# ON WEIGHTED DISTRIBUTION FUNCTIONS OF SEQUENCES 

Rita Giuliano Antonini - Oto Strauch


#### Abstract

In this paper we prove that the set of logarithmically weighted distribution functions of the sequence of iterated logarithm $\log ^{(i)} n \bmod 1, n=$ $n_{i}, n_{i}+1, \ldots$ is the same as the set of classical distribution functions of the sequence $\log ^{(i-1)} n \bmod 1$ for every $i=2,3, \ldots$. Also we prove that $\log (n \log n)$ $\bmod 1$ is logarithmically uniformly distributed. This implies that the sequence $p_{n} / n \bmod 1$, where $p_{n}$ denotes the $n$th prime, is also logarithmically uniformly distributed.


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## 1. Introduction

In this paper we study some connection between the set of weighted distribution functions and the set of classical distribution functions of sequences. For the definitions see Part 2.

In Part 3 we prove Theorems 1, 2 and 3 which give methods for computing the set of weighted distribution functions of some sequences $f(n) \bmod 1, n=$ $1,2, \ldots$, where $f(x)$ increases. These Theorems are a weighted generalization of Koksma's Theorem 7.7 in [KN, p. 58], Theorem 5 in [SB], and extend some results by J. Cigler (1960)[Cig] and J.H.B. Kemperman (1973)[Ke, pp.157-158] (for a discussion see Paragraph 3.1).

In Part 5, (Theorem 6), we give a formula for the set of weighted distribution functions of a given sequence. It involves a sequence having the same set, but formed by classical distribution functions.

[^0]In Theorem 7 we give a condition in order that two sequences $x(n)$ and $y(n)$, $n=1,2, \ldots$ such that $(x(n)-y(n)) \bmod 1 \rightarrow 0$, have the same sets of weighted distribution functions.

Also we prove a simple result (Theorem 8) which gives two conditions in order that the sets of weighted and non-weighted distribution functions of some sequences coincide. An open problem is to find other suitable conditions.

In Theorem 9 a criterion of connectivity of a set of weighted distribution functions is given.

In Part 6 we apply our Theorems 1, 2 and 3 to the set of logarithmically weighted distribution functions of the sequence of iterated logarithm $\log ^{(i)}(n)$ $\bmod 1, n=n_{i}, n_{i}+1, \ldots$ and we prove that it coincides with the set of classical distribution functions of $\log ^{(i-1)}(n) \bmod 1, n=n_{i}, n_{i}+1, \ldots$, for every $i=2,3, \ldots$. This implies, generally speaking, that two sequences having the same distribution functions do not need to have the same weighted distribution functions.

Furthermore, we also prove that the sequence $\log (n \log n) \bmod 1$ is $\operatorname{logarith}-$ mically weighted uniformly distributed and as a consequence, applying Theorem 3 , we prove that $p_{n} / n \bmod 1$ is logarithmically weighted uniformly distributed, where $p_{n}$ denotes the $n$th prime. Note that in [SB] it is proved that the sequence $p_{n} / n \bmod 1, n=1,2, \ldots$, has the same set of distribution functions as the sequence $\log n \bmod 1$.

All proofs are elementary and main parts employ the well-known CauchyStolz lemma, Helly theorems (see [SP, 4. Appendix]) and the Lagrange mean value theorem in the following form:
Cauchy-Stolz lemma: Let $x_{n}$ and $y_{n}, n=1,2, \ldots$, be the real-valued sequences. If $y_{n}$ is strictly monotone, $\left|y_{n}\right| \rightarrow \infty$, and if the limit (finite or infinite) $\lim _{n \rightarrow \infty} \frac{x_{n+1}-x_{n}}{y_{n+1}-y_{n}}$ exists, then the limit of the sequence $\frac{x_{n}}{y_{n}}$ also exists and $\lim _{n \rightarrow \infty} \frac{x_{n}}{y_{n}}=\lim _{n \rightarrow \infty} \frac{x_{n+1}-x_{n}}{y_{n+1}-y_{n}}$.
First Helly theorem: Any sequence $g_{n}$ of distribution functions contains a subsequence $g_{k_{n}}$ such that the sequence $g_{k_{n}}(x)$ converges for every $x \in[0,1]$ and its point limit $\lim _{n \rightarrow \infty} g_{k_{n}}(x)=g(x)$ is also a distribution function.
Second Helly theorem: If we have $\lim _{n \rightarrow \infty} g_{n}(x)=g(x)$ a.e. on $[0,1]$, then for every continuous function $f:[0,1] \rightarrow \mathbb{R}$ we have $\lim _{n \rightarrow \infty} \int_{0}^{1} f(x) \mathrm{d} g_{n}(x)=$ $\int_{0}^{1} f(x) \mathrm{d} g(x)$.
Lagrange theorem: Let $f:[a, b] \rightarrow \mathbb{R}$ be a continuous function in a closed interval $[a, b]$ with a finite or infinite derivative $f^{\prime}(x)$ in each point $x \in(a, b)$. Then there is at least one point $x=\gamma$ inside the interval such that $f(b)-f(a)=$ $f^{\prime}(\gamma)(b-a)$.

## 2. Definitions

Here we refer to the monographs [KN], [DT] and [SP]:

- Given a real number $x,[x]$ denotes the integer part of $x$, and $\{x\}$ is the fractional part of $x$ (i.e., $\{x\}=x \bmod 1$ ).
- A function $g:[0,1] \rightarrow[0,1]$ will be called distribution function (abbreviated d.f.) if $g(0)=0, g(1)=1$ and $g(x)$ is nondecreasing. We shall identify two d.f.s $g(x)$ and $\tilde{g}(x)$ if $g(x)=\tilde{g}(x)$ almost everywhere (i.e., in every point $x$ of common continuity). Note that any d.f. $g(x)$ can be seen as a Borel probability measure $\mu$, where $\mu([0, x))=\lim _{t \rightarrow x-0} g(t)$, but we prefer the notion of d.f.s since the graph of $g(x)$ can be represented by a picture.
- For a sequence $x_{n}, n=1,2, \ldots$ and a positive integer $N$ we define step d.f. $F_{N}(x)$ for $x \in[0,1)$ by $F_{N}(x)=\frac{1}{N} \sum_{n=1}^{N} c_{[0, x)}\left(\left\{x_{n}\right\}\right)$, while $F_{N}(1)=1$. Here $c_{[0, x)}(t)$ is the indicator function of the interval $[0, x)$.
- A d.f. $g(x)$ is called d.f. of the sequence $x_{n} \bmod 1$ if an increasing sequence of positive integer $N_{1}, N_{2}, \ldots$ exists such that $\lim _{k \rightarrow \infty} F_{N_{k}}(x)=g(x)$ holds at every point $x \in[0,1]$ of continuity of $g(x)$ and thus a.e. on $[0,1]$.
- The set of all d.f. of a sequence $x_{n} \bmod 1$ will be denoted by $G\left(x_{n} \bmod 1\right)$.
- Let $f(n)$ and $w(n), n=1,2, \ldots$, be two real-valued sequences. We assume that $w(n)>0$ and $\sum_{n=1}^{N} w(n) \rightarrow \infty$, as $N \rightarrow \infty$. We call $w(n)$ weights and we define the $w(n)$-weighted distribution function (shortly $w(n)$-d.f.) $g(x)$ of the sequence $f(n) \bmod 1, n=1,2, \ldots$, as follows:
- For every positive integer $N$ we consider the $w(n)$-step d.f.

$$
\begin{equation*}
F_{N}(x)=\frac{1}{\sum_{n=1}^{N} w(n)} \sum_{n=1}^{N} w(n) c_{[0, x)}(\{(f(n)\}) \tag{1}
\end{equation*}
$$

Assume that for an increasing sequence of indices $N_{k}, k=1,2, \ldots$, there exists the weak limit $g(x)$ of $F_{N_{k}}(x)$, i.e., $\lim _{i \rightarrow \infty} F_{N_{i}}(x)=g(x)$ for every continuity point $x \in[0,1]$ of $g(x)$. Then the d.f. $g(x)$ is called the d.f. of $f(n) \bmod 1$ with respect to the weights $w(n)$ (i.e. $w(n)$-d.f.). We denote as $G_{w(n)}(f(n) \bmod 1)$ the set of all such possible limits.

