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uniform distribution theory

ON THE DISTRIBUTION MODULO ONE OF THE MEAN VALUES OF SOME ARITHMETICAL FUNCTIONS

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ABSTRACT. Florian Luca asked whether the sequence consisting of the arithmetic (*resp.* geometric) mean values of the Euler φ function is uniformly distributed modulo 1. We show that it is the case for the arithmetic means, and that it the case for the geometric means if and only if the number $e^{-1}\prod_p(1-1/p)^{1/p}$ is irrational. More general arithmetic functions are also considered.

Communicated by Sergei Konyagin

To Gérard Rauzy, with our friendship

1. Introduction

Florian Luca [2] raised the question whether some sequences of mean values of the Euler function φ are uniformly distributed modulo one. The first one of those is the sequence of the arithmetic means, defined by

$$\mathcal{A} = (a_n)_{n\geq 1}, \quad \text{where } a_n = \frac{1}{n}\sum_{m\leq n}\varphi(m). \tag{1}$$

The first result of this paper is to give a positive answer to Luca's original question, showing that the sequence $\mathcal{A}$ defined by (1) is indeed uniformly distributed modulo one. We shall in fact consider a slightly larger class of strongly multiplicative functions which have a linear growth, showing that the sequence of the arithmetic means is uniformly distributed modulo 1 when the leading coefficient is irrational. This is the case for the Euler function, since it is well-known that

$$a_n = \frac{3}{\pi^2}n + O(\log n). \tag{2}$$

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More precisely, we have the following

Theorem 1. *Let $(\nu(n))_n$ be an arithmetic function which is completely multiplicative and satisfies the conditions*

$$|\nu(p)| \leq \nu \quad \text{for some positive number } \nu \text{ and every prime } p, \tag{3}$$

$$\sum_{d \leq x} \mu(d)\nu(d) \ll x \left(\log x\right)^{-A} \quad \text{for every positive } A, \tag{4}$$

where the implied constant depends only on ν and A.
We define the arithmetic function ϕ by

$$\phi(m) = m \prod_{p \mid m} \left(1 - \frac{\nu(p)}{p}\right). \tag{5}$$

Then, if the number

$$\alpha = \frac{1}{2} \prod_{p} \left(1 - \frac{\nu(p)}{p^2}\right) \tag{6}$$

is irrational, the sequence $\mathcal{A} = (a_n)_n$ defined by

$$a_n = \frac{1}{n} \sum_{m \leq n} \phi(m) \tag{7}$$

is uniformly distributed modulo one.

As we already mentioned, Theorem 1 implies a positive answer to the question of Luca on the arithmetical mean of the Euler φ function: indeed, this corresponds to the case when $\nu(n) = 1$ for every n; then (3) is trivially satisfied and (4) is an avatar of the prime number theorem; the uniform distribution of the sequence $(a_n)_n$ modulo one follows from the fact that the corresponding α in (6) is equal to $3/\pi^2$, an irrational number.

Another interesting case is when $\nu = \chi$ is a real primitive character modulo $D > 1$. In this case, (3) is trivially satisfied and (4) is this time an avatar of the prime number theorem for Dirichlet characters; moreover, we have

$$\alpha = \frac{1}{2} L(2, \chi)^{-1}, \tag{8}$$

which is irrational [5].

The number α actually enters the subject through the asymptotic expression

$$a_n = \alpha n + O\left((\log n)^{\nu}\right). \tag{9}$$

The case when this leading coefficient is rational has its own interest and we plan to come back to that topic.

The question we are addressing would reduce to classical arguments in the theory of uniform distribution modulo one if the error term in (9) were, up to $o(1)$, a nice steady function of n. This is not the case: indeed, this error term usually bears high variations of an arithmetic nature and it is precisely the essence of this paper to cope with them. In [1], F. Luca and the first-named author turned out those arithmetic variations to their advantage, when they proved that the sequence $\mathcal{A}$ related to the Euler function is dense modulo one.

