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WEIGHTED SUMS IN FINITE ABELIAN GROUPS

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ABSTRACT. In this note we prove the following weighted generalization of Bollobás and Leader theorem (J. Number Theory **78** (1999), no. 1, 27–35): Let G be an abelian group of order n and k a positive integer. Let $(w_1, w_2, \ldots, w_k)$ be a sequence of integers where each w_i is co-prime to n. Then, given a sequence $(x_1, x_2, \ldots, x_{k+r})$ of elements of G, where $1 \leq r \leq n-1$, if 0 is the most repeated element in the sequence, and $\sum_1^k w_i x_{\sigma(i)} \neq 0$, for all permutations σ of $\{1, 2, \ldots, k+r\}$, we have

$$\left|\left\{\sum_1^k w_i x_{\sigma(i)} : \sigma \text{ is a permutation of } \{1,2,\ldots,k+r\}\right\}\right| \geq r+1\,.$$

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1. Introduction

Let G be a finite abelian group (written additively) of order n. For positive integers t, d, with $t \geq d$, given a sequence $(a_1, a_2, \cdots, a_t)$ of length t, by a *d-sum* of the sequence one means a sum $a_{i_1} + \cdots + a_{i_d}$ of the elements in a subsequence of length d.

A result of Bollobás–Leader [2] is the following:

Theorem A. *Suppose we are given an abelian group G of order n and a sequence $(a_1, a_2, \cdots, a_{n+r})$ of elements of G, where r is a positive integer. Then, if 0 is not an n-sum, the number of distinct n-sums of the sequence is at least $r+1$.*

For $n \geq 3$, taking $r = n-2$ in the above result, one observes that given a sequence of length $2n-2$ of elements of G, if 0 is not an n-sum, then the set of n-sums is the set $G \setminus \{0\}$, that is, any non-zero element of G is an n-sum.

Similarly, for $n \geq 2$, taking $r = n-1$ in Theorem A, it follows that given a sequence of length $2n-1$ of elements of G, 0 must be an n-sum. When G is

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cyclic, this is the content of the well known theorem of Erdős–Ginzburg–Ziv [4] (can also see [1] or [8], for instance).

In the present paper, following the method of a simple proof of the above result of Bollobás and Leader as given by Yu [10], we prove a theorem which will imply (see Corollary 2) a result of Hamidoune [6], which had confirmed a conjecture of Caro [3] (see also [5], for instance) in a special case. For further information regarding these results, we refer to the paper of Grynkiewicz [5], where, among other things, the above mentioned conjecture of Caro has been established in full generality.

In what follows, we shall use the following notations. For a positive integer n, the symbol $[n]$ will denote the set $\{1, 2, \cdots, n\}$ and for a finite set S, $|S|$ will denote the number of elements of S.

Theorem 1. *Let G be an abelian group of order n and k a positive integer. Let $(w_1, w_2, ..., w_k)$ be a sequence of integers where each w_i is co-prime to n. Then, given a sequence $A : (x_1, x_2, ..., x_{k+r})$ of elements of G, where $1 \leq r \leq n-1$, if 0 is the most repeated element in the sequence, and*

$$\sum_{i=1}^{k} w_i x_{\sigma(i)} \neq 0,$$

for all permutations σ of $[k+r]$, we have

$$\left| \left\{ \sum_{i=1}^{k} w_i x_{\sigma(i)} : \sigma \text{ is a permutation of } [k+r] \right\} \right| \geq r+1\,.$$

If in the above statement, instead of 0, x_1 happens to be the most repeated element, then applying the result on the sequence $(a_1, a_2, ..., a_{k+r})$, where $a_i = x_i - x_1$, for all $i = 1, 2, \cdots k+r$, and observing that translation of a subset of G by an element does not change its cardinality, one obtains the following:

