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NOTE ON THE JOINT DISTRIBUTION OF THE WEIGHTED SUM-OF-DIGITS FUNCTION MODULO ONE IN CASE OF PAIRWISE COPRIME BASES

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ABSTRACT. In this note we respond to the open question of Pillichshammer in [*Uniform distribution of sequences connected with the weighted sum-of-digits function*, Uniform Distribution Theory **2** (2007), 1–10.] : Under which conditions on the weight sequences is the multi-dimensional weighted sum-of-digits function for given pairwise coprime bases uniformly distributed modulo one? We do not give a complete answer, but we give a sufficient condition on the weight sequences. Furthermore, we prove that almost all kinds of weight sequences produce a uniformly distributed sequence.

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1. Introduction

For the definition and an introduction into the theory of uniform distribution modulo one we refer to [5] and [2]. Here we consider uniform distribution properties of sequences which are based on the weighted q-ary sum-of-digits function.

Let $\gamma = (\gamma_0, \gamma_1, \ldots)$ be a sequence in $\mathbb{R}$ and let $q \in \mathbb{N}, q \geq 2$. For $n \in \mathbb{N}$ with base q representation $n = n_0 + n_1 q + \cdots$ we define the weighted q-ary sum-of-digits function as

$$s_{q,\gamma}(n) := \sum_{s=0}^{r} \gamma_i \cdot n_i,$$

where $r = [\log_q(n)]$. Here and later on $[x]$ denotes the integer part of a real number x.

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Let $d \in \mathbb{N}$. For $j \in \{1, \ldots, d\}$ let $\gamma^{(j)} = \big(\gamma_0^{(j)}, \gamma_1^{(j)}, \ldots\big)$ be given weight sequences in $\mathbb{R}$ and $q_j \in \mathbb{N}, q_j \geq 2$ be given bases. We define the multi-dimensional weighted sum-of-digits function as

$$s_{q_1,\ldots,q_d,\boldsymbol{\gamma}}(n) := \big(s_{q_1,\gamma^{(1)}}(n), \ldots, s_{q_d,\gamma^{(d)}}(n)\big)\,, \tag{1}$$

where

$$\boldsymbol{\gamma} = (\boldsymbol{\gamma}_0, \boldsymbol{\gamma}_1, \ldots) \quad \text{and} \quad \boldsymbol{\gamma}_k = \big(\gamma_k^{(1)}, \ldots, \gamma_k^{(d)}\big) \quad \text{for} \quad k \in \mathbb{N}_0\,.$$

Pillichshammer stated in [7] the following open question, which appears as Problem 1.22 in [6]:

Let $q_1, \ldots, q_d \geq 2$ be pairwise coprime integers. Under which conditions on the weight sequences $\gamma^{(j)} = \big(\gamma_0^{(j)}, \gamma_1^{(j)}, \ldots\big)$ in $\mathbb{R}$ for $j \in \{1, \ldots, d\}$, is the sequence

$$\big(\big\{s_{q_1,\ldots,q_d,\boldsymbol{\gamma}}(n)\big\}\big)_{n\geq 0} \tag{2}$$

uniformly distributed modulo one. Here $\{x\}$ denotes the fractional part of a real number x and for a vector $\boldsymbol{x}$, $\{\boldsymbol{x}\}$ is understood componentwise.
In the following theorem we find a sufficient condition on the weight sequences, so that uniform distribution of (2) follows. For $x \in \mathbb{R}$ we define $\|x\| := \min_{k\in\mathbb{Z}} |x - k|$.

Theorem 1. *Let $q_1, \ldots, q_d \geq 2$ be pairwise coprime integers and $\gamma^{(1)}, \ldots, \gamma^{(d)}$ be given weight sequences in $\mathbb{R}$. If for each dimension $j \in \{1, \ldots, d\}$ the following sum*

$$\sum_{i=0}^{\infty} \Big\| h\big(\gamma_{2i+1}^{(j)} - q_j \gamma_{2i}^{(j)}\big) \Big\|^2$$

is divergent for every nonzero integer h, then

$$\big(\big\{s_{q_1,\ldots,q_d,\boldsymbol{\gamma}}(n)\big\}\big)_{n\geq 0}$$

is uniformly distributed in $[0,1)^d$.

