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WEIGHTED LIMIT THEOREMS FOR GENERAL DIRICHLET SERIES

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ABSTRACT. Under some hypotheses, weighted limit theorems in the sense of weak convergence of probability measures on the complex plane for general Dirichlet series are obtained. If the system of exponents of the series is linearly independent over the field of rational numbers, then the explicit form of the limit measure is given.

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1. Introduction

An idea of application of probability methods in the theory of Dirichlet series belongs to H. Bohr, and it was implemented in his joint works with B. Jessen. Let R be a closed rectangle on the complex plane with the edges parallel to the axes, and let $L_T(\sigma, R)$ denote the Jordan measure of the set $\{t \in [0, T] : \log\zeta(\sigma + it) \in R\}$, where, as usual, $\zeta(s)$ is the Riemann zeta-function. Then in [3] it was proved that, for $\sigma > 1$, there exists the limit

$$\lim_{T\to\infty} \frac{L_T(\sigma, R)}{T} = W(\sigma, R).$$

In [4], the latter result was extended to the half - plane $\sigma > 1/2$.

Later, the Bohr-Jessen method was developed by B. Jessen and A. Wintner, V. Borchsenius and B. Jessen, A. Selberg, A. Ghosh, P. D. T. A. Elliott and others. In the middle of the last century, the theory of the weak convergence of probability measures was created, and it became convenient to state Bohr--Jessen's type results as limit theorems in the sense of weak convergence of probability measures. For example, we recall a modern version of Bohr-Jessen's theorem for the Riemann zeta-function. Denote by $\mathcal{B}(S)$ the class of Borel sets

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of a metric space S. Let $\operatorname{meas}\{A\}$ denote the Lebesgue measure of a measurable set $A \subset \mathbb{R}$, and let, for $T > 0$,

$$\nu_T(\ldots) = \frac{1}{T}\operatorname{meas}\big\{t \in [0,T] : \ldots\big\},$$

where in place of dots a condition satisfied by t is to be written. Then, see [7], for $\sigma > 1/2$, the probability measure

$$\nu_T\big(\zeta(\sigma+it) \in A\big), \quad A \in \mathcal{B}(\mathbb{C}),$$

converges weakly to some probability measure on $\big(\mathbb{C}, \mathcal{B}(\mathbb{C})\big)$ as $T \to \infty$.

Limit theorems of the above type characterize the asymptotic behaviour of functions given by Dirichlet series.

Other allied results and bibliography can be found in [1], [6], [7], [13] and [14].

Let $\{a_m : m \in \mathbb{N}\}$ be a sequence of complex numbers, and $\{\lambda_m : m \in \mathbb{N}\}$ be an increasing sequence of real numbers such that $\lim\limits_{m\to\infty} \lambda_m = +\infty$. Then the series

$$\sum_{m=1}^{\infty} a_m \mathrm{e}^{-\lambda_m s}, \quad s = \sigma + it, \tag{1}$$

is called a general Dirichlet series. If $\lambda_m = \log m$ for all $m \in \mathbb{N}$, then we have an ordinary Dirichlet series. It is well known that the region of convergence as well as of absolute convergence of Dirichlet series is a half-plane.

In [9] limit theorems in the sense of weak convergence of probability measures on the complex plane $\mathbb{C}$ for the general Dirichlet series were obtained. Suppose that the series (1) converges absolutely for $\sigma > \sigma_a$ and has a sum $f(s)$. Moreover, we assume that the function $f(s)$ admits a meromorphic continuation to the region $\sigma > \sigma_1$, $\sigma_1 < \sigma_a$, all poles in this region are included in a compact set, and that, for $\sigma > \sigma_1$, σ is not the real part of a pole of $f(s)$, the estimates

$$f(\sigma+it) = \mathrm{O}\big(|t|^a\big), \quad a = a(\sigma) > 0, \quad |t| \geq t_0 > 0, \tag{2}$$

and

$$\int_{-T}^{T} \big|f(\sigma+it)\big|^2 \mathrm{d}t = \mathrm{O}(T), \quad T \to \infty, \tag{3}$$

are satisfied.

Define the probability measure

$$P_{T,\sigma}(A) = \nu_T\big(f(\sigma+it) \in A\big), \quad A \in \mathcal{B}(\mathbb{C}).$$

The first theorem of [9] gives the existence of the limit measure for $P_{T,\sigma}$.

