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uniform distribution theory

THE FOUR-DIMENSIONAL DIVISOR PROBLEM

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ABSTRACT. Let $\vec{a}_4 = (a_1, a_2, a_3, a_4)$, where a_i are natural numbers with $1 \le a_1 \le a_2 \le a_3 \le a_4$. The divisor function $d(\vec{a}_4; n)$ counts the numbers of ways of expressing n as the product $n = n_1^{a_1} n_2^{a_2} n_3^{a_3} n_4^{a_4}$. A new proof for the representation of the remainder term in the asymptotic formula for the summatory function of the four-dimensional divisor function is given.

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1. Some remarks to divisor problems

Let $p \ge 2$ and $\vec{a}_p = (a_1, a_2, \dots, a_p)$. If $a_1, a_2, \dots, a_p$ are natural numbers with $1 \le a_1 \le a_2 \le \dots \le a_p$ the divisor function $d(\vec{a}_p; n)$ denotes the number of representations of an integer $n \ge 1$ in the form $n = n_1^{a_1} n_2^{a_2} \cdots n_p^{a_p}$ with natural numbers $n_1, n_2, \dots, n_p$. We are interested in the asymptotic behaviour of the summatory function

$$D(\vec{a}_p; x) = \sum_{n \le x} d(\vec{a}_p; n).$$

We can also write

$$D(\vec{a}_p; x) = \#\left\{(n_1, n_2, \dots, n_p) \in \mathbb{N}^p : n_1^{a_1} n_2^{a_2} \cdots n_p^{a_p} \le x\right\}.$$

We see that now it is not required that $a_1, a_2, \dots, a_p$ are natural numbers. Therefore, we shall always assume that $a_1, a_2, \dots, a_p$ are real numbers with $1 \le a_1 \le a_2 \le \dots \le a_p$. We put $A_r = a_1 + a_2 + \dots + a_r$ for $r = 1, 2, \dots, p$.

It is known that the summatory function has the representation

$$D(\vec{a}_p; x) = H(\vec{a}_p; x) + \Delta(\vec{a}_p; x)$$

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with the main term $H(\vec{a}_p;x)$ and a remainder term $\Delta(\vec{a}_p;x)$. The main term is given by

$$H(\vec{a}_p;x) = \sum_{k=1}^{p} \left(\prod_{\substack{i=1\\ i\neq k}}^{p} \zeta\left(\frac{a_i}{a_k}\right) \right) x^{\frac{1}{a_k}},$$

provided that all the numbers a_k are distinct. In case of some equalities we may take the appropriate limit values in the sum. $\zeta(s)$ denotes the RIEMANN zetafunction.

In general, the remainder $\Delta(\vec{a}_p, x)$ will be estimated by means of the analytic theory of the RIEMANN zetafunction. But for small values of p, especially for $p = 2, 3, 4$, we also may apply the theory of estimations of exponential sums. For this purpose we obtain for the remainder the representation

$$\Delta(\vec{a}_p;x) = -\sum_{\vec{k}\in\pi(p)} \sum\nolimits_1 \psi\left(\left(\frac{x}{n_1^{a_{k_1}} n_2^{a_{k_2}} \cdots n_{p-1}^{a_{k_{p-1}}}}\right)^{\frac{1}{a_{k_p}}}\right) + O\left(x^{\frac{p-2}{A_p}}\right). \quad (1)$$

ψ denotes the function $\psi(t) = t - [t] - \frac{1}{2}$. $\vec{k} \in \pi(p)$ means that the p-tuple $\vec{k} = (k_1, k_2, \ldots, k_p)$ is a permutation of the numbers $1, 2, \ldots, p$. Then the sum is extended over all permutations. The summation condition SC of $\sum_1$ is given by

$$SC\left(\sum\nolimits_1\right) : n_1^{a_{k_1}} \cdots n_{p-2}^{a_{k_{p-2}}} n_{p-1}^{a_{k_{p-1}}+a_{k_p}} \leq x, \quad n_1(\leq)n_2(\leq)\cdots(\leq)n_{p-1}.$$

$n_i(\leq)n_{i+1}$ means that $n_i \leq n_{i+1}$ for $k_i < k_{i+1}$ and $n_i < n_{i+1}$ for $k_i > k_{i+1}$.