As a consequence of this theorem we get the following corollary:

Corollary 1. *The multi-dimensional weighted sum-of-digits function,*

$$s_{q_1,\ldots,q_d,\boldsymbol{\gamma}}(n) = \big(s_{q_1,\gamma^{(1)}}(n), \ldots, s_{q_d,\gamma^{(d)}}(n)\big)\,,$$

in given pairwise coprime bases is uniformly distributed modulo one for almost all weight sequences $\boldsymbol{\gamma} : \mathbb{N}_0 \to [0,1)^d$.

Before we prove our statements in the next section we mention some previous results. Pillichshammer [7] investigated the distribution of the sequence (2) in case of fixed, equal bases and proved the following result.

Theorem 2 (Pillichshammer). *For any arbitrary $\boldsymbol{\gamma}$ and $q_1 = \cdots = q_d = q$, where $q \geq 2, q \in \mathbb{N}$ the sequence (2) is uniformly distributed in $[0,1)^d$ if and only if for every $\boldsymbol{h} \in \mathbb{Z}^d \setminus \{\mathbf{0}\}$ one of the following conditions holds:*
Either

$$\sum_{\substack{k=0 \\ \langle \boldsymbol{h}, \boldsymbol{\gamma}_k \rangle q \notin \mathbb{Z}}}^{\infty} \left\| \langle \boldsymbol{h}, \boldsymbol{\gamma}_k \rangle \right\|^2 = \infty$$

or there exists a $k \in \mathbb{N}_0$ such that $\langle \boldsymbol{h}, \boldsymbol{\gamma}_k \rangle \notin \mathbb{Z}$ and $\langle \boldsymbol{h}, \boldsymbol{\gamma}_k \rangle q \in \mathbb{Z}$.

If the weighted q-ary sum-of-digits function is replaced by the weighted digit-block-counting-function, a similar criterion for uniform distribution of the corresponding sequence can be found in [3].

At least one of the conditions above in the one-dimensional case is of course a necessary condition on each weight sequence $\gamma^{(j)}$, $j \in \{1, \ldots, d\}$, for uniform distribution of (2). It can be easily checked, that if our sufficient condition of Theorem 1 holds for a weight sequence $\gamma^{(j)}$ and base q_j, then the first condition above for $d = 1$ and $q = q_j$ will be fulfilled as well for this weight sequence.

In the following we list some examples of special weight sequences, for which the question of Pillichshammer has been answered already.

Example 1. If $\gamma_k^{(j)} = q_j^k \alpha_j$ for all $j \in \{1, \ldots, d\}$ and all $k \in \mathbb{N}_0$, we get the d-dimensional Kronecker sequence $\big(\{n(\alpha_1, \ldots, \alpha_d)\}\big)_{n \in \mathbb{N}_0}$, which is well known to be uniformly distributed in $[0,1)^d$ if and only if $1, \alpha_1, \ldots, \alpha_d$ are linearly independent over $\mathbb{Q}$.

Example 2. For pairwise coprime bases and $\gamma_k^{(j)} = q_j^{-k-1}$ for all $j \in \{1, \ldots, d\}$ the sequence (2) is called the d-dimensional van der Corput-Halton sequence, which is uniformly distributed in $[0,1)^d$.

Example 3. For pairwise coprime bases and constant weight sequences $\boldsymbol{\gamma}_k := (\alpha_1, \ldots, \alpha_d)$ for all $k \in \mathbb{N}_0$ Drmota and Larcher [1] found, that the sequence (2) is uniformly distributed modulo one if and only if $\alpha_1, \ldots, \alpha_d$ are irrational numbers.

Our sufficient condition is a generalization of the sufficient condition on the special weight sequences of Drmota and Larcher given in Example 3. As will be pointed out later our method neither leads to a sufficient condition on the class of weight sequences in Example 1, nor reproves Example 2.

Throughout the paper let the dimension $d \in \mathbb{N}$ and the bases $q_1, \ldots, q_d \geq 2$, pairwise coprime integers, be fixed.

2. Proofs

Here we generalize the method of Kim [4] and especially the method of Drmota and Larcher [1] from completely q-additive functions to q-additive functions.

Let $q \geq 2$ be an integer. A function $f : \mathbb{N}_0 \to \mathbb{R}$ is called q-additive if $f(0) = 0$ and if for any nonnegative integers a, b, j with $0 \leq b \leq q^j - 1$ the relation $f(aq^j + b) = f(aq^j) + f(b)$ holds. f is called completely q-additive if $f(aq^j) = f(a)$ is true for all nonnegative integers a, j in addition.

Note that the weighted q-ary sum-of-digits function is q-additive and only in case of constant weights it is completely q-additive.