- If the set $G_{w(n)}(f(n) \bmod 1)$ is a singleton, say $G_{w(n)}(f(n) \bmod 1)=\{g(x)\}$, then $g(x)$ is said to be the $w(n)$-weighted asymptotic distribution function (shortly $w(n)$-a.d.f.) of the sequence $f(n) \bmod 1, n=1,2, \ldots$, and if $g(x)=x$ we say that the sequence $f(n) \bmod 1$ is $w(n)$-weighted uniformly distributed (shortly $w(n)$-u.d.).
- If $w(n)=1$ for $n=1,2, \ldots$, then in all the above notation the symbol $w(n)$ is omitted.
- If $w(n)=1 / n$, then the weights $w(n)$ are called logarithmic weights.
- The $w(n)$-weighted lower d.f. $\underline{g}(x)$ and the $w(n)$-weighted upper d.f. $\bar{g}(x)$ are defined as

$$
\underline{g}(x)=\inf _{g \in G_{w(n)}(f(n) \bmod 1)} g(x), \quad \bar{g}(x)=\sup _{g \in G_{w(n)}(f(n) \bmod 1)} g(x) .
$$

- $c_{u}(x)$ is the one-step d.f., i.e., $c_{u}(x)=0$ if $0 \leq x<u$ and $c_{u}(x)=1$ if $u \leq x \leq 1$. $h_{\beta}(x)$ is the constant d.f., i.e., $h_{\beta}(x)=\beta$ if $x \in(0,1)$. In all cases $c_{u}(0)=h_{\beta}(0)=0$ and $c_{u}(1)=h_{\beta}(1)=1$.
- Two sequences $f(n) \bmod 1, h(n) \bmod 1, n=1,2, \ldots$ are statistically independent if every two-dimensional d.f. $g(x, y) \in G((f(n) \bmod 1, h(n) \bmod 1))$ has the form $g(x, y)=g(x, 1) \cdot g(1, y)$ for every continuity point $(x, y) \in[0,1]^{2}$ of $g(x, y)$ (see [SP, p. 1-17, 1.8.9.]).


## 3. Main results

In the following, let
(I) $f(x)$ and $w(x)>0$ be two real-valued functions defined for $x \geq 1$ such that $f(x)$ is strictly increasing with its inverse function $f^{-1}(x)$.
(II) Put $\int_{1}^{x} w(t) \mathrm{d} t=M(x)$ for $x \geq 1$ and express $\sum_{n \in[x, y)} w(n)=M(y)-$ $M(x)+\theta(x, y)$ for $1 \leq x<y$.
Assume that the following limits exist:
(III) $\lim _{k \rightarrow \infty} \frac{M\left(f^{-1}(k+x)\right)-M\left(f^{-1}(k)\right)}{M\left(f^{-1}(k+1)\right)-M\left(f^{-1}(k)\right)}=\tilde{g}(x)$ for each $x \in[0,1]$, point of continuity of $\tilde{g}(x)$;
(IV) $\lim _{k \rightarrow \infty} \frac{M\left(f^{-1}(k+u)\right.}{M\left(f^{-1}(k)\right)}=\psi(u)$ for each $u \in[0,1]$, point of continuity of $\psi(u)$, or $\psi(u)=\infty$ for $u>0$;
(V) $\lim _{k \rightarrow \infty} M\left(f^{-1}(k+1)\right)-M\left(f^{-1}(k)\right)=\infty$ and $|\theta(x, y)| \leq c$ for $1 \leq x \leq y$; or alternatively
$\left(V^{\prime}\right) \lim _{k \rightarrow \infty} M\left(f^{-1}(k+1)\right)-M\left(f^{-1}(k)\right)=c>0$ and $\lim _{x \rightarrow \infty} \theta(x, y)=0$.
For computing $G_{w(n)}(f(n) \bmod 1)$ we can use the following three theorems.

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Theorem 1. If $1<\psi(1)<\infty$ and $f^{\prime}(x) \rightarrow 0$ as $x \rightarrow \infty$, then

$$
\begin{equation*}
G_{w(n)}(f(n) \bmod 1)=\left\{g_{u}(x)=\frac{\min (\psi(x), \psi(u))-1}{\psi(u)}+\frac{1}{\psi(u)} \tilde{g}(x) ; u \in[0,1]\right\}, \tag{2}
\end{equation*}
$$

where $\tilde{g}(x)=\frac{\psi(x)-1}{\psi(1)-1}$ and $F_{N_{i}}(x) \rightarrow g_{u}(x)$ as $i \rightarrow \infty$ if and only if $f\left(N_{i}\right) \bmod 1 \rightarrow u$. The $w(n)$-lower d.f. $g(x)$ and the $w(n)$-upper d.f. $\bar{g}(x)$ of $f(n) \bmod 1$ are

$$
\underline{g}(x)=\tilde{g}(x), \quad \bar{g}(x)=1-\frac{1}{\psi(x)}(1-\tilde{g}(x)) .
$$

Furthermore $\underline{g}(x)=g_{0}(x)=g_{1}(x)$ belongs to $G_{w(n)}(f(n) \bmod 1)$ but $\bar{g}(x)=$ $g_{x}(x)$ does not.

Theorem 2. If $\psi(1)=1$, then the sequence $f(n) \bmod 1, n=1,2, \ldots$ has $w(n)$-a.d.f. $\tilde{g}(x)$, i.e.

$$
\begin{equation*}
G_{w(n)}(f(n) \bmod 1)=\{\tilde{g}(x)\} . \tag{3}
\end{equation*}
$$

Theorem 3. Let $\psi(u)=\infty$, for every $u>0$ and assume that for $u=0$ the limit $\psi(u)$ is not defined in the way that for every $t \in[0, \infty)$ there exists a sequence $u(k) \rightarrow 0$ such that
(i) $\lim _{k \rightarrow \infty} \frac{M\left(f^{-1}(k+u(k))\right)}{M\left(f^{-1}(k)\right)}=t$. Moreover assume that
(ii) $\lim _{x \rightarrow \infty} \frac{w(x)}{M(x)}=0$.

Then we have

$$
\begin{equation*}
G_{w(n)}(f(n) \bmod 1)=\left\{c_{u}(x) ; u \in[0,1]\right\} \cup\left\{h_{\beta}(x) ; \beta \in[0,1]\right\} \tag{4}
\end{equation*}
$$

where $F_{N_{i}} \rightarrow c_{u}(x)$ if and only if $f\left(N_{i}\right) \bmod 1 \rightarrow u>0$ and $F_{N_{i}} \rightarrow h_{\beta}(x)$ if and only if $f\left(N_{i}\right) \bmod 1 \rightarrow 0$ and $\frac{M\left(f^{-1}\left(\left[f\left(N_{i}\right)\right]\right)\right)}{M\left(N_{i}\right)} \rightarrow 1-\beta$.