The case $p = 2$ can be found as Theorem 5.1 in [1]. The proof is easy. The case $p = 3$ is given by Theorem 6.1 in [1]. The complicated proof is based on a proof of the special case $\vec{a}_3 = (1, 2, 3)$, given by P.G. SCHMIDT [3]. M. VOGTS [5] has given a proof of (1) for $p \geq 4$ with a complicated error term. For the error term in (1) see [2]. The proof of M. VOGTS is very complicated and difficult and not clearly arranged. Therefore, one can be doubtful on the correctness of the proof. Hence, in this paper a new and sufficiently simple proof for the important case $p = 4$ will be given applying the method of proof of Theorem 6.1 [1].

THEOREM 1. *The representation (1) holds in the special case* $p = 4$ *under the condition* $2a_4 < A_4$.

The condition $2a_4 < A_4$ ensures that the error term is smaller than the smallest term in the main term $H(\vec{a}_4;x)$.

We mention some *important estimates of the remainder*, which can be derived from (1). Besides of the trivial estimate

$$\Delta(\vec{a}_4;x) \ll x^{\frac{3}{A_4}} \quad \text{for} \quad 3a_4 < A_4 \quad (2)$$

the general estimate

$$\Delta(\vec{a}_4;x) \ll x^{\frac{3}{a_1+A_4}} (\log x)^4 \quad \text{for} \quad 3a_4 < a_1 + A_4 \tag{3}$$

holds under a weak condition. Further we mention two pairs of estimates.

Firstly

$$\Delta(\vec{a}_4;x) \ll x^{\frac{31}{13}\cdot\frac{1}{A_4}} (\log x)^4 \quad \text{for} \quad \frac{31}{13}\, a_4 < A_4 \le \frac{31}{7}\, a_1 \tag{4}$$

and

$$\Delta(\vec{a}_4;x) \ll x^{\frac{17}{5a_1+6A_4}} (\log x)^4 \tag{5}$$

for

$$17a_4 < 5a_1 + 6A_4, \quad 15a_1 \le 4A_3$$

and

$$\frac{A_3}{15} \le a_1 \le \frac{A_2}{5} \quad \text{or} \quad \frac{A_2}{5} \le a_1 \quad \text{and} \quad 4a_3 \le 7a_2.$$

Secondly

$$\Delta(\vec{a}_4;x) \ll x^{\frac{19}{8}\cdot\frac{1}{A_4}} (\log x)^5 \quad \text{for} \quad \frac{19}{6}\, a_4 \le A_4 \le \frac{19}{4}\, a_1 \tag{6}$$

and

$$\Delta(\vec{a}_4;x) \ll x^{\frac{11}{3a_1+4A_4}} (\log x)^5 \tag{7}$$

for

$$11a_4 < 3a_1 + 4A_4 \le 22a_2$$

and

$$19a_1 < 4A_4 < 30a_1, \quad 12a_1 + 11a_4 \le 6A_4.$$

The estimates (2), (3), (4), (5) are proved in [2] and the estimates (6), (7) in [4]. In both papers there are a lot of further estimations.

2. Proof of the theorem

For the sake of simplicity we assume $1 \le a_1 < a_2 < a_3 < a_4$. We write for the summatory function

$$D(\vec{a}_4;x) = \sum_{\vec{k}\in\pi(4)} \sum 1,$$

$$SC\big(\textstyle\sum\big) : n_1^{a_{k_1}} n_2^{a_{k_2}} n_3^{a_{k_3}} n_4^{a_{k_4}} \le x, \quad n_1(\le)n_2(\le)n_3(\le)n_4.$$

The meaning of the sign $(\le)$ ensures that each equality $n_i = n_{i+1}$ occurs exactly once. Further let $\varepsilon_{i,i+1} = 1$ for $k_i < k_{i+1}$ and $\varepsilon_{i,i+1} = 0$ for $k_i > k_{i+1}$. Summing over n_4 we obtain

$$D(\vec{a}_4; x) = \sum_{\vec{k}\in\pi(4)} \sum\nolimits_1 \left\{ \left[\left(\frac{x}{n_1^{a_{k_1}} n_2^{a_{k_2}} n_3^{a_{k_3}}} \right)^{\frac{1}{a_{k_4}}} \right] - n_3 + \varepsilon_{3,4} \right\},$$

$$SC\left(\sum\nolimits_1\right) : n_1^{a_{k_1}} n_2^{a_{k_2}} n_3^{a_{k_3}+a_{k_4}} \le x, \quad n_1(\le)n_2(\le)n_3.$$