Proof of Theorem 1. We fix the weight sequences $\gamma^{(1)}, \dots, \gamma^{(d)}$ and define $g(n) := \prod_{j=0}^{d} g_j(n)$, where $g_j(n) := e\big(h_j s_{q_j,\gamma^{(j)}}(n)\big), j \in \{1, \dots, d\}$ for any $(h_1, \dots, h_d) \in \mathbb{Z}^d \setminus \{\mathbf{0}\}$. Here and later on $e(x)$ denotes $e^{2\pi i x}$. As in [4] and [1] we use Weyl's inequality

$$\left| \sum_{n=0}^{N-1} g(n) \right|^2 \leq \frac{2N^2}{K} + \frac{4N}{K} \sum_{k=1}^{K} \left| \sum_{n=0}^{N-k-1} \overline{g(n)} g(n+k) \right|, \tag{3}$$

for $K = \left[N^{1/(3d)}\right]$, to estimate exponential sums.

We generalize Lemma 6 in [4] from completely q-additive functions to q-additive functions and get the following auxiliary result, which is proved as the original one.

Let f be a q-additive function. Let t and k be positive integers with $0 \leq r < q^t - k$. Then we have, for all nonnegative integers n satisfying $n \equiv r \bmod q^t$,

$$f(n+k) - f(n) = f(r+k) - f(r).$$

Hence, as in [4], we obtain

$$\left| \sum_{n=0}^{N-1} g(n) \right|^2 \leq 4N^2 \prod_{j=1}^{d} \left(\frac{1}{K} \sum_{k=1}^{K} \left| \frac{1}{Q_j} \sum_{r_j=0}^{Q_j-1} \overline{g_j(r_j)} g_j(r_j+k) \right|^2 \right)^{\frac{1}{d+1}} + O\left(\frac{2N^2}{K} \right), \tag{4}$$

where $Q_j = q_j^{t_j}$ with $t_j = [2 \log_{q_j}(K)]$ for $j \in \{1, \dots, d\}$.

We refer to [4, p. 329–332] for more detailed information.

To prove uniform distribution of (2), it suffices to show that for every $(h_1, \dots, h_d) \in \mathbb{Z}^d \setminus \{\mathbf{0}\}$ at least one factor of the upper bound above tends to zero as N (and therefore K) increases. Note that each factor is bounded by 1. In order to do this we prove for arbitrary $j \in \{1, \dots, d\}$, if $h_j \neq 0$ and if the weight sequence fulfills the condition, $\sum_{i=0}^{\infty}\left\|h_j\left(\gamma_{2i+1}^{(j)} - q_j\gamma_{2i}^{(j)}\right)\right\|^2$ is divergent, the corresponding factor on the right side of (4) tends to zero as K increases.

For simplicity of notation we restrict to $d = 1$ in the following. We omit the index j and the superscript (j) and fix integers $q \geq 2$ and $h \neq 0$.
For arbitrary positive integers N, K and $r \in \{0, 1\}$ we define the correlation functions

$$\begin{aligned}\Phi_N(k) &= \frac{1}{N}\sum_{n=0}^{N-1}\overline{g(n)}g(n+k)\,,\\ \Phi_{K,N}(r) &= \frac{1}{K}\sum_{k=0}^{K-1}\overline{\Phi_N(k)}\Phi_N(k+r)\end{aligned}$$

for $g(n) = e\big(hs_{q,\gamma}(n)\big)$.

Note that each factor in (4), apart from the exponent, is similar to $\Phi_{K,Q_j}(0)$, where $g_j(n) = e\big(h_j s_{q_j,\gamma^{(j)}}(n)\big)$ and it suffices to show that $\Phi_{K,Q_j}(0)$ tends to zero as K increases in order to show this asymptotic behavior for the corresponding factor. In the following we compute $\Phi_{K,N}(0)$. Our method is similar to the one of Kim [4] and Drmota and Larcher [1], but we have to take care always when completely q-additive properties are applied. Therefore we define the superscript $^{(l)}$ for $l \in \mathbb{N}_0$, which changes the argument in an arithmetic function from n to $q^l n$. Note that if $f(n) = s_{q,\gamma}(n)$, $f^{(l)}(n)$ remains a weighted q-ary sum-of-digits function $s_{q,\gamma'}(n)$ with the new weight sequence $(\gamma_i')_{i\geq 0} = (\gamma_{i+l})_{i\geq 0}$ for all $i \in \mathbb{N}_0$. For the correlation functions the superscript $^{(l)}$ denotes

$$\begin{aligned}\Phi_N^{(l)}(k) &= \frac{1}{N}\sum_{n=0}^{N-1}\overline{g^{(l)}(n)}g^{(l)}(n+k)\,,\\ \Phi_{K,N}^{(l)}(r) &= \frac{1}{K}\sum_{k=0}^{K-1}\overline{\Phi_N^{(l)}(k)}\Phi_N^{(l)}(k+r)\,,\end{aligned}$$

where $g^{(l)}(n) = g(q^{\,l}n)$, since $g : \mathbb{N}_0 \to \mathbb{C}$ is an arithmetic function.
If $l = 0$ we will often omit the superscript $^{(l)}$ for simplicity.

