$$

we get

$$\begin{aligned} &|e(< m, f_{\nu^*,n^*} + R_{\nu^*,n^*} >) - e(< m, f_{\nu^*,n^*} >)| = |(e(< m, R_{\nu^*,n^*} > -1))| \\ &\quad \leq 2\pi | < m, R_{\nu^*,n^*} > | \leq 2\pi |m| |R_{\nu^*,n^*}| \leq 2\pi s p^j |R_{\nu^*,n^*}| = O(p^{-j}). \end{aligned}$$

By (1) we have the trivial estimate

$$LD\left((x_n)_{n=J}^{J+L-1} \right) \leq L, \quad L = 1, 2, \ldots$$

Hence by (23), (24) and (6), we get

$$\begin{aligned} &(n_{j+1} - n_j) D(\{\alpha A^k\}_{k=n_j}^{n_{j+1}-1}) \\ &\leq \sum_{l^*=0}^{k_0 j - 1} \sum_{\nu^*=0}^{2} \left((p^j - 2) D((f_{\nu^*,n^*} + R_{\nu^*,n^*}))_{n^*=1}^{p^j-2} + 2 \right) \end{aligned}$$

$$\leq \sum_{l^*=0}^{k_0 j-1} \sum_{\nu^*=0}^{2} \left(p^j D((f_{\nu^*,n^*} + R_{\nu^*,n^*}))_{n^*=0}^{p^j-1} + 4 \right) \leq \left(\frac{3}{2}\right)^s$$

$$\times \left(\frac{p^j}{M} + 12k_0 j + \sum_{\nu^*=0}^{2} \sum_{l^*=0}^{k_0 j-1} \sum_{0<max|m_i|\leq M} \frac{\left|\sum_{n^*=0}^{p^j-1} e(< m, (f_{\nu^*,n^*} + R_{\nu^*,n^*}) >)\right|}{\overline{m_1} \dots \overline{m_s}} \right)$$

$$= O\left(j + \sum_{l^*=0}^{k_0 j-1} \sum_{\nu^*=0}^{2} \sum_{0<max|m_i|\leq M} \frac{\left|\sum_{n^*=0}^{p^j-1} e(< m, f_{\nu^*,n^*} >)\right| + 1}{\overline{m_1} \dots \overline{m_s}} \right)$$

with $M = [p^j/2]$. According to (9), we obtain

$$(n_{j+1} - n_j) D(\{\alpha A^k\}_{k=n_j}^{n_{j+1}-1})$$

$$= O\left(j + \sum_{0<max|m_i|\leq M} \sum_{l^*=0}^{k_0 j-1} \sum_{\nu^*=0}^{2} \frac{|S_\nu(m, l, p^j - 1, b_{0,j}, b_{1,j}, b_{2,j})|}{\overline{m_1} \dots \overline{m_s}} \right).$$

Applying Lemma 4, (16), (17) and Lemma 6, we get

$$(n_{j+1} - n_j) D\Big(\{\alpha A^k\}_{k=n_j}^{n_{j+1}-1}\Big) = O(j^{2s+2})$$

for any full interval $[n_j, n_{j+1})$, $\quad 2 \leq j \leq r-1$.
Consider the not full interval $[n_r, N]$. Using Lemma 5 and Lemma 6 we get, similarly, that

$$(N - n_r + 1) D\left(\left\{\alpha A^k\right\}_{k=n_r}^{N}\right) = O(r^{2s+3}).$$

So, finally, we have the following:

$$\begin{aligned} ND\left(\left\{\alpha A^k\right\}_{k=1}^{N}\right) &\leq (n_2 - 1) D(\{\alpha A^k\}_{k=1}^{n_2}) \\ &+ \sum_{j=2}^{r-1} (n_{j+1} - n_j) D(\{\alpha A^k\}_{k=n_j}^{n_{j+1}}) + (N - n_r + 1) D(\{\alpha A^k\}_{k=n_r}^{N}) \\ &= O(1) + \sum_{j=2}^{r-1} O(j^{2s+2}) + O(r^{2s+3}) = O(r^{2s+3}). \end{aligned}$$

Now by (22), we obtain

$$D(\{\alpha A^k\}_{k=1}^{N}) = O\left(N^{-1} (\log N)^{2s+3}\right).$$

The Theorem is proved. □

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Mordechay B. Levin
Department of Mathematics
Bar-Ilan University
Ramat-Gan, 52900, ISRAEL
E-mail: mlevin@math.biu.ac.il

Irina L. Volinsky
Department of Mathematics
Bar-Ilan University
Ramat-Gan, 52900, ISRAEL
E-mail: volinskaya_i@yahoo.com