Now we put $[t] = t - \psi(t) - \frac{1}{2}$. Then the sum over $\varepsilon_{3,4} - \frac{1}{2}$ vanishes, because in this sum k_3 and k_4 may be exchanged. Hence

$$D(\vec{a}_4; x) = K(\vec{a}_4; x) + S(\vec{a}_4; x), \tag{8}$$

$$K(\vec{a}_4; x) = \sum_{\vec{k}\in\pi(4)} \sum\nolimits_1 \left\{ \left(\frac{x}{n_1^{a_{k_1}} n_2^{a_{k_2}} n_3^{a_{k_3}}} \right)^{\frac{1}{a_{k_4}}} - n_3 \right\}, \tag{9}$$

$$S(\vec{a}_4; x) = -\sum_{\vec{k}\in\pi(4)} \sum\nolimits_1 \psi \left(\left(\frac{x}{n_1^{a_{k_1}} n_2^{a_{k_2}} n_3^{a_{k_3}}} \right)^{\frac{1}{a_{k_4}}} \right). \tag{10}$$

Apart from the error term $S(\vec{a}_4; x)$ is equal to $\Delta(\vec{a}_4; x)$.

In (9) we develop the sum over n_3 by means of the Euler-Maclaurin sum formula

$$\sum_{a<n\le b} f(n) = \int_a^b f(t)\mathrm{d}t - \psi(b)f(b) + \psi(a)f(a) + \int_a^b f'(t)\psi(t)\mathrm{d}t. \tag{11}$$

We put

$$f(t) = \left(\frac{x}{n_1^{a_{k_1}} n_2^{a_{k_2}} t^{a_{k_3}}} \right)^{\frac{1}{a_{k_4}}} - t,$$

$$a = n_2, \qquad b = \left(\frac{x}{n_1^{a_{k_1}} n_2^{a_{k_2}}} \right)^{\frac{1}{a_{k_3}+a_{k_4}}}.$$

It is seen that $\psi(a) = -\frac{1}{2}$, $f(b) = 0$. Hence, we obtain from (9) and (10)

$$K(\vec{a}_4; x) = K_0 + K_1 + K_3 + K_4 + K_5,$$

$$\begin{aligned}
K_0 &= \sum_{\vec{k}\in\pi(4)} \sum\nolimits_2 \left(\frac{a_{k_4}}{a_{k_4}-a_{k_3}} - \frac{1}{2}\right)\left(\frac{x}{n_1^{a_{k_1}} n_2^{a_{k_2}}}\right)^{\frac{2}{a_{k_3}+a_{k_4}}}, \\
K_1 &= \sum_{\vec{k}\in\pi(4)} \sum\nolimits_2 \left\{\frac{a_{k_4}}{a_{k_3}-a_{k_4}} n_2 \left(\frac{x}{n_1^{a_{k_1}} n_2^{a_{k_2}+a_{k_3}}}\right)^{\frac{1}{a_{k_4}}} + \frac{1}{2} n_2^2\right\}, \qquad (12) \\
K_3 &= \sum_{\vec{k}\in\pi(4)} \sum\nolimits_2 \left(\varepsilon_{2,3} - \frac{1}{2}\right)\left\{\left(\frac{x}{n_1^{a_{k_1}} n_2^{a_{k_2}+a_{k_3}}}\right)^{\frac{1}{a_{k_4}}} - n_2\right\}, \\
K_4 &= -\sum_{\vec{k}\in\pi(4)} \sum\nolimits_2 \int \frac{a_{k_3}}{a_{k_4}} \cdot \frac{1}{t_3} \left(\frac{x}{n_1^{a_{k_1}} n_2^{a_{k_2}} t_3^{a_{k_3}}}\right)^{\frac{1}{a_{k_4}}} \psi(t_3)\,\mathrm{d}t_3, \\
K_5 &= -\sum_{\vec{k}\in\pi(4)} \sum\nolimits_2 \int \psi(t_3)\,\mathrm{d}t_3
\end{aligned}$$