Corollary 1. *Let G be an abelian group of order n and k a positive integer. Let $(w_1, w_2, ..., w_k)$ be a sequence of integers where each w_i is co-prime to n. Then, given a sequence $A : (x_1, x_2, ..., x_{k+r})$ of elements of G, where $1 \leq r \leq n-1$, if x_1 is the most repeated element in the sequence, and*

$$\sum_{i=1}^{k} w_i x_{\sigma(i)} \neq \left(\sum_{i=1}^{k} w_i \right) x_1\,,$$

for all permutations σ of $[k+r]$, we have

$$\left| \left\{ \sum_{i=1}^{k} w_i x_{\sigma(i)} : \sigma \text{ is a permutation of } [k+r] \right\} \right| \geq r+1\,.$$

In Corollary 1 above, taking $r = n-2$ (that is, when A is of length $k+n-2$), and writing

$$\alpha = \left(\sum_{i=1}^{k} w_i\right) x_1 \,,$$

if

$$\sum_{i=1}^{k} w_i x_{\sigma(i)} \neq \alpha \,,$$

for all permutations σ of $[k+n-2]$, we have

$$\left\{\sum_{i=1}^{k} w_i x_{\sigma(i)} : \sigma \text{ is a permutation of } [k+n-2]\right\} = G \setminus \{\alpha\} \,.$$

Similarly, taking $r = n-1$ in Corollary 1, one obtains the following:

Corollary 2 (Hamidoune)**.** *Let G be an abelian group of order n and k a positive integer. Let $(w_1, w_2, ..., w_k)$ be a sequence of integers where each w_i is coprime to n. Then, given a sequence $A : (x_1, x_2, ..., x_{k+n-1})$ of elements of G, if x_1 is the most repeated element in the sequence, we have*

$$\sum_{i=1}^{k} w_i x_{\sigma(i)} = \left(\sum_{i=1}^{k} w_i\right) x_1 \,,$$

for some permutation σ of $[k+n-1]$. Hence, if the weights w_i satisfy

$$\sum_{i=1}^{k} w_i \equiv 0 \pmod{n} \,, \quad \textit{then we have} \quad \sum_{i=1}^{k} w_i x_{\sigma(i)} = 0 \,,$$

for some permutation σ of $[k+n-1]$.

2. Proof of Theorem 1

We need the following result of Scherk [9] (see also [7]).

Lemma 1. *Let B and C be two subsets of an abelian group G of order n. Suppose $0 \in B \cap C$ and suppose that the only solution of*

$$b + c = 0 \,, \quad b \in B \,, \quad c \in C \quad \textit{is} \quad b = c = 0 \,.$$

Then

$$|B + C| \geq \min(n, |B| + |C| - 1) \,.$$

Proof of Theorem 1. Let $L = \{i : x_i = 0\}$ and $|L| = l$. By our assumption, $l \le k-1$.

Let $S \subset [k+r] \setminus L$ be such that $|S| = s$ is maximal subject to the conditions $s \le k-1$ and

$$\sum_{i \in S} w_{f(i)} x_i = 0 \tag{1}$$

for some injective map $f : S \to [k]$. It is possible that S is empty.

We note that

$$l + s \le k - 1\,. \tag{2}$$

For, if $l + s \ge k$, then $l \ge k - s$ and hence writing $S' = \{j \in [k] : j \ne f(i) \text{ for } i \in S\}$,

$$\sum_{j \in S'} w_j x_{i_j} = 0\,, \tag{3}$$

where x_{i_j}'s run over a subsequence of $(x_i : i \in L)$ and from (1) and (3), we have

$$\sum_{i=1}^{k} w_i x_{\sigma(i)} = 0\,,$$

for some permutation σ of $[k+r]$, contradicting our assumption.