Although Theorem 1 only deals with the value of the ν function at square-free integers, we define it as a completely multiplicative function, for convenience in the proof. We take this opportunity to mention that our method can be extended to study more general arithmetical functions, not necessarily strongly multiplicative.

We now turn our attention to the geometric mean values of the ϕ function. In the fifth section, we prove the following

Theorem 2. *Let $(\nu(n))_n$ be a completely multiplicative function such that*

$$-\nu \leq \nu(p) < \min\{p, \nu\} \text{ for some positive } \nu \text{ and every prime } p \tag{10}$$

and such that there exist real numbers β and λ such that

$$\prod_{p \leq n} \left(1 - \frac{\nu(p)}{p}\right) = \beta (\log n)^{-\lambda} \left(1 + O\left(\frac{1}{\log n}\right)\right), \tag{11}$$

where the implied constant depends only on ν.
We define the strongly multiplicative function ϕ as in Theorem 1 by

$$\phi(m) = m \prod_{p \mid m} \left(1 - \frac{\nu(p)}{p}\right),$$

and we let

$$\alpha = \frac{1}{e} \prod_{p} \left(1 - \frac{\nu(p)}{p}\right)^{\frac{1}{p}}. \tag{12}$$

If α is irrational, then the sequence $\mathcal{A} = (a_n)_n$ defined by

$$a_n = \left(\prod_{m \leq n} \phi(m)\right)^{\frac{1}{n}} \tag{13}$$

is uniformly distributed modulo one.

Condition (11) which, in the geometric context, replaces condition (4) is another avatar of the prime number theorem with a Mertens flavor.

In the archetypal case when the ϕ function is Euler's φ function, i.e. when $\nu(p)$ is equal to 1 for every p, the discriminating constant α is equal to

$$\alpha = \frac{1}{e}\prod_p \left(1-\frac{1}{p}\right)^{\frac{1}{p}}. \tag{14}$$

This constant is very likely to be irrational: Richard Bumby showed that if α is rational, then its denominator has at least 20 decimal digits. A special case of the next theorem, proved in the last section, shows that if the constant in (14) is rational, then the sequence $\left(\prod_{m\le n}\varphi(m)\right)^{\frac{1}{n}}$ is not uniformly distributed modulo 1.

Theorem 3. *Let the arithmetical functions ν, ϕ and a satisfy the conditions of Theorem 2. If α defined in* (12) *is rational and ν takes only algebraic values, then the sequence $\mathcal{A} = (a_n)_n$ defined by* (13) *is not uniformly distributed modulo one.*

We finally remark that our analysis can be extended to more general mean values, such as those considered in [2] and [1].

In the whole text, the symbol $B(\nu)$ denotes a constant that depends only on ν, the value of which may change from one occurrence to the other.

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2. Evaluation of the arithmetic mean

In this section, we evaluate the elements of our sequence $\mathcal{A}$ in order to give an explicit expression in terms of the leading expression αn in (9) and the remaining part. Let us first fix some notation. We let ψ be the *saw* function

$$\psi(x) = x - \lfloor x \rfloor - \frac{1}{2}, \tag{15}$$

where $\lfloor x \rfloor$ denotes the integral part of the real number x. For a real $z \ge 2$, we denote by $P(z)$, or simply P, the product

$$P = P(z) = \prod_{p<z} p. \tag{16}$$

For given real t, $z \geq 2$ and D satisfying

$$P(z) < D, \tag{17}$$

we define the quantities $r_t(D), \rho_t(z)$ and $\rho_t(D,z)$ by

$$r_t(D) = \sum_{d \leq D} \mu(d) \frac{\nu(d)}{d} \psi\left(\frac{t}{d}\right), \tag{18}$$

$$\rho_t(z) = \sum_{d \mid P(z)} \mu(d) \frac{\nu(d)}{d} \psi\left(\frac{t}{d}\right) \tag{19}$$

and

$$\rho_t(D,z) = \sum_{\substack{d \leq D \\ d \nmid P(z)}} \mu(d) \frac{\nu(d)}{d} \psi\left(\frac{t}{d}\right), \tag{20}$$