We get the following estimates for the correlation functions, similar to Lemma 9, 10 and 11 [4].

- For any integers $k \geq 0$ and $0 \leq r < q$ we have

$$\Phi_{qN}^{(l)}(qk+r) = \alpha_r^{(l)}\Phi_N^{(l+1)}(k) + \beta_r^{(l)}\Phi_N^{(l+1)}(k+1)\,,$$

where

$$\begin{aligned}\alpha_r^{(l)} &= \frac{1}{q}\sum_{i=0}^{q-r-1}\overline{g^{(l)}(i)}g^{(l)}(i+r)\,,\\ \beta_r^{(l)} &= \frac{1}{q}\sum_{i=q-r}^{q-1}\overline{g^{(l)}(i)}g^{(l)}(i+r-q)\,.\end{aligned}$$

The quantities $\alpha_r^{(l)}$ and $\beta_r^{(l)}$ satisfy

$$\left|\alpha_r^{(l)}\right| \leq \frac{q-r}{q}, \quad \left|\beta_r^{(l)}\right| \leq \frac{r}{q}\,.$$

Note that it is possible to extend the domain of r to $0 \leq r \leq q$ by setting $\alpha_q^{(l)} = 0$ and $\beta_q^{(l)} = 1$.

- For $r \in \{0,1\}$ we have

$$\Phi_{qK,qN}^{(l)}(r) = \lambda_r^{(l)}\Phi_{K,N}^{(l+1)}(0) + \mu_r^{(l)}\Phi_{K,N}^{(l+1)}(1) + \nu_r^{(l)}\overline{\Phi_{K,N}^{(l+1)}(0)} + E_{K,N}^{(l+1)}(r)\,,$$

where $\left|E_{K,N}^{(l+1)}(r)\right| \leq 2/K$ and

$$\begin{aligned}\lambda_r^{(l)} &= \frac{1}{q}\sum_{i=0}^{q-1}\left(\overline{\alpha_i^{(l)}}\alpha_{i+r}^{(l)} + \overline{\beta_i^{(l)}}\beta_{i+r}^{(l)}\right),\\ \mu_r^{(l)} &= \frac{1}{q}\sum_{i=0}^{q-1}\overline{\alpha_i^{(l)}}\beta_{i+r}^{(l)}\,,\\ \nu_r^{(l)} &= \frac{1}{q}\sum_{i=0}^{q-1}\overline{\beta_i^{(l)}}\alpha_{i+r}^{(l)}\,.\end{aligned}$$

Here $\alpha_i^{(l)}$ and $\beta_i^{(l)}$ are given as above. Furthermore, for $r \in \{0,1\}$ we have

$$\left|\lambda_r^{(l)}\right| + \left|\mu_r^{(l)}\right| + \left|\nu_r^{(l)}\right| \leq 1.$$

- For $r \in \{0,1\}$ we have

$$\left|\Phi_{q^2K,q^2N}^{(l)}(r)\right| \leq \rho_r^{(l)}\left|\Phi_{K,N}^{(l+2)}(0)\right| + \sigma_r^{(l)}\left|\Phi_{K,N}^{(l+2)}(1)\right| + \frac{7}{K}\,, \tag{5}$$

where

$$\begin{aligned}\rho_r^{(l)} &= \left|\lambda_r^{(l)}\lambda_0^{(l+1)} + \mu_r^{(l)}\lambda_1^{(l+1)} + \nu_r^{(l)}\overline{\lambda_1^{(l+1)}}\right|, \\ \sigma_r^{(l)} &= \left|\lambda_r^{(l)}\mu_0^{(l+1)} + \mu_r^{(l)}\mu_1^{(l+1)} + \nu_r^{(l)}\overline{\nu_1^{(l+1)}}\right| \\ &\quad + \left|\lambda_r^{(l)}\nu_0^{(l+1)} + \mu_r^{(l)}\nu_1^{(l+1)} + \nu_r^{(l)}\overline{\mu_1^{(l+1)}}\right|,\end{aligned}$$

where $\lambda_r^{(l)}, \mu_r^{(l)}, \nu_r^{(l)}$ are given as above.