### 3.1. Comments

1. In the case $\psi(1)=1$ in Theorem 2 , the limit (III) can be any d.f. $\tilde{g}(x)$. To check this, put $H(x)=M\left(f^{-1}(x)\right)$ and $H(k+x)=k+\tilde{g}(x)$ for $x \in[0,1]$. Then
(IV) $\frac{H(k+1)}{H(k)}=\frac{k+1}{k} \rightarrow 1$, and
(III) $\frac{H(k+x)-H(k)}{H(k+1)-H(k)}=\tilde{g}(x)$. Similarly, for $H(k+x)=(k+\tilde{g}(x))^{2}$.
2. The situation is different if we replace the limit
(III) $\lim _{k \rightarrow \infty} \frac{M\left(f^{-1}(k+x)\right)-M\left(f^{-1}(k)\right)}{M\left(f^{-1}(k+1)\right)-M\left(f^{-1}(k)\right)}=\tilde{g}(x)$ for $x \in[0,1]$ by the
(III') $\lim _{t \rightarrow \infty} \frac{M\left(f^{-1}(t+x)\right)-M\left(f^{-1}(t)\right)}{M\left(f^{-1}(t+1)\right)-M\left(f^{-1}(t)\right)}=\tilde{g}(x)$ for $x \in[0,1]$, where $t$ is a real variable.

This limit was introduced by J. Cigler (1960)[Cig], generalizing J.F. Koksma's (1936) result on lower and upper d.f.s of $f(n) \bmod 1$, see [Ko, p. 88]. J.H.B. Kemperman (1973)[Ke, Th. 11] proved that in (III') exactly one of the following relations must hold:
(a) $\tilde{g}(x)=x$ for $x \in[0,1]$;
(b) $\tilde{g}(x)=\frac{e^{c x}-1}{e^{c}-1}$ for $x \in[0,1]$;
(c) $\tilde{g}(x)=0$ for $x \in(0,1)$.

Furthermore he remarked that
(a) In the case $\tilde{g}(x)=x$, the sequence $f(n) \bmod 1$ is u.d.
(b) In the case $\tilde{g}(x)=\frac{e^{c x}-1}{e^{c}-1}$ the set $G_{w(n)}(f(n) \bmod 1)$ is described by the following Theorem 4, where $t=1 / c$.
(c) The case $\tilde{g}(x)=0$ is equivalent to $\lim _{t \rightarrow \infty} \frac{M\left(f^{-1}(t+x)\right)}{M\left(f^{-1}(t)\right)}=\infty$ and the set $G_{w(n)}(f(n) \bmod 1)$ contains only one-step d.f.s $c_{u}(x)$, where $F_{N_{i}}(x) \rightarrow$ $c_{u}(x)$ if and only if $f\left(N_{i}\right) \bmod 1 \rightarrow u$.
3. For the sequence $f(n) \bmod 1$ with increasing $f(x)$ Kemperman $[\mathrm{Ke}]$ proved the following two theorems which are different in nature from our Theorems 1, 2 and 3.

Theorem 4 (Kemperman [Ke, Th.9]). Assume that
(j) $\lim _{n \rightarrow \infty} f(n+1)-f(n)=0$;
(jj) $\lim _{n \rightarrow \infty} \frac{w(1)+w(2)+\cdots+w(n)}{w(n)}(f(n+1)-f(n))=t$.
Then $G_{w(n)}(f(n) \bmod 1)$ contains only d.f.s of the type

$$
g_{u}(x)= \begin{cases}\frac{e^{(1+x-u) / t}-e^{(1-u) / t}}{e^{1 / t}-1} & \text { if } 0 \leq x \leq u  \tag{5}\\ 1-\frac{e^{(1-u) / t}-e^{(x-u) / t}}{e^{1 / t}-1} & \text { if } u<x \leq 1\end{cases}
$$

Here the density $g_{u}^{\prime}(x)$ of $g_{u}(x)$ has the form

$$
g_{u}^{\prime}(x)=\frac{e^{\{x-u\} / t}}{t\left(e^{1 / t}-1\right)}
$$

Furthermore $F_{N_{i}}(x) \rightarrow g_{u}(x)$ if and only if $f\left(N_{i}\right) \bmod 1 \rightarrow u$.
Theorem 5 (Kemperman [Ke, p. 148, Coroll. 1]). Assume that
(j) $\lim _{n \rightarrow \infty} f(n+1)-f(n)=0$;
(jj) $\lim _{n \rightarrow \infty} \frac{w(1)+w(2)+\cdots+w(n)}{w(n)}(f(n+1)-f(n))=+\infty$.
(jjj) $\frac{f(n+1)-f(n)}{w(n)}$ is monotone in $n$.
Then the sequence $f(n) \bmod 1, n=1,2, \ldots$ is $w(n)-u . d$.

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## 4. Proofs of Theorems 1, 2 and 3

For a positive integer $N$ define

- $K=K(N)=[f(N)]$,
- $u(N)=\{f(N)\}$,
- $S_{N}([x, y))=\sum_{n=1, f(n) \in[x, y)}^{N} w(n)$,
- $W_{N}=\sum_{n=1}^{N} w(n)$.

Clearly $f^{-1}(K+u(N))=N$ and for every $x \in[0,1]$ by the very definition of $w(n)$-step d.f. $F_{N}(x)$ we have ${ }^{1}$

$$
\begin{aligned}
F_{N}(x) & =\frac{\sum_{k=0}^{K-1} S_{N}([k, k+x))+S_{N}([K, K+x) \cap[K, K+u(N)))}{W_{N}} \\
& +\frac{O\left(S_{N}([0, x))\right.}{W_{N}}
\end{aligned}
$$

From the monotonicity of $f(x)$ (assumption (I)) it follows that $S_{N}([x, y))=$ $\sum_{n=1, n \in\left[f^{-1}(x), f^{-1}(y)\right)}^{N} w(n)$, and $W_{N}=S_{N}([0, K+u(N))$ and we have

- $S_{N}([x, y))=M\left(f^{-1}(y)\right)-M\left(f^{-1}(x)\right)+\theta\left(f^{-1}(x), f^{-1}(y)\right)$,
- $W_{N}=M\left(f^{-1}(K+u(N))\right)+\theta\left(f^{-1}(0), f^{-1}(K+u(N))\right)$
and thus

$$
\begin{aligned}
F_{N}(x) & =\frac{\sum_{k=0}^{K-1}\left(M\left(f^{-1}(k+x)\right)-M\left(f^{-1}(k)\right)\right)}{W_{N}} \\
& +\frac{\min \left(M\left(f^{-1}(K+x)\right), M\left(f^{-1}(K+u(N))\right)\right)-M\left(f^{-1}(K)\right)}{W_{N}} \\
& +\frac{O\left(\sum_{k=0}^{K} \theta\left(f^{-1}(k), f^{-1}(k+x)\right)\right.}{W_{N}}+\frac{O\left(\theta\left(f^{-1}(K), f^{-1}(K+u(N))\right)\right)}{W_{N}} .
\end{aligned}
$$

Assumption (V) implies that $1 /\left(M\left(f^{-1}(k+1)\right)-M\left(f^{-1}(k)\right)\right) \rightarrow 0$; this in turn yields that $K / M\left(f^{-1}(K)\right) \rightarrow 0$ by the Cauchy-Stolz lemma, which gives

$$
\frac{O\left(\sum_{k=0}^{K} \theta\left(f^{-1}(k), f^{-1}(k+x)\right)\right)}{W_{N}}=\frac{O(K)}{W_{N}} \rightarrow 0 .
$$