so that we have

$$r_t(d) = \rho_t(z) + \rho_t(D,z). \tag{21}$$

The main result of this section is the following

Lemma 4. *Assume that the multiplicative function ν satisfies* (3) *and* (4)*. Let $2 \leq z \leq D < n$ be such that* (17) *holds. Then, for any $A > 0$, we have*

$$a_n = \alpha n - \rho_n(z) - \rho_n(D,z) + O\left(\frac{D}{n} (\log D)^{B(\nu)} + \frac{n}{D} (\log D)^{-A}\right), \tag{22}$$

where $\rho_n(z)$ and $\rho_n(D,z)$ are respectively given by (19) *and* (20)*, and where the constant implied in the O symbol depends only on ν and A.*

Proof. Let $2 \leq z < D < n$ be such that (17) holds. We have

$$\begin{aligned} a_n &= \frac{1}{n} \sum_{ad \leq n} a\mu(d)\nu(d) \\ &= \frac{1}{2n} \sum_{d \leq D} \mu(d)\nu(d) \left\lfloor \frac{n}{d} \right\rfloor \left(\left\lfloor \frac{n}{d} \right\rfloor + 1 \right) + \frac{1}{n} \sum_{a \leq \frac{n}{D}} a \sum_{D < d \leq \frac{n}{a}} \mu(d)\nu(d). \end{aligned} \tag{23}$$

Here D is at our disposal and it will be later chosen to be quite large, comparable to n. By (4), we have

$$\frac{1}{n} \sum_{a \leq \frac{n}{D}} a \sum_{D < d \leq \frac{n}{a}} \mu(d)\nu(d) \ll \frac{n}{D} (\log D)^{-A}. \tag{24}$$

In the first term of (23), we express the integral parts in terms of the *saw* function ψ. The first term in (23) is equal to

$$\frac{1}{2n}\sum_{d\le D}\mu(d)\nu(d)\left(\frac{n}{d}-\frac{1}{2}-\psi\left(\frac{n}{d}\right)\right)\left(\frac{n}{d}+\frac{1}{2}-\psi\left(\frac{n}{d}\right)\right)$$

$$=\frac{n}{2}\sum_{d\le D}\mu(d)\nu(d)d^{-2}-\sum_{d\le D}\mu(d)\frac{\nu(d)}{d}\psi\left(\frac{n}{d}\right)+O\left(\frac{D}{n}\left(\log D\right)^{B(\nu)}\right). \tag{25}$$

Relation (4) permits to deal with the first term of (25). We have

$$\frac{n}{2}\sum_{d\le D}\mu(d)\nu(d)d^{-2}=\alpha n+O\left(\frac{n}{D}\left(\log D\right)^{-A}\right).$$

Expressing the second term in (25) in terms of the quantities $\rho_n(z)$ and $\rho_n(D,z)$ (cf. (19) and (20)) leads to the proof of Lemma 4. □

The next section is devoted to the estimation of $\rho_n(D,z)$.

3. A mean-square bound for $\rho_n(D,z)$

We introduce the classical generalized divisor function τ_z, defined for complex z, such that $\tau_z(n)$ is the n-th coefficient in the Dirichlet expansion of ζ^z; relevant properties of τ_z can be found in [4], ch. 3. First, we establish the following result.

Lemma 5. *Let ν be a positive real number. For any complex numbers $c(d)$ with $|c(d)|\le\tau_\nu(d)$ we have*

$$\int_{-T}^{T}\Big|\sum_{z\le d\le D}\frac{c(d)}{d}\psi\left(\frac{t}{d}\right)\Big|^2\mathrm{d}t\ll Tz^{-1}\left(\log z\right)^{B(\nu)}+D\left(\log D\right)^{B(\nu)}, \tag{26}$$

where the implied constants depend on ν only.