For the sake of completeness we prove the first estimate in order to show how to use the superscript $^{(l)}$. Note that,

$$g^{(l)}(aq+b) = g^{(l)}(aq)g^{(l)}(b) = g^{(l+1)}(a)g^{(l)}(b),$$

for any integers $a \geq 0$ and $0 \leq b \leq q-1$, since g is the exponential of a q-additive function. Analogously to Kim [4, p. 317] we get the following chain of equalities:

$$\begin{aligned} qN\Phi_{qN}^{(l)}(qk+r) &= \sum_{i=0}^{q-1}\sum_{n=0}^{N-1} \overline{g^{(l)}(qn+i)}g^{(l)}(qn+i+qk+r) \\ &= \sum_{i=0}^{q-r-1}\sum_{n=0}^{N-1} \overline{g^{(l)}(qn)g^{(l)}(i)}g^{(l)}\big(q(n+k)\big)g^{(l)}(i+r) \\ &\quad + \sum_{i=q-r}^{q-1}\sum_{n=0}^{N-1} \overline{g^{(l)}(qn)g^{(l)}(i)}g^{(l)}\big(q(n+k+1)\big)g^{(l)}(i+r-q) \\ &= \sum_{i=0}^{q-r-1} \overline{g^{(l)}(i)}g^{(l)}(i+r) \sum_{n=0}^{N-1} \overline{g^{(l+1)}(n)}g^{(l+1)}(n+k) \\ &\quad + \sum_{i=q-r}^{q-1} \overline{g^{(l)}(i)}g^{(l)}(i+r-q) \sum_{n=0}^{N-1} \overline{g^{(l+1)}(n)}g^{(l+1)}(n+k+1) \\ &= qN\left(\alpha_r^{(l)}\Phi_N^{(l+1)}(k) + \beta_r^{(l)}\Phi_N^{(l+1)}(k+1)\right). \end{aligned}$$

The estimates for $\alpha_r^{(l)}$ and $\beta_r^{(l)}$ follow immediately since $g^{(l)}(n)$ has absolute value 1 for all nonnegative integers n.

The proofs of the other estimates are similar. For the interested reader we refer to [4, p. 318–320] and to the definition of the superscript $^{(l)}$. Every time completely q-additive properties are applied we have to increase the superscript $^{(l)}$ as in the proof above.

The following estimate is similar to (7) in [1] and in case of constant weights $\gamma_i = \alpha$ for all $i \in \mathbb{N}_0$ they are equal.

- For $r \in \{0,1\}$ the quantities $\rho_r^{(l)}$ and $\sigma_r^{(l)}$, defined as above, satisfy

$$\rho_r^{(l)} + \sigma_r^{(l)} \leq 1 - \frac{\left\|h(\gamma_{l+1} - \gamma_l q)\right\|^2}{4}\,. \tag{6}$$

In the following we sketch the proof of (6).

As $g^{(l)}(n) = e(h\gamma_l n)$ for $n \in \{0,1,\ldots,q-1\}$, it is easy to check that

$$\begin{aligned} \alpha_r^{(l)} &= \frac{q-r}{q} e\big(h(\gamma_l r)\big)\,, \\ \beta_r^{(l)} &= \frac{r}{q} e\big(h\gamma_l(r-q)\big)\,, \end{aligned}$$

where $0 \leq r < q$. Note that these equations can be used for $r = q$ as well.

Using these equations leads to the following formulas for $r \in \{0,1\}$ after little exhausting computations

$$\begin{aligned} \lambda_r^{(l)} &= e(h\gamma_l r)\frac{2q^2 - 3r + 1}{3q^2} \\ \mu_r^{(l)} &= e\big(h\gamma_l(r-q)\big)\frac{q^2 + 3qr + 3r - 1}{6q^2} \\ \nu_r^{(l)} &= e\big(h\gamma_l(r+q)\big)\frac{q^2 - 3qr + 3r - 1}{6q^2}\,. \end{aligned}$$

The further steps are along the lines of [1, p. 93–94]:

For $r \in \{0,1\}$ we estimate $\rho_r^{(l)}$ first.