with the summation and integration conditions

$$\begin{aligned}
SC\big(\textstyle\sum_2\big) &: \; n_1^{a_{k_1}} n_2^{a_{k_2}+a_{k_3}+a_{k_4}} \le x, \quad n_1(\le)n_2, \\
IC\big(\textstyle\int\big) &: \; n_1^{a_{k_1}} n_2^{a_{k_2}} t_3^{a_{k_3}+a_{k_4}} \le x, \quad n_2 \le t_3.
\end{aligned}$$

In K_0 we have

$$\frac{a_{k_4}}{a_{k_4}-a_{k_3}} - \frac{1}{2} = \frac{1}{2} \cdot \frac{a_{k_4}+a_{k_3}}{a_{k_4}-a_{k_3}}.$$

Changing k_3 and k_4 it is seen that $K_0 = 0$. Analogously, changing k_2 and k_3 it follows that $K_3 = 0$. In K_5 the integral over the ψ-function is of order 1. Therefore

$$K_5 \ll \sum_{\vec{k}\in\pi(4)} \sum\nolimits_{2,1} 1,$$

$$SC\big(\textstyle\sum_{2,1}\big) : n_1^{a_{k_1}} n_2^{a_{k_2}+a_{k_3}+a_{k_4}} \le x, \quad n_1 \le n_2,$$

and thus

$$K_5 \ll \sum_{\vec{k}\in\pi(4)} \sum_{n_1^{A_4}\le x} \left(\frac{x}{n_1^{a_{k_1}}}\right)^{\frac{1}{a_{k_2}+a_{k_3}+a_{k_4}}}.$$

Because of $2a_4 < A_4$ it is $a_{k_1} < a_{k_2} + a_{k_3} + a_{k_4}$. This gives

$$K_5 \ll x^{\frac{2}{A_4}}.$$

With regard to K_4 let us consider

$$K_{4,1} = -\sum_{\vec{k}\in\pi(4)} \sum\nolimits_2 \int_1 \frac{a_{k_3}}{a_{k_4}} \cdot \frac{1}{t_3} \left(\frac{x}{n_1^{a_{k_1}} n_2^{a_{k_2}} t_3^{a_{k_3}}}\right)^{\frac{1}{a_{k_4}}} \psi(t_3)\,\mathrm{d}t_3,$$

now with the integration condition

$$IC\left(\textstyle\int_1\right) : n_1^{a_{k_1}} n_2^{a_{k_2}} t_3^{a_{k_3}+a_{k_4}} > x.$$

Using partial integration it follows that

$$K_{4,1} \ll \sum_{\vec{k}\in\pi(4)} \sum\nolimits_2 1 \ll x^{\frac{2}{A_4}}.$$

Hence

$$K(\vec{a}_4; x) = K_1 + K_2 + O\left(x^{\frac{2}{A_4}}\right), \tag{13}$$

where K_1 is given by (12) and

$$K_2 = -\sum_{\vec{k}\in\pi(4)} \frac{a_{k_3}}{a_{k_4}} \sum_{n_{k_1}(\leq)n_{k_2} n_{k,2}} \int^{\infty} \frac{1}{t_3} \left(\frac{x}{n_1^{a_{k_1}} n_2^{a_{k_2}} t_3^{a_{k_3}}}\right)^{\frac{1}{a_{k_4}}} \psi(t_3)\,\mathrm{d}t_3.$$

It is seen at once that

$$K_2 = \sum_{k=1}^{4} c_{1,k} x^{\frac{1}{a_k}}. \tag{14}$$

We omit the detailed computation of the coefficients $c_{1,k}$.