P r o o f. We first notice that the value of the left hand side of (26) is unchanged if one replaces the ψ saw function by

$$\psi^*(x)=\begin{cases}x-\lfloor x\rfloor-\frac{1}{2} & \text{if } x \text{ is not an integer,}\\ 0 & \text{if } x \text{ is an integer,}\end{cases} \tag{27}$$

which is the sum of its boundedly convergent Fourier expansion

$$\psi^*(x)=-\sum_{k\ne 0}\frac{1}{2\pi ik}e(kx). \tag{28}$$

Let χ be a smooth function which majorizes the characteristic function of the segment $|t| \le T$, with

$$\hat{\chi}(u) \ll \min\{T, |u|^{-1}\}.$$

Then, the integral (26) is bounded by

$$\int_{-\infty}^{+\infty} \chi(t) \left| \sum_{z \le d \le D} \frac{c(d)}{d} \psi^* \left(\frac{t}{d}\right) \right|^2 \mathrm{d}t$$

$$\le \sum\sum_{z \le d_1,\, d_2 \le D} \frac{|c(d_1)c(d_2)|}{d_1 d_2} \left| \int_{-\infty}^{+\infty} \chi(t) \psi^* \left(\frac{t}{d_1}\right) \psi^* \left(\frac{t}{d_2}\right) \mathrm{d}t \right|.$$

Using the Fourier expansion (28) we see that the last integral equals

$$-\sum\sum_{k_1 k_2 \ne 0} \frac{1}{4\pi^2 k_1 k_2} \hat{\chi}\left(\frac{k_1}{d_1} + \frac{k_2}{d_2}\right),$$

which is

$$\ll T \sum\sum_{\substack{k_1 k_2 \ne 0 \\ k_1 d_2 + k_2 d_1 = 0}} \frac{1}{|k_1 k_2|} + \sum\sum_{\substack{k_1 k_2 \ne 0 \\ k_1 d_2 + k_2 d_1 \ne 0}} \frac{d_1 d_2}{|k_1 k_2 (k_1 d_2 + k_2 d_1)|}.$$

We put $m_1 = k_2 d_1$ and $m_2 = k_1 d_2$. The integral in (26) is bounded by

$$T \sum_{m \ge z} \frac{\tau_{\nu+1}^2(m)}{m^2} + \sum\sum_{\substack{m_1, m_2 > 0 \\ m_1 \ne m_2}} \frac{1}{m_1 m_2 |m_1 - m_2|} \left(\sum_{\substack{d_1 | m_1 \\ d_1 \le D}} \tau_\nu(d_1) d_1 \right) \left(\sum_{\substack{d_2 | m_2 \\ d_2 \le D}} \tau_\nu(d_2) d_2 \right).$$

The first term is $O\left(T z^{-1} (\log z)^{B(\nu)}\right)$. The second one is bounded by

$$\sum_{m > 0} \frac{1}{m} \left(\sum_{\substack{d | m \\ d \le D}} \tau_\nu(d) d \right)^2 \sum_{\substack{\ell > 0 \\ \ell \ne m}} \frac{1}{\ell |\ell - m|}.$$

But we have

$$\sum_{\substack{\ell > 0 \\ \ell \ne m}} \frac{1}{\ell |\ell - m|} \ll \frac{\log(2m)}{m}.$$

Hence we get

$$\begin{aligned}\sum_{m>0}\frac{\log 2m}{m^2}\left(\sum_{\substack{d|m\\ d\le D}}\tau_\nu(d)d\right)^2 &\ll (\log D)\sum_{d_1,d_2\le D}\sum\frac{\tau_\nu(d_1)\tau_\nu(d_2)}{d_1d_2}(d_1,d_2)^2\\ &\ll (\log D)\left(\sum_{\delta\le D}\tau_\nu^2(\delta)\right)\left(\sum_{d\le D}\frac{\tau_\nu(d)}{d}\right)^2\\ &\ll D\left(\log D\right)^{B(\nu)}.\end{aligned}$$

This completes the proof of Lemma 5. □

Next, we establish a discrete version of Lemma 5.