$$\begin{aligned} \rho_r^{(l)} &= \left|\lambda_r^{(l)}\lambda_0^{(l+1)} + \mu_r^{(l)}\lambda_1^{(l+1)} + \nu_r^{(l)}\overline{\lambda_1^{(l+1)}}\right| \\ &= \Big||\lambda_r^{(l)}||\lambda_0^{(l+1)}|e(h(\gamma_l r)) + |\mu_r^{(l)}||\lambda_1^{(l+1)}|e(h(\gamma_l(r-q) + \gamma_{l+1})) \\ &\quad + |\nu_r^{(l)}||\overline{\lambda_1^{(l+1)}}|e(h(\gamma_l(r+q) - \gamma_{l+1}))\Big| \\ &\leq \Big||\lambda_r^{(l)}||\lambda_0^{(l+1)}| + |\mu_r^{(l)}||\lambda_1^{(l+1)}|e(h(\gamma_{l+1} - \gamma_l q))\Big| + |\nu_r^{(l)}||\overline{\lambda_1^{(l+1)}}|. \end{aligned}$$

We observe that $|\lambda_r^{(l)}||\lambda_0^{(l+1)}| \geq |\mu_r^{(l)}||\lambda_1^{(l+1)}|$, use the inequality $\big|a + be(\theta)\big| \leq a + b - 4b\|\theta\|^2$ (for $0 \leq b \leq a$) and obtain

$$\begin{aligned} &\Big||\lambda_r^{(l)}||\lambda_0^{(l+1)}| + |\mu_r^{(l)}||\lambda_1^{(l+1)}|e(h(\gamma_{l+1} - \gamma_l q))\Big| \\ &\leq |\lambda_r^{(l)}||\lambda_0^{(l+1)}| + |\mu_r^{(l)}||\lambda_1^{(l+1)}| - \tfrac{\|h(\gamma_{l+1} - \gamma_l q)\|^2}{4}, \end{aligned}$$

where we also used the relations $|\mu_r^{(l)}| \geq |\mu_0^{(l)}| \geq 1/8$ and $|\lambda_1^{(l+1)}| \geq 1/2$.

Considering these inequalities for $\rho_r^{(l)}$, the formula for $\sigma_r^{(l)}$ and that $|\lambda_r^{(l)}| + |\mu_r^{(l)}| + |\nu_r^{(l)}| \le 1$ for $r \in \{0,1\}$ leads after some straightforward estimates to the desired result.

For the next step we use a generalization of Lemma 5 in [4]. Let a_i, b_i, c_i, d_i for $i \in \mathbb{N}$ be nonnegative reals satisfying $a_i + b_i \le 1 - \epsilon_i$ and $c_i + d_i \le 1 - \epsilon_i$ for some $\epsilon_i > 0$. Let $m \ge 1$ and

$$\prod_{i=1}^{m} \begin{pmatrix} a_i & b_i \\ c_i & d_i \end{pmatrix} = \begin{pmatrix} A_m & B_m \\ C_m & D_m \end{pmatrix}.$$

Then we have

$$A_m + B_m \le \prod_{i=1}^{m} (1-\epsilon_i) \le e^{-\sum_{i=1}^{m} \epsilon_i}, \quad C_m + D_m \le \prod_{i=1}^{m} (1-\epsilon_i) \le e^{-\sum_{i=1}^{m} \epsilon_i}.$$

This is easily proved by induction and the fact that $1 - x \le e^{-x}$ for all reals x.

Now we use the equations (5) and (6) and the mentioned auxiliary result step-by-step to obtain the following estimate.

- For $r \in \{0,1\}$ and any nonnegative integer t we have

$$\Phi_{q^{2t}K, q^{2t}N}(r) \le e^{-\sum_{i=0}^{t-1} \frac{\|h(\gamma_{2i+1} - q\gamma_{2i})\|^2}{4}} \left(1 + \frac{7q^2}{K}\right).$$

To prove this we define

$$M^{(2l)} := \begin{pmatrix} \rho_0^{(2l)} & \sigma_0^{(2l)} \\ \rho_1^{(2l)} & \sigma_1^{(2l)} \end{pmatrix}$$

and $P_{2t} := \left|\Phi_{q^{2t}K, q^{2t}N}(0)\right|$ and $Q_{2t} := \left|\Phi_{q^{2t}K, q^{2t}N}(1)\right|$. The superscript $^{(l)}$ for these quantities is defined by the obvious way. We estimate P_{2t} and Q_{2t} step-by-step and get