We now deal with K_1 in (13). In the representation (12) of K_1 we develop the sum over n_2 by means of the Euler-Maclaurin sum formula (11). We put

$$f(t) = \frac{a_{k_4}}{a_{k_3} - a_{k_4}}\, t \left(\frac{x}{n_1^{a_{k_1}} t^{a_{k_2}+a_{k_3}}}\right)^{\frac{1}{a_{k_4}}} + \frac{1}{2}\, t^2,$$

$$a = n_1, \quad b = \left(\frac{x}{n_1^{a_{k_1}}}\right)^{\frac{1}{a_{k_2}+a_{k_3}+a_{k_4}}}.$$

Again it is $\psi(a) = -\frac{1}{2}$. Hence, we obtain from (11) and (12)

$$\begin{aligned} K_1 &= K_{1,1} + K_{1,2} + K_{1,3} + K_{1,4} + K_{1,5}, \\ K_{1,1} &= K_{1,1}^{(1)} + K_{1,1}^{(2)}, \end{aligned}$$

$$\begin{aligned}
K_{1,1}^{(1)} &= \sum_{\vec{k}\in\pi(4)} \sum_{n_1^{A_4}\le x} \int\limits_a^b \frac{a_{k_4}}{a_{k_3}-a_{k_4}}\, t \left(\frac{x}{n_1^{a_{k_1}} t^{a_{k_2}+a_{k_3}}}\right)^{\frac{1}{a_{k_4}}} \mathrm{d}t, \\
K_{1,1}^{(2)} &= \sum_{\vec{k}\in\pi(4)} \sum_{n_1^{A_4}\le x} \frac{1}{6} \left\{ \left(\frac{x}{n_1^{a_{k_1}}}\right)^{\frac{3}{a_{k_2}+a_{k_3}+a_{k_4}}} - n_1^3 \right\}, \\
K_{1,2} &= -\sum_{\vec{k}\in\pi(4)} \sum_{n_1^{A_4}\le x} \left(\frac{a_{k_4}}{a_{k_3}-a_{k_4}} + \frac{1}{2}\right) b\psi(b) \\
&= 0, \\
K_{1,3} &= \sum_{\vec{k}\in\pi(4)} \sum_{n_1^{A_4}\le x} (\varepsilon_{1,2} - \frac{1}{2}) \left\{ \frac{a_{k_4}}{a_{k_3}-a_{k_4}}\, n_1^2 \left(\frac{x}{n_1^{A_4}}\right)^{\frac{1}{a_{k_4}}} + \frac{1}{2}\, n_1^2 \right\} \\
&= 0, \\
K_{1,4} &= \sum_{\vec{k}\in\pi(4)} \sum_{n_1^{A_4}\le x} \int_2 \left(\frac{a_{k_2}}{a_{k_4}-a_{k_3}} - 1\right) \left(\frac{x}{n_1^{a_{k_1}} t_2^{a_{k_2}+a_{k_3}}}\right)^{\frac{1}{a_{k_4}}} \psi(t_2)\, \mathrm{d}t_2, \\
K_{1,5} &= \sum_{\vec{k}\in\pi(4)} \sum_{n_1^{A_4}\le x} \int_2 t_2 \psi(t_2)\, \mathrm{d}t_2 \\
&\ll x^{\frac{2}{A_4}}.
\end{aligned}$$

$$IC\big(\textstyle\int_2\big) : n_1^{a_{k_1}} t_2^{a_{k_2}+a_{k_3}+a_{k_4}} \le x, \quad n_1 \le t_2.$$

We obtain for $K_{1,4}$ as before

$$K_{1,4} = \sum_{\vec{k}\in\pi(4)} \sum \int \left(\frac{a_{k_2}}{a_{k_4}-a_{k_3}} - 1\right) \left(\frac{x}{n_1^{a_{k_1}} t_2^{a_{k_2}+a_{k_3}}}\right)^{\frac{1}{a_{k_4}}} \psi(t_2)\, \mathrm{d}t_2 + O\left(x^{\frac{2}{A_4}}\right),$$

$$SC\big(\textstyle\sum\int\big) : 1 \le n_1 \le t_2 < \infty.$$

Therefore

$$K_{1,4} = \sum_{k=1}^{4} c_{2,k} x^{\frac{1}{a_k}} + O\left(x^{\frac{2}{A_4}}\right).$$

Now we consider the term $K_{1,1}$. At first we remark that in $K_{1,1}^{(1)}$ the special case $2a_{k_4} = a_{k_2} + a_{k_3}$ may arise. If we exchange k_2 and k_3 in this case it is seen that the sum of the two terms in question vanishes. Consequently we assume