Lemma 6. *Let ν be a positive real number. For any complex numbers $c(d)$ with $|c(d)|\le\tau_\nu(d)$, we have*

$$\sum_{|n|\le T}\Big|\sum_{z\le d\le D}\frac{c(d)}{d}\psi\left(\frac{n}{d}\right)\Big|^2 \ll Tz^{-1}\left(\log z\right)^{B(\nu)}+D\left(\log D\right)^{B(\nu)},$$

where the implied constant depends only on ν.

Proof. For $0\le t<1$, we have

$$\psi\left(\frac{n+t}{d}\right)-\psi\left(\frac{n}{d}\right)=t/d.$$

Hence our sum is estimated by the corresponding integral, up to the correction

$$T\left(\sum_{d\ge z}\frac{|c(d)|}{d^2}\right)^2\ll Tz^{-2}\left(\log z\right)^{B(\nu)}.$$

This completes the proof of Lemma 6. □

We summarize the results of this section in the following proposition.

Proposition 7. *Let z,D satisfy* (17) *and recall that $\rho_n(D,z)$ is defined in* (20). *Then, for $X\ge D$, we have*

$$\sum_{n\le X}|\rho_n(D,z)|^2\ll Xz^{-1}(\log z)^{B(\nu)}+D(\log D)^{B(\nu)},$$

where the implied constants depend only on ν, the parameter introduced in (3).

4. Proof of Theorem 1

We assume in this section that the number α defined in (6), the leading coefficient involved in (9), is irrational.

In order to show that the sequence $\mathcal{A}$ is uniformly distributed modulo one, we use Weyl's criterion, i.e. we show that for every positive integer h, we have

$$\sum_{n\le X} e(ha_n) = o(X), \text{ as } X\to\infty. \tag{29}$$

By Lemma 4, the above sum equals

$$\sum_{n\le X} e(h(\alpha n-\rho_n(z)))+O\left(\sum_{n\le X}|\rho_n(D,z)| + D(\log X)^{B(\nu)} + X^2D^{-1}(\log X)^{-A}\right), \tag{30}$$

where A is arbitrary and the constant implied in the O symbol depends only on A and ν.

By Proposition 7 and Cauchy's inequality, we obtain

$$\sum_{n\le X}|\rho_n(D,z)| \ll Xz^{-\frac{1}{2}}(\log z)^B + \sqrt{XD}(\log X)^B. \tag{31}$$

Hence, by an appropriate choice of A and D, we arrive at

$$\sum_{n\le X} e(ha_n) = \sum_{n\le X} e(h(\alpha n-\rho_n(z))) + O\left(Xz^{-\frac{1}{3}}\right) \tag{32}$$

for every $2\le z\le \log X$, where the implied constant depends only on h and ν.

We wish to use the irrationality of α and the fact that $\rho_n(z)$ is periodic in n with period $P=P(z)$. First, using the irrationality of α, we can find a function $\varepsilon(z,X)$ such that

$$\left|\sum_{\substack{n\le X\\ n\equiv b (\mathrm{mod}\, P)}} e(h\alpha n)\right| \le \varepsilon(z,X)X, \tag{33}$$

uniformly in b, where, for fixed z, the function $\varepsilon(z,X)$ tends to 0 as X tends to infinity. Summing over the classes modulo P, we obtain

$$\left|\sum_{n\le X} e(ha_n)\right| \le \varepsilon(z,X)P(z)X + O(Xz^{-\frac{1}{3}}), \tag{34}$$

which implies (29) and the uniform distribution modulo one of $\mathcal{A}$.