Assumption $\left(\mathrm{V}^{\prime}\right)$ implies that $1 /\left(M\left(f^{-1}(k+1)\right)-M\left(f^{-1}(k)\right)\right) \rightarrow(1 / c)$; this gives that $K / M\left(f^{-1}(K)\right) \rightarrow(1 / c)$ by the Cauchy-Stolz lemma, so that

$$
\frac{O\left(\sum_{k=0}^{K} \theta\left(f^{-1}(k), f^{-1}(k+x)\right)\right)}{W_{N}}=\frac{O\left(\sum_{k=0}^{K} \theta\left(f^{-1}(k), f^{-1}(k+x)\right)\right)}{K} \cdot \frac{K}{W_{N}} \rightarrow 0 .
$$

[^1]Thus in both cases

$$
F_{N}(x)=F_{N}^{(1)}(x)+F_{N}^{(2)}(x)+o(1),
$$

where

$$
\begin{aligned}
F_{N}^{(1)}(x) & =\frac{\sum_{k=0}^{K-1}\left(M\left(f^{-1}(k+x)\right)-M\left(f^{-1}(k)\right)\right)}{M\left(f^{-1}(K+u(N))\right)}, \\
F_{N}^{(2)}(x) & =\frac{\min \left(M\left(f^{-1}(K+x)\right), M\left(f^{-1}(K+u(N))\right)\right)-M\left(f^{-1}(K)\right)}{M\left(f^{-1}(K+u(N))\right)} .
\end{aligned}
$$

We shall express the first term $F_{N}^{(1)}(x)$ as

$$
F_{N}^{(1)}(x)=\frac{\sum_{k=0}^{K-1}\left(M\left(f^{-1}(k+x)\right)-M\left(f^{-1}(k)\right)\right)}{\sum_{k=0}^{K-1}\left(M\left(f^{-1}(k+1)\right)-M\left(f^{-1}(k)\right)\right)} \cdot \frac{M\left(f^{-1}(K)\right)-M\left(f^{-1}(0)\right)}{M\left(f^{-1}(K+u(N))\right)}
$$

and by the Cauchy-Stolz lemma and assumption (III),

$$
\lim _{K \rightarrow \infty} \frac{\sum_{k=0}^{K-1}\left(M\left(f^{-1}(k+x)\right)-M\left(f^{-1}(k)\right)\right)}{\sum_{k=0}^{K-1}\left(M\left(f^{-1}(k+1)\right)-M\left(f^{-1}(k)\right)\right)}=\tilde{g}(x) .
$$

Now, let $F_{N_{i}}(x) \rightarrow g(x)$. Then there exists a subsequence $N_{i}^{\prime}$ of $N_{i}$ such that $u\left(N_{i}^{\prime}\right)=u_{i}^{\prime} \rightarrow u^{\prime}$ for some $u^{\prime} \in[0,1]$. Thus, in the following we can assume that $F_{N_{i}}(x) \rightarrow g(x)$, and
$K_{i}=\left[f\left(N_{i}\right)\right]$, and $u\left(N_{i}\right)=u_{i} \rightarrow u$, simultaneously.
We shall prove Theorems 1, 2, and 3 one by one.
Proof of Theorem 1. In this case $\tilde{g}(x)=\frac{\psi(x)-1}{\psi(1)-1}$, and the relation $u_{i} \rightarrow u$ implies ${ }^{2}$

$$
\begin{aligned}
& \frac{M\left(f^{-1}\left(K_{i}\right)\right)-M\left(f^{-1}(0)\right)}{M\left(f^{-1}\left(K_{i}+u_{i}\right)\right)} \rightarrow \frac{1}{\psi(u)} ; \\
& F_{N_{i}}^{(1)}(x) \rightarrow \frac{\psi(x)-1}{\psi(1)-1} \cdot \frac{1}{\psi(u)} ; \\
& F_{N_{i}}^{(2)}(x) \rightarrow \frac{\min (\psi(x), \psi(u))-1}{\psi(u)} .
\end{aligned}
$$

Thus the $w(n)$-d.f. $g(x)$ has the form $g(x)=g_{u}(x)$. On the other hand, for every $u \in[0,1]$ there exists an increasing sequence of indices $N_{i}$ such that $u\left(N_{i}\right)=$ $u_{i} \rightarrow u$. It follows from the assumption $f^{\prime}(x) \rightarrow 0$ because for some $\varepsilon_{i} \rightarrow 0$ we can find $N_{i} \in f^{-1}\left(\left(K_{i}+u-\varepsilon_{i}, K_{i}+u+\varepsilon_{i}\right)\right)$.

[^2]
## on weighted distribution functions of sequences

We find the $w(n)$-lower and upper d.f. by computing $g_{u}(x)$ for fixed $x \in[0,1]$ and $u \in[0,1]$

$$
g_{u}(x)= \begin{cases}1-\frac{1}{\psi(u)}\left(1-\frac{\psi(x)-1}{\psi(1)-1}\right) & \text { if } u \leq x \\ \frac{\psi(1)}{\psi(u)} \frac{\psi(x)-1}{\psi(1)-1} & \text { if } u \geq x\end{cases}
$$

thus $\inf _{u \leq x} g_{u}(x)=g_{0}(x), \inf _{u \geq x} g_{u}(x)=g_{1}(x)$ and since $g_{0}(x)=g_{1}(x)$ we have $\underline{g}(x)=g_{0}(x)=g_{1}(x)=\frac{\psi(x)-1}{\psi(1)-1}$. Similarly, $\sup _{u \leq x} g_{u}(x)=g_{x}(x), \sup _{u \geq x} g_{u}(x)=$ $g_{x}(x)$, and thus $\bar{g}(x)=g_{x}(x) \notin G_{\omega(n)}(f(n) \bmod 1)$.

Proof of Theorem 2. In this case we cannot use the relation $\tilde{g}(x)=\frac{\psi(x)-1}{\psi(1)-1}$ and moreover $\psi(u)=1$ for all $u \in[0,1]$. Then

$$
\begin{aligned}
& F_{N_{i}}^{(1)}(x) \rightarrow \tilde{g}(x) \frac{1}{1} \\
& F_{N_{i}}^{(2)}(x) \rightarrow \frac{\min (1,1)-1}{1}
\end{aligned}
$$

and thus $g(x)=\tilde{g}(x)$ for $x \in[0,1)$.
Proof of Theorem 3. In this case

$$
\tilde{g}(x)=\lim _{k \rightarrow \infty} \frac{M\left(f^{-1}(k+x)\right)-M\left(f^{-1}(k)\right)}{M\left(f^{-1}(k+1)\right)-M\left(f^{-1}(k)\right)}=0
$$

for every $x \in[0,1)$, thus $F_{N_{i}}^{(1)}(x) \rightarrow 0$. To compute the limit of $F_{N_{i}}^{(2)}(x)$ we distinguish the two following cases, where $u_{i} \rightarrow u$.
$1^{0}$. If $u>0$, then
$\lim _{i \rightarrow \infty} F_{N_{i}}^{(2)}(x)=\frac{\min \left(M\left(f^{-1}\left(K_{i}+x\right)\right), M\left(f^{-1}\left(K_{i}+u_{i}\right)\right)\right)-M\left(f^{-1}\left(K_{i}\right)\right)}{M\left(f^{-1}\left(K_{i}+u_{i}\right)\right)}=c_{u}(x)$.
$2^{0}$. If $u=0$, then

$$
F_{N_{i}}^{(2)}(x)=1-\frac{M\left(f^{-1}\left(K_{i}\right)\right)}{M\left(f^{-1}\left(K_{i}+u_{i}\right)\right)}
$$