$2a_{k_4} \neq a_{k_2} + a_{k_3}$. Also in this case we exchange k_2 and k_3. Then

$$\begin{aligned} K_{1,1}^{(1)} &= \sum_{\vec{k}\in\pi(4)} \sum_{n_1^{A_4}\le x} b(k_i)\cdot \\ &\cdot\left\{\left(\frac{x}{n_1^{a_{k_1}}}\right)^{\frac{3}{a_{k_2}+a_{k_3}+a_{k_4}}} - n_1^2\left(\frac{x}{n_1^{a_{k_1}+a_{k_2}+a_{k_3}}}\right)^{\frac{1}{a_{k_4}}}\right\}, \end{aligned}$$

$$\begin{aligned} b(k_i) &= \frac{a_{k_4}^2}{2}\left(\frac{1}{a_{k_3}-a_{k_4}} + \frac{1}{a_{k_2}-a_{k_4}}\right)\frac{1}{2a_{k_4}-a_{k_2}-a_{k_3}} \\ &= -\frac{a_{k_4}^2}{2}\cdot\frac{1}{(a_{k_4}-a_{k_2})(a_{k_4}-a_{k_3})}. \end{aligned}$$

Now we see from this representation that the two terms in question also vanish in this case. Of course, the case $2a_{k_4} = a_{k_2} + a_{k_3}$ has been addressed before. Hence, we use this representation in both cases. Further we exchange k_4 and k_2 and also k_4 and k_3 with respect to the first term. Then we obtain

$$\begin{aligned} K_{1,1}^{(1)} &= \sum_{\vec{k}\in\pi(4)} \sum_{n_1^{A_4}\le x} \left\{ c(k_i)\left(\frac{x}{n_1^{a_{k_1}}}\right)^{\frac{3}{a_{k_2}+a_{k_3}+a_{k_4}}} - \right. \\ &\left. -b(k_i)\, n_1^2\left(\frac{x}{n_1^{a_{k_1}+a_{k_2}+a_{k_3}}}\right)^{\frac{1}{a_{k_4}}}\right\}, \end{aligned}$$

$$\begin{aligned} c(k_i) &= -\frac{1}{6}\left(\frac{a_{k_4}^2}{(a_{k_4}-a_{k_2})(a_{k_4}-a_{k_3})} + \frac{a_{k_2}^2}{(a_{k_2}-a_{k_4})(a_{k_2}-a_{k_3})} + \right. \\ &\left. +\frac{a_{k_3}^2}{(a_{k_3}-a_{k_2})(a_{k_3}-a_{k_4})}\right) \\ &= -\frac{1}{6}. \end{aligned}$$

Then we obtain for the sum $K_{1,1}^{(1)} + K_{1,1}^{(2)}$

$$K_{1,1} = \sum_{\vec{k}\in\pi(4)} \sum_{n_1^{A_4}\le x} \left\{ -b(k_i)\, n_1^2 \left(\frac{x}{n_1^{a_{k_1}+a_{k_2}+a_{k_3}}}\right)^{\frac{1}{a_{k_4}}} - \frac{1}{6}\, n_1^3 \right\}. \tag{15}$$

Finally, all things considered, we obtain

$$K_1 = \sum_{k=1}^{4} c_{2,k} x^{\frac{1}{a_k}} + K_{1,1} + O\left(x^{\frac{2}{A_4}}\right) \tag{16}$$

with the representation (15) for $K_{1,1}$.

In the end we use the EULER-MACLAURIN sum formula (11) in (15) with

$$f(t) = -b(k_i)\, t^2 \left(\frac{x}{n_1^{a_{k_1}+a_{k_2}+a_{k_3}}}\right)^{\frac{1}{a_{k_4}}} - \frac{1}{6}\, t^3,$$