$$\begin{aligned} \begin{pmatrix} P_{2t} \\ Q_{2t} \end{pmatrix} &\le M^{(0)} \begin{pmatrix} P_{2(t-1)}^{(2)} \\ Q_{2(t-1)}^{(2)} \end{pmatrix} + \frac{7}{q^{2(t-1)}K} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \\ &\le \prod_{l=0}^{t-1} M^{(2l)} \begin{pmatrix} P_0^{(2t)} \\ Q_0^{(2t)} \end{pmatrix} + \begin{pmatrix} R_t \\ S_t \end{pmatrix}, \end{aligned}$$

where

$$\begin{pmatrix} R_t \\ S_t \end{pmatrix} = \sum_{j=1}^{t} \frac{7}{q^{2(j-1)}K} \prod_{l=0}^{t-j-1} M^{(2l)} \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

Here and later on inequalities for vectors are read componentwise.

The trivial bounds $P_0^{(2t)} \leq 1$ and $Q_0^{(2t)} \leq 1$ and the auxiliary result imply the desired inequality for the first term

$$\prod_{l=0}^{t-1} M^{(2l)} \begin{pmatrix} P_0^{(2t)} \\ Q_0^{(2t)} \end{pmatrix} \leq \begin{pmatrix} e^{-\sum_{l=0}^{t-1} \frac{\|h(\gamma_{2l+1}-\gamma_{2l}q)\|^2}{4}} \\ e^{-\sum_{l=0}^{t-1} \frac{\|h(\gamma_{2l+1}-\gamma_{2l}q)\|^2}{4}} \end{pmatrix}.$$

It remains to estimate the second term:

$$\sum_{j=1}^{t} \frac{7}{q^{2(j-1)}K} \prod_{l=0}^{t-j-1} M^{(2l)} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \leq$$

$$\leq \sum_{j=1}^{t} \frac{7}{q^{2(j-1)}K} \prod_{l=0}^{t-j-1} \left(1 - \frac{\|h(\gamma_{2l+1} - \gamma_{2l}q)\|^2}{4}\right) \begin{pmatrix} 1 \\ 1 \end{pmatrix} \leq$$

$$\leq e^{-\sum_{l=0}^{t-1} \frac{\|h(\gamma_{2l+1}-\gamma_{2l}q)\|^2}{4}} \frac{7q^2}{K} \cdot \underbrace{\sum_{j=1}^{t} \frac{1}{q^{2j} \prod_{l=t-j}^{t-1}(1 - \frac{\|h(\gamma_{2l+1}-\gamma_{2l}q)\|^2}{4})}}_{(*)} \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

Since $\left(1 - \frac{\|h(\gamma_{2l+1}-\gamma_{2l}q)\|^2}{4}\right) \geq 1/2$, we observe that $(*) \leq 1$ and the estimate above follows.

To estimate $\Phi_{K,N}(0)$ we use the following result of Kim [4, p. 327]. For $\sqrt{N} \leq K \leq N$, $N \geq q^{10}$, $t := [\log_q(N)/5]$ $M \geq 1$, $L \geq 1$, $0 \leq R, S < q^{2t}$, such that $N = q^{2t}M + R, K = q^{2t}L + S$, we have

$$\Phi_{K,N}(0) = \Phi_{q^{2t}L, q^{2t}M}(0) + O\left(\frac{q^{2t}}{\sqrt{N}}\right).$$

By the estimate above we have

$$\Phi_{q^{2t}L, q^{2t}M}(0) = O\left(e^{-\sum_{i=0}^{t-1} \frac{\|h(\gamma_{2i+1}-q\gamma_{2i})\|^2}{4}}\right)$$

and since $q^{2t} \leq q^{2\frac{\log(N)}{5\log(q)}} \leq N^{\frac{2}{5}}$, we get for $N^{2/5}/N^{1/2} = N^{-1/10} = o(1)$. Since $K \leq Q_j \leq K^2$ for all $j \in \{1, \ldots, d\}$, we can apply this result for Q_j instead of N.