From $K_{i}$ we select $K_{i}^{\prime}$ (i.e. from $N_{i}$ we select a subsequence $N_{i}^{\prime}$ ) such that

$$
\begin{equation*}
\lim _{i \rightarrow \infty} \frac{M\left(f^{-1}\left(K_{i}^{\prime}\right)\right)}{M\left(f^{-1}\left(K_{i}^{\prime}+u_{i}^{\prime}\right)\right)}=t \in[0,1] \tag{6}
\end{equation*}
$$

where again $K_{i}^{\prime}=\left[f\left(N_{i}^{\prime}\right)\right]$ and $u_{i}^{\prime}=\left\{f\left(N_{i}^{\prime}\right)\right\}$. Then we have $F_{N_{i}^{\prime}}^{(2)}(x) \rightarrow h_{\beta}(x)$, where $\beta=1-t$ and $h_{\beta}(x)=\beta$ for $x \in(0,1)$. On the other hand (by assumptions (i) and (ii)) for any given $t \in[0,1]$ there exists a sequence of positive integers $K_{i}$ and real numbers $u_{i} \rightarrow 0$ such that $\lim _{i \rightarrow \infty} \frac{M\left(f^{-1}\left(K_{i}\right)\right)}{M\left(f^{-1}\left(K_{i}+u_{i}\right)\right)}=t$. Then there
exist integers $K_{i}^{\prime}\left(K_{i}^{\prime}=K_{i}\right.$ for almost all $\left.i\right)$ and real numbers $u_{i}^{\prime} \in(0,1)$ such that $N_{i}^{\prime}=f^{-1}\left(K_{i}^{\prime}+u_{i}^{\prime}\right)$ is an integer and

$$
\left|f^{-1}\left(K_{i}+u_{i}\right)-f^{-1}\left(K_{i}^{\prime}+u_{i}^{\prime}\right)\right|<1 .
$$

Thus $\left|M\left(f^{-1}\left(K_{i}+u_{i}\right)\right)-M\left(f^{-1}\left(K_{i}^{\prime}+u_{i}^{\prime}\right)\right)\right| \leq w\left(N_{i}^{\prime}\right)+O(1)$ and applying the relation $\frac{w(x)}{M(x)} \rightarrow 0$ we have (6) again.

## 5. Further results

Theorem 6. Let the function $f(x)$ and weights $w(x)$ satisfy assumptions (I)(V) in Part 3 and either assumptions of Theorem 1 or Theorem 2 or Theorem 3. In case Theorem 1 we assume in addition that $\lim _{x \rightarrow \infty} \frac{f^{\prime}(x)}{M^{\prime}(x)}=0$. Then we have

$$
\begin{equation*}
G_{w(n)}(f(n) \bmod 1)=G\left(f\left(M^{-1}(n)\right) \bmod 1\right) . \tag{7}
\end{equation*}
$$

Proof. Denote $h(x)=f\left(M^{-1}(x)\right)$. Let $F_{N}(x)$ be the step d.f. of the sequence $f(n) \bmod 1$ with respect to the weights $w(n)$ and let $F_{N}^{*}(x)$ be the step d.f. of the sequence $h(n) \bmod 1$ with respect to the weights 1 . For a positive integer $N$ define again

- $K=K(N)=[f(N)], K^{*}=K^{*}(N)=[h(N)] ;$
- $u(N)=\{f(N)\}, u^{*}(N)=\{h(N)\}$;
- $S_{N}([x, y))=\sum_{n=1, f(n) \in[x, y)}^{N} w(n), A_{N}([x, y))=\sum_{n=1, h(n) \in[x, y)}^{N} 1$;
- $W_{N}=\sum_{n=1}^{N} w(n)$.

By the very definition we have

$$
\begin{aligned}
F_{N}(x) & =\frac{\sum_{k=0}^{K-1} S_{N}([k, k+x))+S_{N}([K, K+x) \cap[K, K+u(N)))}{W_{N}}+\frac{O(1)}{W_{N}} \\
F_{N}^{*}(x) & =\frac{\sum_{k=0}^{K^{*}-1} A_{N}([k, k+x))+A_{N}\left(\left[K^{*}, K^{*}+x\right) \cap\left[K^{*}, K^{*}+u^{*}(N)\right)\right)}{N} \\
& +\frac{O(1)}{N}
\end{aligned}
$$

Assumption (V) and the relations

$$
\begin{aligned}
& h^{-1}(x)=M\left(f^{-1}(x)\right) ; \\
& \sum_{n \in[x, y)} 1=y-x+\theta^{\prime}(x, y), \text { where }\left|\theta^{\prime}(x, y)\right| \leq 1 ; \\
& W_{N}=h^{-1}(K+u(N))+O(1) ; \\
& N=h^{-1}\left(K^{*}+u^{*}(N)\right),
\end{aligned}
$$

imply

$$
\begin{aligned}
F_{N}(x) & =\frac{\sum_{k=0}^{K-1}\left(h^{-1}(k+x)-h^{-1}(k)\right)}{W_{N}} \\
& +\frac{\min \left(h^{-1}(K+x), h^{-1}(K+u(N))\right)-h^{-1}(K)}{W_{N}}+\frac{O(K)}{W_{N}} \\
F_{N}^{*}(x) & =\frac{\sum_{k=0}^{K^{*}-1}\left(h^{-1}(k+x)-h^{-1}(k)\right)}{N} \\
& +\frac{\min \left(h^{-1}\left(K^{*}+x\right), h^{-1}\left(K^{*}+u^{*}(N)\right)\right)-h^{-1}\left(K^{*}\right)}{N}+\frac{O\left(K^{*}\right)}{N}
\end{aligned}
$$

and

$$
\frac{O(K)}{W_{N}} \rightarrow 0, \quad \frac{O\left(K^{*}\right)}{N} \rightarrow 0
$$

Note that we cannot use ( $\mathrm{V}^{\prime}$ ) since this implies $\frac{O(K)}{W_{N}} \rightarrow 0$ but not $\frac{O\left(K^{*}\right)}{N} \rightarrow 0$ because we have only $\left|\theta^{\prime}(x, y)\right| \leq 1$.

Theorem 7. Let $x(n)$ and $y(n)$ be two given real sequences and $w(n)>0$, $n=1,2, \ldots$ be a sequence of weights such that $W_{N}=\sum_{n=1}^{N} w(n) \rightarrow \infty$ as $N \rightarrow \infty$. Assume that all the d.f.s in $G_{w(n)}(x(n) \bmod 1)$ are continuous at 0 and 1. Then

$$
\begin{equation*}
\lim _{n \rightarrow \infty}(x(n)-y(n)) \bmod 1=0 \Longrightarrow G_{w(n)}(x(n) \bmod 1)=G_{w(n)}(y(n) \bmod 1) \tag{8}
\end{equation*}
$$

The same implication follows from the continuity of d.f.s in $G_{w(n)}(y(n) \bmod 1)$ at 0 and 1 .