$$a = 1, \qquad b = x^{\frac{1}{A_4}}.$$

Then

$$K_{1,1} = K_{1,1,1} + K_{1,1,2} + K_{1,1,3} + K_{1,1,4} + K_{1,1,5},$$

$$\begin{aligned}
K_{1,1,1} &= \sum_{\vec{k}\in\pi(4)} \left\{-b(k_i)\frac{a_{k_4}}{4a_{k_4} - A_4} - \frac{1}{24}\right\} x^{\frac{4}{A_4}},\\
K_{1,1,2} &= \sum_{\vec{k}\in\pi(4)} \left\{b(k_i)\frac{a_{k_4}}{4a_{k_4} - A_4} x^{\frac{1}{a_{k_4}}} + \frac{1}{24}\right\},\\
K_{1,1,3} &= \sum_{\vec{k}\in\pi(4)} \left\{-b(k_i) - \frac{1}{6}\right\} x^{\frac{3}{A_4}}\, \psi\left(x^{\frac{1}{A_4}}\right)\\
&= 0,\\
K_{1,1,4} &= -\frac{1}{2}\sum_{\vec{k}\in\pi(4)} \left\{b(k_i)\frac{a_{k_4}}{4a_{k_4} - A_4} x^{\frac{1}{a_{k_4}}} - \frac{1}{6}\right\},\\
K_{1,1,5} &= \sum_{\vec{k}\in\pi(4)} \int_1^{x^{1/A_4}} \left\{b(k_i)\frac{3a_{k_4} - A_4}{a_{k_4}}\, t_1 \left(\frac{x}{t_1^{A_4 - a_{k_4}}}\right)^{\frac{1}{a_{k_4}}} - \frac{t_1^2}{2}\right\} \psi(t_1)\, \mathrm{d}t_1\\
&\ll x^{\frac{2}{A_4}}.
\end{aligned}$$

In the representation of $K_{1,1,1}$ we exchange a_{k_1} with a_{k_2} and a_{k_1} with a_{k_4}. Then

$$K_{1,1,1} = \sum_{\vec{k}\in\pi(4)} \left(d(k_i) - \frac{1}{24} \right) x^{\frac{4}{A_4}},$$

$$d(k_i) = \frac{a_{k_4}^3}{6} \cdot \frac{1}{(a_{k_4} - a_{k_1})(a_{k_4} - a_{k_2})(a_{k_4} - a_{k_3})}.$$

Further we work as in the representation of $K_{1,1}^{(1)}$ and exchange a_{k_4} with a_{k_1} and a_{k_2} with a_{k_3}. Then it is easily seen that $K_{1,1,1} = 0$. Hence, we obtain

$$K_{1,1} = \sum_{k=1}^{4} c_{3,k} x^{\frac{1}{a_k}} + O\left(x^{\frac{2}{A_4}} \right)$$

and by inserting into (16)

$$K_1 = \sum_{k=1}^{4} c_{4,k} x^{\frac{1}{a_k}} + O\left(x^{\frac{2}{A_4}} \right). \tag{17}$$

After all we obtain from (9), (13), (14) and (17)

$$K(\vec{a}_4; x) = \sum_{k=1}^{4} C_k x^{\frac{1}{a_k}} + O\left(x^{\frac{2}{A_4}} \right)$$

and, by means of (8),

$$D(\vec{a}_4; x) = \sum_{k=1}^{4} C_k x^{\frac{1}{a_k}} + S(\vec{a}_4; x) + O\left(x^{\frac{2}{A_4}} \right).$$

The exact calculation of the coefficients C_k seems to be superflous. It must be

$$H(\vec{a}_4; x) = \sum_{k=1}^{4} C_k x^{\frac{1}{a_k}}.$$

Further it is

$$\Delta(\vec{a}_4; x) = S(\vec{a}_4; x) + O\left(x^{\frac{2}{A_4}} \right).$$

This proves the theorem.

REFERENCES

[1] KRÄTZEL, E.: *Lattice Points*, Dt. Verlag d. Wiss., Berlin und Kluwer Academic Publishers Dordrecht, Boston, London, 1988.

[2] KRÄTZEL, E.: *Estimates in the General Divisor Problem*, Abh. Math. Sem. Univ. Hamburg **62** (1992), 191–206.

[3] SCHMIDT, P.G.: *Zur Anzahl Abelscher Gruppen gegebener Ordnung*, J. Reine Angew. Math. **229** (1968), 34–42.
[4] SEIBOLD, R. – KRÄTZEL, E.: *Exponentialsummen und unsymmetrische Teilerprobleme*, Analysis **18** (1998), 201–215.
[5] VOGTS, M.: *Many-dimensional generalized divisor problems*, Math. Nachr. **124** (1985), 103–121.

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