Now, from (2),

$$\big|[k+r] \setminus L \cup S\big| = k + r - (l+s) \ge k + r - (k-1) = r + 1\,.$$

Therefore, there is $T \subset [k+r] \setminus L \cup S$ such that $|T| = r$. Let h be the maximum number of repetition of any element in the subsequence $X : (x_i, i \in T)$. By our choice of L , $h \le l$ and hence by (2)

$$h + s \le l + s \le k - 1\,. \tag{4}$$

Let $X = X_1 \cup X_2 \cup \cdots \cup X_h$ be a partition of X into non-empty subsets, that is, in a particular X_i no element is repeated. More precisely, this is done in the following way. Let x be an element of X which is repeated h times. Then we put x in each X_i . Any other element, say y, occurring in X appears $m \le h$ times and we put y in X_i , $1 \le i \le m$. Thus,

$$|X_1| + |X_2| + \cdots + |X_h| = r\,. \tag{5}$$

From (4), $h < k-s$. Let $w'_1, \cdots, w'_{k-s}$ be the subsequence $(w_i : i \in S')$, where

$$S' = \{j \in [k] : j \ne f(i) \text{ for } i \in S\}\,.$$

We claim that for $1 \le j \le h$,

$$0 \notin w'_1 X_1 + w'_2 X_2 + \cdots + w'_j X_j\,. \tag{6}$$

If possible, suppose

$$0 = w'_1 x_{i_1} + w'_2 x_{i_2} + \cdots + w'_j x_{i_j} \,,$$

where $x_{i_t} \in X_t$, for $t = 1, 2, \cdots, j$. Then, appending $w'_1 x_{i_1} + w'_2 x_{i_2} + \cdots + w'_j x_{i_j}$ to the left hand side of (1), since $|S \cup \{i_1, i_2, \cdots, i_j\}| = s + j \le h + s \le k - 1$, by (4), we are led to a contradiction to the maximality of S.

This establishes the claim.

Writing $X'_t = X_t \cup \{0\}$, for $t = 1, 2, \cdots, h$, and observing that $|cX'_i| = |X'_i|$, for any integer c co-prime to n , from (5), we have

$$\sum_{i=1}^{h} |w'_i X'_i| = r + h \,.$$

Therefore, by repeated application of Lemma 1 (observe that by (6), the condition of the lemma is satisfied), we have

$$\begin{aligned} \left| \sum_{i=1}^{h} w'_i X'_i \right| &\ge \min\left\{ n, \sum_{i=1}^{h} |w'_i X'_i| - (h-1) \right\} \\ &= \min\{n, r+1\} = r + 1 \,. \end{aligned}$$

Therefore, $X : (x_i, i \in T)$ with h zeros from $(x_i, i \in L)$ has at least $(r + 1)$ h-sums with weights w'_i. So adding a weighted sum of the remaining $k + r - (r + h) = k - h$ elements of the sequence A with the remaining $k - h$ weights to each of the above $(r + 1)$ h-sums, we get at least $(r + 1)$ k-sums with the given weights. □

REFERENCES

[1] ADHIKARI, S. D.: *Aspects of Combinatorics and Combinatorial Number Theory*, Narosa Publishing House, New Delhi, 2002.

[2] BOLLOBÁS, B. – LEADER, I.: *The number of k-sums modulo k*, J. Number Theory **78** (1999), no. 1, 27–35.

[3] CARO, Y.: *Zero-sum problems - a survey*, Discrete Math. **152**(1996), 93–113.

[4] ERDŐS, P. – GINZBURG, A. – ZIV, A.: *Theorem in the additive number theory*, Bull. Res. Council Israel **10** (F) (1961), 41-43.

[5] GRYNKIEWICZ, D. J.: *A weighted Erdős-Ginzburg-Ziv theorem*, Combinatorica **26** (2006), no. 4, 445-453.

[6] HAMIDOUNE, Y.O.: *On weighted sums in abelian groups*, Discrete Math. **162** (1996), 127-132.

[7] KEMPERMAN, J.H.B. – SCHERK, P.: *Complexes in abelian groups*, Canad. J. Math. **6** (1954), 230–237.

[8] MELVYN B. NATHANSON: *Additive Number Theory. Inverse Problems and the Geometry of Sumsets*, Grad. Texts in Math., 165. Springer-Verlag, New York, 1996.

[9] SCHERK, P.: *Solution to Problem 4466*, Amer. Math. Monthly **62**, (1955) 46–47.

[10] YU, H.B.: *A simple proof of a theorem of Bollobás and Leader*, Proc. Amer. Math. Soc. **131** (2003) no. 9, 2639 -2640.

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