Proof. We shall adapt a proof of a similar result (Theorem 1 of [SB]) in which the weights $w(n)=1$ were used. Assume that, for an increasing sequence of positive integers $N_{k}, k=1,2, \ldots$,

$$
\begin{aligned}
& F_{N_{k}}(x)=\frac{1}{W_{N_{k}}} \sum_{n=1}^{N_{k}} w(n) c_{[0, x)}(\{x(n)\}) \rightarrow g(x), \\
& \tilde{F}_{N_{k}}(x)=\frac{1}{W_{N_{k}}} \sum_{n=1}^{N_{k}} w(n) c_{[0, x)}(\{y(n)\}) \rightarrow \tilde{g}(x)
\end{aligned}
$$

for every continuity point $x \in[0,1]$. By Riemann-Stieltjes integration and the well-known second Helly theorem

$$
\begin{aligned}
& \frac{1}{W_{N_{k}}} \sum_{n=1}^{N_{k}} w(n) e^{2 \pi i h x(n)}=\int_{0}^{1} e^{2 \pi i h x} \mathrm{~d} F_{N_{k}}(x) \rightarrow \int_{0}^{1} e^{2 \pi i h x} \mathrm{~d} g(x), \\
& \frac{1}{W_{N_{k}}} \sum_{n=1}^{N_{k}} w(n) e^{2 \pi i h y(n)}=\int_{0}^{1} e^{2 \pi i h x} \mathrm{~d} \tilde{F}_{N_{k}}(x) \rightarrow \int_{0}^{1} e^{2 \pi i h x} \mathrm{~d} \tilde{g}(x) .
\end{aligned}
$$

Since,

$$
\begin{aligned}
& \left|\frac{1}{W_{N_{k}}} \sum_{n=1}^{N_{k}} w(n)\left(e^{2 \pi i h x(n)}-e^{2 \pi i h y(n)}\right)\right| \\
= & \left|\frac{1}{W_{N_{k}}} \sum_{n=1}^{N_{k}} w(n)\left(e^{2 \pi i h(x(n)-[x(n)-y(n)])}-e^{2 \pi i h y(n)}\right)\right| \\
\leq & \frac{2 \pi h}{W_{N_{k}}} \sum_{n=1}^{N_{k}} w(n)|x(n)-[x(n)-y(n)]-y(n)|=\frac{2 \pi h}{W_{N_{k}}} \sum_{n=1}^{N_{k}} w(n)\{x(n)-y(n)\}
\end{aligned}
$$

the relation $\{x(n)-y(n)\} \rightarrow 0$ implies that $\int_{0}^{1} e^{2 \pi i h x} \mathrm{~d} g(x)=\int_{0}^{1} e^{2 \pi i h x} \mathrm{~d} \tilde{g}(x)$ for every $h= \pm 1, \pm 2, \ldots$. Thus for every continuous $h:[0,1] \rightarrow \mathbb{R}, h(0)=h(1)$, we have

$$
\int_{0}^{1} h(x) \mathrm{d} g(x)=\int_{0}^{1} h(x) \mathrm{d} \tilde{g}(x), \text { i.e. } \int_{0}^{1} g(x) \mathrm{d} h(x)=\int_{0}^{1} \tilde{g}(x) \mathrm{d} h(x) .
$$

For two common points $0<x_{1}<x_{2}<1$ of continuity of $g(x)$ and $\tilde{g}(x)$ and for sufficiently small $\Delta>0$, define

$$
\begin{aligned}
& h(x)=0 \text { for } x \in\left[0, x_{1}-\Delta\right], \\
& h^{\prime}(x)=1 / \Delta \text { for }\left(x_{1}-\Delta, x_{1}\right), \\
& h(x)=1 \text { for }\left[x_{1}, x_{2}-\Delta\right], \\
& h^{\prime}(x)=-1 / \Delta \text { for }\left(x_{2}-\Delta, x_{2}\right), \text { and } \\
& h(x)=0 \text { for }\left[x_{2}, 1\right] .
\end{aligned}
$$

Then

$$
\begin{aligned}
& \int_{0}^{1} g(x) \mathrm{d} h(x)=\frac{1}{\Delta} g\left(x_{1}\right) \Delta-\frac{1}{\Delta} g\left(x_{2}\right) \Delta+O(\Delta), \\
& \int_{0}^{1} \tilde{g}(x) \mathrm{d} h(x)=\frac{1}{\Delta} \tilde{g}\left(x_{1}\right) \Delta-\frac{1}{\Delta} \tilde{g}\left(x_{2}\right) \Delta+O(\Delta),
\end{aligned}
$$

and $\Delta \rightarrow 0$ gives

$$
g\left(x_{1}\right)-\tilde{g}\left(x_{1}\right)=g\left(x_{2}\right)-\tilde{g}\left(x_{2}\right) .
$$

Now, assume that $g\left(x_{1}\right) \neq \tilde{g}\left(x_{1}\right)$ and $g\left(x_{2}\right) \neq \tilde{g}\left(x_{2}\right)$. Fixing $x_{1}$ and letting $x_{2} \rightarrow 1$ we have that one of $g, \tilde{g}$ must be discontinuous at 1 , and fixing $x_{2}$, letting $x_{1} \rightarrow 0$, then one of $g, \tilde{g}$ must be discontinuous at 0 .

Remark 1. By the referee, if $x(n), y(n) \in[0,1), n=1,2, \ldots$, then the assumption that all the d.f.s in $G_{w(n)}(x(n) \bmod 1)$ are continuous at 0 and 1 (in Theorem 7) can be omitted. The referee's proof: Given a sequence $x=(x(n))$, $y=(y(n))$ and a sequence of weights $w=(w(n))$ we can define the measures $\mu(x, w, N)=\frac{1}{W_{n}} \sum_{n=1}^{N} w(n) \delta_{x(n)}$ where, as usual, $\delta_{x(n)}$ denotes the point measure in the point $x(n)$. Let $d$ be a compatible metric on the space of all measures (equivalently: distribution functions). Then (by uniform continuity) $|x(n)-y(n)| \rightarrow 0$ implies $d\left(\delta_{x(n)}, \delta_{y(n)}\right) \rightarrow 0$. Thus the assumption $W_{N} \rightarrow \infty$ easily yields $d(\mu(x, w, N), \mu(y, w, N)) \rightarrow 0$. As a consequence the corresponding sets of d.f.s have to coincide.

Theorem 8. Let $f(x), w(x) \in[0,1)$ be two functions defined for $x \geq 1$. If
(i) the sequences $f(n)$ and $w(n)$ are statistically independent, and
(ii) $\sum_{n=1}^{N} w(n) \geq c . N$ for $N=1,2, \ldots$, where $c$ is a positive constant,
then

$$
G_{w(n)}(f(n))=G(f(n))
$$

Proof. Let $h(x)$ be a continuous function defined on $[0,1]$ and $F_{N}(x)$ be the step $w(n)$-weighted d.f. of the sequence $f(n), F_{N}^{(1)}(x)$ be the step d.f. of $f(n)$ with weights 1 and $F_{N}^{(2)}(x)$ be the step d.f. of the sequence $w(n)$ again with weights 1. Assume that for the sequence $N_{i}$ of indices we have

$$
\begin{aligned}
& F_{N_{i}}(x) \rightarrow g(x), \\
& F_{N_{i}}^{(1)}(x) \rightarrow g_{1}(x), \\
& F_{N_{i}}^{(2)}(x) \rightarrow g_{2}(x),
\end{aligned}
$$

in the points $x$ of continuity. By (i), (ii) and by Helly's second theorem we have

$$
\begin{aligned}
& \frac{1}{\sum_{n=1}^{N_{i}} w(n)} \sum_{n=1}^{N_{i}} w(n) h(f(n)) \rightarrow \int_{0}^{1} h(x) \mathrm{d} g(x) \\
& \frac{1}{\frac{1}{N_{i}} \sum_{n=1}^{N_{i}} w(n)} \frac{1}{N_{i}} \sum_{n=1}^{N_{i}} w(n) h(f(n)) \rightarrow \frac{1}{\int_{0}^{1} x \mathrm{~d} g_{2}(x)} \int_{0}^{1} x \mathrm{~d} g_{2}(x) \int_{0}^{1} h(x) \mathrm{d} g_{1}(x)
\end{aligned}
$$

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Thus $\int_{0}^{1} h(x) \mathrm{d} g(x)=\int_{0}^{1} h(x) \mathrm{d} g_{1}(x)$ for any continuous $h(x)$, thus $g(x)=g_{1}(x)$ for the common points $x$ of continuity.