5. Proof of Theorem 2

Let a_n be defined by (13). We have

$$a_n = (n!)^{\frac{1}{n}} \prod_{p \le n} \left(1 - \frac{\nu(p)}{p}\right)^{\frac{1}{n}\left\lfloor \frac{n}{p} \right\rfloor}.$$

We use Stirling's formula

$$(n!)^{\frac{1}{n}} = \frac{n}{e} + \frac{1}{e} \log \sqrt{2\pi n} + O\left(\frac{(\log n)^2}{n}\right),$$

the *saw* function as

$$\frac{1}{n}\left\lfloor \frac{n}{p} \right\rfloor = \frac{1}{p} - \frac{1}{2n} - \frac{1}{n}\psi\left(\frac{n}{p}\right),$$

and the estimation

$$\prod_{p > n} \left(1 - \frac{\nu(p)}{p}\right)^{\frac{1}{p}} = 1 + O\left(\frac{1}{n \log n}\right),$$

where the constant implied in the O symbol depends on ν only. We thus get

$$a_n = \left(\frac{n}{e} + \frac{1}{2e}\log(2\pi n) + O\left(\frac{(\log n)^2}{n}\right)\right) \times \\ \left(e\alpha + O\left(\frac{1}{n\log n}\right)\right)\left(\prod_{p \le n}\left(1 - \frac{\nu(p)}{p}\right)^{\frac{-1}{2n}}\right)\left(\prod_{p \le n}\left(1 - \frac{\nu(p)}{p}\right)^{\frac{-1}{n}\psi\left(\frac{n}{p}\right)}\right). \tag{35}$$

By (11), we get

$$\begin{aligned} \prod_{p \le n}\left(1 - \frac{\nu(p)}{p}\right)^{\frac{-1}{2n}} &= \exp\left(-\frac{1}{2n}\log\left(\prod_{p \le n}\left(1 - \frac{\nu(p)}{p}\right)\right)\right) \\ &= \exp\left(-\frac{1}{2n}\left(\log\beta - \lambda\log\log n + O\left(\frac{1}{\log n}\right)\right)\right) \\ &= 1 - \frac{\log\beta}{2n} + \frac{\lambda}{2n}\log\log n + O\left(\frac{1}{n\log n}\right). \end{aligned} \tag{36}$$

Using the fact that ψ is bounded, we get

$$\prod_{p\le n}\left(1-\frac{\nu(p)}{p}\right)^{\frac{-1}{n}\psi\left(\frac{n}{p}\right)} = \exp\left(-\frac{1}{n}\sum_{p\le n}\psi\left(\frac{n}{p}\right)\log\left(1-\frac{\nu(p)}{p}\right)\right)$$
$$= 1-\frac{1}{n}\sum_{p\le n}\psi\left(\frac{n}{p}\right)\log\left(1-\frac{\nu(p)}{p}\right)+O\left(\frac{(\log\log n)^2}{n^2}\right). \quad (37)$$

Putting together (35), (36) and (37), we get

$$a_n = \alpha\left(n+\frac{1}{2}\log(2\pi n/\beta)+\frac{1}{2}\lambda\log\log n\right)$$
$$-\alpha\sum_{p\le n}\psi\left(\frac{n}{p}\right)\log\left(1-\frac{\nu(p)}{p}\right)+O\left(\frac{1}{\log n}\right). \quad (38)$$

For $X>1$, and $2\le z<D\le X(\log X)^{-A}$, where the positive number A is to be chosen later, we define

$$\Psi_n(z)=\sum_{p<z}\psi\left(\frac{n}{p}\right)\log\left(1-\frac{\nu(p)}{p}\right),$$

$$\Psi_n(z,D)=\sum_{z\le p<D}\psi\left(\frac{n}{p}\right)\log\left(1-\frac{\nu(p)}{p}\right).$$

From (38), we obtain for $X<n\le 2X$

$$a_n = \alpha\left(n+\frac{1}{2}\log(2\pi n/\beta)+\frac{1}{2}\lambda\log\log X\right)$$
$$-\alpha\Psi_n(z)-\alpha\Psi_n(z,D)+O\left(\frac{\log\log X}{\log X}\right). \quad (39)$$