For arbitrary $(h_1, \ldots, h_d) \in \mathbb{Z}^d \setminus \{\mathbf{0}\}$ we have at least one $h_j \neq 0$ with $j \in \{1, \ldots, d\}$. Since $\sum_{i=0}^{\infty} \|h_j(\gamma_{2i+1}^{(j)} - q_j\gamma_{2i}^{(j)})\|^2 = \infty$, the j-th factor on the right of (4) tends to zero as N, thus K as well, increases. Thus, the right side of (4) is $o(N^2)$ and uniform distribution follows by Weyl's criterion, which completes the proof of Theorem 1. $\square$

Proof of Corollary 1. We mimic the proof of Corollary 1 in [7]. Let $j \in \{1, \ldots, d\}$. We get for the sequence of independent uniformly distributed random variables $X_0, X_1, X_2, \ldots \in [0,1)$ and arbitrary but fixed $q_j \geq 2, q_j \in \mathbb{N}$, $h \in \mathbb{Z}\backslash\{0\}$ the expected value $\mathbb{E}\left(\left\|h(X_{2i+1} - q_j X_{2i})\right\|^2\right) = \frac{1}{12}$. From Kolmogorov's strong law of large numbers it follows that for $n \to \infty$

$$\frac{\left\|h(X_1 - q_j X_0)\right\|^2 + \cdots + \left\|h(X_{2n+1} - q_j X_{2n})\right\|^2}{n+1} \to \frac{1}{12} \quad a.e.$$

This leads to

$$\sum_{i=0}^{\infty} \left\|h\left(\gamma_{2i+1}^{(j)} - q_j \gamma_{2i}^{(j)}\right)\right\|^2 = \infty$$

for almost all weight sequences $\gamma^{(j)} : \mathbb{N}_0 \to [0,1)$ and hence

$$\sum_{i=0}^{\infty} \left\|h\left(\gamma_{2i+1}^{(j)} - q_j \gamma_{2i}^{(j)}\right)\right\|^2 = \infty \quad \forall h \in \mathbb{Z}\backslash\{0\}$$

for almost all weight sequences $\gamma^{(j)} : \mathbb{N}_0 \to [0,1)$. □

3. Concluding remarks

Note that Corollary 1 proves uniform distribution for almost all weight sequences, however the sufficient condition for uniform distribution given in Theorem 1 is certainly not a necessary one.

For example the van der Corput-Halton sequence, mentioned already in Example 2, does not satisfy our condition. Here we have $\gamma_{2i} = \frac{1}{q^{2i+1}}$ and $\gamma_{2i+1} = \frac{1}{q^{2i+2}}$ for any arbitrary nonnegative integer i, hence $\sum_{i=0}^{\infty}\left\|h(\gamma_{2i+1} - q\gamma_{2i})\right\|^2$ converges for every integer h.

Also the d-dimensional Kronecker sequence, which is uniformly distributed if and only if $1, \alpha_1, \ldots, \alpha_s$ are linear independent over $\mathbb{Q}$, does not satisfy our condition. In this case we have

$$\gamma_{2i} = \alpha q^{2i}, \gamma_{2i+1} = \alpha q^{2i+1}$$

hence

$$\sum_{i=0}^{\infty}\left\|h(\gamma_{2i+1} - q\gamma_{2i})\right\|^2 = 0$$

for every integer h.

Why does our method fail for these sequences? Let us briefly consider the following question: What properties should a sufficient as well as necessary list of conditions on the weight sequences at least have? First of all, for $d = 1$ the list of conditions should be equivalent to the two conditions in Theorem 2. Due to Example 2 and 3 one may assume, that it suffices to show uniform distribution of each component, if we have given weighted q-ary sum-of-digits functions in pairwise coprime bases. We restrict to sequences with this property by using inequality (4), which contains a product of terms depending on one component of the multi-dimensional weighted sum-of-digits function. A counter example of this assumption is the Kronecker sequence, mentioned in Example 1. The one-dimensional Kronecker sequence $(n\alpha)_{n\geq 0}$ is uniformly distributed modulo one if and only if $\alpha \in \mathbb{R}\backslash\mathbb{Q}$. For the d-dimensional Kronecker sequence irrationality of α_j for all $j \in \{1, \ldots, d\}$ does not suffice. For d joint one-dimensional Kronecker sequences we get the following sufficient as well as necessary condition for uniform distribution modulo one: $1, \alpha_1, \ldots, \alpha_d$ have to be linear independent over $\mathbb{Q}$. Obviously our method fails for all sequences with "interdependent components" and an appropriate method of proof should not lead to one-dimensional estimates. This makes clear, why our method fails for the d-dimensional Kronecker sequence.

Concerning the van der Corput-Halton sequence: To prove uniform distribution we use Weyl's inequality (3) to estimate exponential sums and we need that the upper bounds for these exponential sums are of order $o(N^2)$. However, neither for the van der Corput-Halton sequence, nor for the Kronecker sequence the right side of (3) is of the form $o(N^2)$.

This leads us to the conclusion, that a quite different method has to be used to obtain a complete solution of the open question in [7].

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