It is well known that $G(f(n) \bmod 1)$ is nonempty, closed and connected in the weak topology (see [W] and [SP, p. 1-9]) and this topology is metrisable by the $L^{2}$ metric

$$
\rho\left(g_{1}, g_{2}\right)=\left(\int_{0}^{1}\left(g_{1}(x)-g_{2}(x)\right)^{2} \mathrm{~d} x\right)^{1 / 2} .
$$

The Helly selection principle (i.e. the Helly first theorem in Part 1, also see [SP, p.4-5]) implies nonemptiness and closedness of $G_{w(n)}(f(n) \bmod 1)$ for all weights $w(n)$. For connectivity we have:
Theorem 9. Assume that $\frac{w(N)}{\sum_{n=1}^{N} w(n)} \rightarrow 0$ as $N \rightarrow \infty$. Then the set

$$
G_{w(n)}(f(n) \bmod 1)
$$

of all weighted d.f.s is connected, for every real-valued sequence $f(n), n=1,2, \ldots$
Proof. For simplicity, let $f(n) \in[0,1)$. A proof follows from the following theorem:

Theorem 10 (H.G. Barone [B]). If $t_{n}$ is a sequence in a metric space ( $X, \rho$ ) satisfying
(i) any subsequence of $t_{n}$ contains a convergent subsequence, and
(ii) $\lim _{n \rightarrow \infty} \rho\left(t_{n+1}, t_{n}\right)=0$,
then the set of all limit points of $t_{n}$ is connected in $(X, \rho)$.
Now, put $X=$ the set of all d.f.s on $[0,1], t_{N}=F_{N}(x)$, and $\rho=$ the $L^{2}$ metric on $X$, then the Helly selection principle implies (i), and thus (ii) $\rho\left(t_{n+1}, t_{n}\right) \rightarrow 0$ implies the connectivity of $G_{w(n)}(f(n))$.

Putting $g_{1}(x)=F_{N+1}(x)$ and $g_{2}(x)=F_{N}(x)$ where

$$
F_{N}(x)=\frac{1}{\sum_{n=1}^{N} w(n)} \sum_{n=1}^{N} w(n) c_{[0, x)}(f(n))
$$

and applying the formula [SP, p. 4-11]

$$
\begin{aligned}
\int_{0}^{1}\left(g_{1}(x)\right. & \left.-g_{2}(x)\right)^{2} \mathrm{~d} x=\int_{0}^{1} \int_{0}^{1}|x-y| \mathrm{d} g_{1}(x) \mathrm{d} g_{2}(x) \\
& -\frac{1}{2} \int_{0}^{1} \int_{0}^{1}|x-y| \mathrm{d} g_{1}(x) \mathrm{d} g_{1}(x)-\frac{1}{2} \int_{0}^{1} \int_{0}^{1}|x-y| \mathrm{d} g_{2}(x) \mathrm{d} g_{2}(x)
\end{aligned}
$$

we find

$$
\begin{aligned}
& \int_{0}^{1}\left(F_{N+1}(x)-F_{N}(x)\right)^{2} \mathrm{~d} x= \\
& -\frac{1}{2}\left(\frac{w(N+1)}{\sum_{n=1}^{N} w(n) \sum_{n=1}^{N+1} w(n)}\right)^{2} \sum_{m, n=1}^{N}|f(m)-f(n)| w(m) w(n) \\
& +\left(\frac{w(N+1)}{\sum_{n=1}^{N+1} w(n)}\right)^{2} \frac{1}{\sum_{n=1}^{N} w(n)} \sum_{m=1}^{N}|f(m)-f(N+1)| w(m) \\
& \leq\left(\frac{w(N+1)}{\sum_{n=1}^{N+1} w(n)}\right)^{2}
\end{aligned}
$$

and the proof is finished.

## 6. Applications

In the following paragraphs I, II we use weights $w(x)=1 / x^{\alpha}, 0<\alpha \leq 1$. In III we define a set $\Omega \subset[0,1]^{2}$ for given $f(n)$ and $w(n)$.
I. Put $w(x)=1 / x$. Thus $M(x)=\log x$ and $\lim _{x \rightarrow \infty} \theta(x, y)=0$. For $f(x)$ we investigate the following cases:

1. If $f(x)=\log x$, then we have $f^{-1}(x)=e^{x}, M\left(f^{-1}(x)\right)=x$, and
(III) $\lim _{k \rightarrow \infty} \frac{M\left(f^{-1}(k+x)\right)-M\left(f^{-1}(k)\right)}{M\left(f^{-1}(k+1)\right)-M\left(f^{-1}(k)\right)}=\tilde{g}(x)=x$;
(IV) $\lim _{k \rightarrow \infty} \frac{M\left(f^{-1}(k+u(k))\right.}{M\left(f^{-1}(k)\right)}=\psi(u)=1$.

Thus by Theorem 2 we have the well known result that the sequence $\log n \bmod 1$ is logarithmically u.d. (see M. Tsuji $[\mathrm{T}](1952)$ ). For this result we cannot use Theorem 6, since $\lim _{k \rightarrow \infty} M\left(f^{-1}(k+1)\right)-M\left(f^{-1}(k)\right)=1$.
2. If $f(x)=\log \log x$, then we have $f^{-1}(x)=e^{e^{x}}, M\left(f^{-1}(x)\right)=e^{x}$, and we have the case of Theorem 1 with $\psi(x)=e^{x}$. Since
(V) $\lim _{k \rightarrow \infty} M\left(f^{-1}(k+1)\right)-M\left(f^{-1}(k)\right)=\infty$
and $\lim _{x \rightarrow \infty} \frac{f^{\prime}(x)}{M^{\prime}(x)}=0$ we can apply Theorem 6 , thus the logarithmic d.f.s of $\log \log n \bmod 1$ are the same as the classical d.f.s of $f\left(M^{-1}(n)\right)=\log n \bmod 1$, $n=1,2, \ldots$ Similarly, for $f(x)=\log ^{(i)} x, i=2,3, \ldots$, we have the case Theorem 3 and by Theorem 6

$$
G_{1 / n}\left(\log ^{(i)} n \bmod 1\right)=G\left(\log ^{(i-1)} n \bmod 1\right) \text { for } i=2,3, \ldots
$$

Furthermore, Theorem 3 implies (see also $[\mathrm{S}]$ )
$G\left(\log ^{(i)} n \bmod 1\right)=G\left(\log ^{(i-1)} n \bmod 1\right)=\left\{c_{u}(x) ; x \in[0,1]\right\} \cup\left\{h_{\beta}(x) ; \beta \in[0,1]\right\}$ for $i=3,4, \ldots$ Thus in general, the implication
$G(f(n) \bmod 1)=G(h(n) \bmod 1) \Longrightarrow G_{w(n)}(f(n) \bmod 1)=G_{w(n)}(h(n) \bmod 1)$
does not hold, e.g., put $f(n)=\log \log \log n$ and $h(n)=\log \log n$.
3. If $f(x)=\log (x \log x)$, then
(III) $\lim _{k \rightarrow \infty} \frac{M\left(f^{-1}(k+x)\right)-M\left(f^{-1}(k)\right)}{M\left(f^{-1}(k+1)\right)-M\left(f^{-1}(k)\right)}=\tilde{g}(x)=x$;
(IV) $\lim _{k \rightarrow \infty} \frac{M\left(f^{-1}(k+u(k))\right.}{M\left(f^{-1}(k)\right)}=\psi(u)=1$,
and by Theorem 2 the sequence $\log (n \log n) \bmod 1, n=2,3, \ldots$, is logarithmically u.d. As for the case $\log n \bmod 1$, also in this case we cannot use Theorem 6 , since $\lim _{k \rightarrow \infty} M\left(f^{-1}(k+1)\right)-M\left(f^{-1}(k)\right)=1$. This limit and also the limit (III) follows from the Lagrange mean value theorem and from the relation $\lim _{x \rightarrow \infty}\left(M\left(f^{-1}(x)\right)\right)^{\prime}=1$. The proof of (ii) follows from the inequality