At that point, we should remark the analogy between (39) and (22), pointing out that the expressions Ψ in (39) are not more complex than the expressions ρ in (22) - sums over primes instead of sums over integers weighted by the Möbius function; this implies that proving that the sequence $(a_n-(\alpha/2)\log n)_n$

is uniformly distributed modulo one can be performed by the arguments used in section 4. Indeed, for an integer n, let us write

$$b_n = a_n - \frac{\alpha}{2}\log n. \tag{40}$$

For a given positive positive integer h, we consider the Weyl's sum associated to the sequence $(b_n)_n$, namely

$$W'_h(X) = \sum_{X<n\le 2X} e(hb_n).$$

By (39), we have for $2 \le z < D \le X(\log X)^{-A}$,

$$W'_h(X) = L_h(X) \sum_{X<n\le 2X} e(\alpha h(n - \Psi_n(z) - \Psi_n(z,D))) + O\left(X\frac{\log\log X}{\log X}\right),$$

where

$$L_h(X) = e\left(\frac{\alpha h}{2}(\log(2\pi/\beta) + \lambda\log\log X)\right).$$

We remove the term $\Psi_n(z,D)$ by using Lemma 6. We argue as in the proof of Theorem 1, select appropriately A and D and obtain the following formula, closely related to (32), namely

$$W'_h(X) = L_h(X) \sum_{X<n\le 2X} e\left(\alpha hn - \alpha h\Psi_n(z)\right) + O\left(Xz^{-\frac{1}{3}}\right) \tag{41}$$

for every $2 \le z \le \log X$, where the implied constant depends only on h and ν.

As in the proof of Theorem 1, we use the above expression for the Weyl's sums, the periodicity in n (with period $P(z)$) of $\Psi_n(z)$ and the irrationality of α to show that for every positive h the quantity $W'_h(X)/X$ tends to 0 as X tends to ∞. By Weyl's criterion, this implies the uniform distribution of the sequence $(b_n)_n$ modulo one.

By a general argument, the sequence $(u_n + \kappa\log n)_n$ is uniformly distributed modulo 1 if and only if it is the case for the sequence $(u_n)_n$ (this result is due to G. Rauzy; cf. [3], Énoncé 11, p. 153). Thus, the sequence $(a_n)_n$ is uniformly distributed when α is irrational. This ends the proof of Theorem 2.

6. Proof of Theorem 3

We keep the notation introduced in Section 5. Thanks to the previous remark, in order to show that the sequence $(a_n)_n$ is not uniformly distributed modulo 1, it is enough to show that this is not the case for the sequence $(b_n)_n$ introduced in (40). Thus, it is enough to find a positive integer h such that the Weyl's sum $W'_h(X)$ is not $o(X)$. We choose h as the denominator of α, say $\alpha = a/h$ where a is an integer. By (41), and since $|L_h(X)| = 1$, it is enough to show that

$$S(X) := \sum_{X<n\leq 2X} e\left(a\Psi_n(z)\right) \text{ is not } o(X).$$

By the definition of $\Psi_n(z)$ and the Chinese remainder theorem, we have

$$\begin{aligned} S(X) &= \frac{X}{P} \sum_{n \bmod P} \prod_{p<z} e\left(a\psi\left(\frac{n}{p}\right)\log\left(1-\frac{\nu(p)}{p}\right)\right) + o(X) \\ &= X \prod_{p<z} \frac{1}{p} \sum_{n \bmod p} e\left(a\psi\left(\frac{n}{p}\right)\log\left(1-\frac{\nu(p)}{p}\right)\right) + o(X). \end{aligned} \tag{42}$$

When p is sufficiently large, each sum in (42) is equal to $p + O(1/p)$, and so the product in (42) absolutely converges; since z tends to infinity with X, we have

$$S(X) = X \prod_{p \text{ prime}} \frac{1}{p} \sum_{n \bmod p} e\left(a\psi\left(\frac{n}{p}\right)\log\left(1-\frac{\nu(p)}{p}\right)\right) + o(X).$$

By our assumption, for any p, the value $\nu(p)$ is algebraic, so that $\log(1-\nu(p)/p)$ is either 0 or irrational: in either case, the factor corresponding to p in the infinite product is different from 0, and so $S(X)$ is not $o(X)$. This ends the proof of Theorem 3.

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