$$
0<\log f^{-1}(k+1)-\log f^{-1}(k)<f\left(f^{-1}(k+1)\right)-f\left(f^{-1}(k)\right)=1
$$

which implies $\lim _{k \rightarrow \infty} \frac{\log f^{-1}(k+1)}{\log f^{-1}(k)}=1$.
4. Let $p_{n}, n=1,2, \ldots$, be the increasing sequence of all primes. By an old result of M. Cipolla (1902)[C] (cf. P. Ribenboim (1995)[R, p. 249])

$$
p_{n}=n \log n+n(\log \log n-1)+o\left(\frac{n \log \log n}{\log n}\right)
$$

and thus

$$
\lim _{n \rightarrow \infty}\left(\frac{p_{n}}{n}-\log (n \log n)\right) \bmod 1=0
$$

Since $\log (n \log n) \bmod 1, n=2,3, \ldots$ is logarithmically u.d., Theorem 3 implies that $p_{n} / n \bmod 1$ is also logarithmically u.d. The logarithmic u.d. of $\log (n \log n) \bmod 1$ also follows from Kemperman's Theorem 5 and from Theorem 2 in Y. Ohkubo [O].
II. Let $0<\alpha<1$. In the following we put $w(x)=1 / x^{\alpha}$. Thus $M(x)=\int_{0}^{x} \frac{\mathrm{~d} t}{t^{\alpha}}=$ $\frac{x^{1-\alpha}}{1-\alpha}$ and $\lim _{x \rightarrow \infty} \theta(x, y)=0$.
5. Put $f(x)=\log \log x$, then $f^{-1}(x)=e^{e^{x}}, M\left(f^{-1}(x)\right)=\frac{e^{e^{x}(1-\alpha)}}{1-\alpha}$ and
(IV) $\lim _{k \rightarrow \infty} \frac{M\left(f^{-1}(k+u(k))\right.}{M\left(f^{-1}(k)\right)}=\lim _{k \rightarrow \infty} e^{e^{k}(1-\alpha)\left(e^{u(k)}-1\right)}=\psi(u)=\infty$,
as $u(k) \rightarrow u$ and $u>0$. If $u=0, \psi(0)$ is not defined and for every $t \in[0, \infty)$ there exists $u(k) \rightarrow 0$ such that $e^{e^{k}(1-\alpha)\left(e^{u(k)}-1\right)} \rightarrow t$. Applying Theorem 3 we have $G_{1 / n^{\alpha}}(\log \log n \bmod 1)=G(\log \log n \bmod 1)$. Since $\left(\right.$ see 1.) $G_{1 / n}(\log \log n \bmod$ 1) $=G(\log n \bmod 1)$, the set $\lim _{\alpha \rightarrow 1} G_{1 / n^{\alpha}}(f(n) \bmod 1)$ "does not converge" (in some sense) to $G_{1 / n}(f(n) \bmod 1)$.
6. Putting $f(x)=\log x$, we have $f^{-1}(x)=e^{x}$ and $M\left(f^{-1}(x)\right)=\frac{e^{(1-\alpha) x}}{1-\alpha}$. Then
(III) $\lim _{k \rightarrow \infty} \frac{M\left(f^{-1}(k+x)\right)-M\left(f^{-1}(k)\right)}{M\left(f^{-1}(k+1)\right)-M\left(f^{-1}(k)\right)}=\frac{e^{(1-\alpha) x}-1}{e^{1-\alpha}-1}=\tilde{g}(x)$,
(IV) $\lim _{k \rightarrow \infty} \frac{M\left(f^{-1}(k+u)\right.}{M\left(f^{-1}(k)\right)}=e^{(1-\alpha) u}=\psi(u)$.

Since $e^{1-\alpha}>1$ and $f^{\prime}(x) \rightarrow 0$, the case of Theorem 1 implies

$$
\begin{aligned}
& G_{1 / n^{\alpha}}(\log n \bmod 1) \\
& \quad=\left\{g_{u}(x)=\frac{e^{(1-\alpha) \min (x, u)}-1}{e^{(1-\alpha) u}}+\frac{1}{e^{(1-\alpha) u}} \frac{e^{(1-\alpha) x}-1}{e^{1-\alpha}-1} ; u \in[0,1]\right\}
\end{aligned}
$$

By the same Theorem 1 and putting $w(n)=1$ we have the well-known result (see [KN, pp. 58-59])

$$
G(\log n \bmod 1)=\left\{g_{u}(x)=\frac{e^{\min (x, u)}-1}{e^{u}}+\frac{1}{e^{u}} \frac{e^{x}-1}{e-1} ; u \in[0,1]\right\}
$$

III. Define the two-dimensional set $\Omega_{w(n)}(f(n) \bmod 1)$ in the unit square $[0,1]^{2}$ as

$$
\begin{aligned}
& \Omega_{w(n)}(f(n) \bmod 1)= \\
& \left\{(x, y) \in[0,1]^{2} ; \exists g \in G_{w(n)}(f(n) \bmod 1)(y=g(x) \text { or } y \in[g(x-0), g(x+0])\}\right.
\end{aligned}
$$

It is known that for every infinite subset of positive integers the lower asymptotic density $\leq$ the lower logarithmic density $\leq$ the upper logarithmic density $\leq$ the upper asymptotic density. Then

$$
\Omega_{1 / n}(f(n) \bmod 1) \subset \Omega(f(n) \bmod 1)
$$

for every real-valued sequence $f(n), n=1,2, \ldots$ By Theorem 9 , both subsets of $[0,1]^{2}, \Omega_{1 / n}(f(n) \bmod 1)$ and $\Omega(f(n) \bmod 1)$ are connected.

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RITA GIULIANO ANTONINI - OTO STRAUCH

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Rita Giuliano Antonini
Dipartimento di Matematica "L. Tonelli"
Largo B. Pontecorvo, 5
I-56127 Pisa, ITALY
E-mail: giuliano@dm.unipi.it

## Oto Strauch

Mathematical Institute
Slovak Academy of Sciences
SK-814 73 Bratislava, SLOVAKIA
E-mail: strauch@mat.savba.sk


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[^1]:    ${ }^{1}$ Without loss of generality we assume that $f(1) \in[0,1)$.

[^2]:    ${ }^{2}$ Note that if $u$ is a point of continuity of $\psi(u)$, the monotonicity of $M\left(f^{-1}(x)\right)$ implies the relation $\frac{M\left(f^{-1}\left(K_{i}+u_{i}\right)\right)}{M\left(f^{-1}\left(K_{i}\right)\right)} \rightarrow \psi(u)$, because $\frac{M\left(f^{-1}\left(K_{i}+u-\varepsilon\right)\right)}{M\left(f^{-1}\left(K_{i}\right)\right)} \leq \frac{M\left(f^{-1}\left(K_{i}+u_{i}\right)\right)}{M\left(f^{-1}\left(K_{i}\right)\right)} \leq$ $\frac{M\left(f^{-1}\left(K_{i}+u+\varepsilon\right)\right)}{M\left(f^{-1}\left(K_{i}\right)\right)}$ for $u_{i} \in(u-\varepsilon, u+\varepsilon)$.