Theorem 1. [9]. *Suppose that $\sigma > \sigma_1$ and for the function $f(s)$ the conditions (2) and (3) are satisfied. Then on $\big(\mathbb{C}, \mathcal{B}(\mathbb{C})\big)$, there exists a probability measure P_σ such that the measure $P_{T,\sigma}$ converges weakly to P_σ as $T \to \infty$.*

The second theorem of [9] contains the explicit form of the limit measure P. To state the theorem, we need one topological structure. Denote by γ the unit circle $\big\{s \in \mathbb{C} : |s| = 1\big\}$ on the complex plane, and let

$$\Omega = \prod_{m=1}^{\infty} \gamma_m,$$

where $\gamma_m = \gamma$ for all $m \in \mathbb{N}$. With the product topology and pointwise multiplication, the torus Ω is a compact topological Abelian group. Therefore, on $\big(\Omega, \mathcal{B}(\Omega)\big)$ the probability Haar measure m_H exists, and we obtain a probability space $\big(\Omega, \mathcal{B}(\Omega), m_H\big)$. Let $\omega(m)$ denote the projection of $\omega \in \Omega$ to the coordinate space γ_m. Suppose that

$$\lambda_m \geq c(\log m)^\delta \tag{4}$$

with some positive constants c and δ, and on the probability space $\big(\Omega, \mathcal{B}(\Omega), m_H\big)$ define the complex-valued random element $f(\sigma, \omega)$ by

$$f(\sigma, \omega) = \sum_{m=1}^{\infty} a_m \omega(m) \mathrm{e}^{-\lambda_m \sigma}, \quad \sigma > \sigma_1, \quad \omega \in \Omega.$$

Theorem 2. [9]. *Let $\sigma > \sigma_1$. Suppose that the system $\{\lambda_m : m \in \mathbb{N}\}$ is linearly independent over the field of rational numbers and inequality (4) holds. If the function $f(s)$ satisfies (2) and (3), then the probability measure $P_{T,\sigma}$ converges weakly to the distribution of the random element $f(\sigma, \omega)$ as $T \to \infty$.*

Our aim is a generalization of Theorems 1 and 2. Let $w(t)$ be a positive function of bounded variation on $[T_0, \infty)$, $T_0 > 0$. Moreover, let

$$U = U(T, w) = \int_{T_0}^{T} w(t) \mathrm{d}t,$$

and suppose that $\lim\limits_{T \to \infty} U(T, w) = +\infty$. We also require that for, $\sigma > \sigma_1$, σ is not the real part of a pole of $f(s)$, and all $v \in \mathbb{R}$, the estimate

$$\int_{T_0+v}^{T+v} w(t-v) \big|f(\sigma + it)\big|^2 \mathrm{d}t \ll U\big(1 + |v|\big) \tag{5}$$

should be satisfied. Here and in the sequel, $g(x) \ll h(x)$, $h(x) > 0$, $x \in X$, means that there exists a constant $C > 0$ such that $\big|g(x)\big| \ll Ch(x)$ for all $x \in X$.

It is easily seen that in the case $w(t)=t^{-1}$ the estimate (5) is a corollary of (3). Define the probability measure

$$P_{T,\sigma,w}(A)=\frac{1}{U}\int\limits_{T_0}^{T} w(t)I_{\{t:f(\sigma+it)\in A\}}\mathrm{d}t,\quad A\in\mathcal{B}(\mathbb{C}),$$

where I_A denotes the indicator function of the set A.

Theorem 3. *Suppose that $\sigma>\sigma_1$ and the function $f(s)$ satisfies the conditions (2) and (5). Then on $(\mathbb{C},\mathcal{B}(\mathbb{C}))$, there exists a probability measure P_σ such that the measure $P_{T,\sigma,w}$ converges weakly to P_σ as $T\to\infty$.*

For identification of the limit measure P_σ, we need an additional restriction for the weight function $w(t)$. Let $X(t,\omega)$ be an arbitrary ergodic process, $\mathbb{E}\big|X(t,\omega)\big|<\infty$, with sample paths integrable almost surely in the Riemann sense over every finite interval. Here $\mathbb{E}X$ denotes the expectation of X. We suppose that

$$\frac{1}{U}\int\limits_{T_0}^{T} w(t)X(t+v,\omega)\mathrm{d}t=\mathbb{E}X(0,\omega)+r_T\big(1+|v|\big)^{\alpha}\tag{6}$$

almost surely for all $v\in\mathbb{R}$ with some $\alpha>0$, where $r_T\to 0$ as $T\to\infty$.

Note that if in (6) $v=0$ and $w(t)\equiv 1$, then we have the classical Birkhoff-Khintchine theorem.

Theorem 4. *Let $\sigma>\sigma_1$. Suppose that the system $\{\lambda_m : m\in\mathbb{N}\}$ is linearly independent over the field of rational numbers and inequality (4) holds. Moreover, suppose that the weight function $w(t)$ satisfies (6), and for the function $f(s)$ estimates (2) and (4) hold. Then the probability measure $P_{T,w}$ converges weakly to the distribution $P_{f,\sigma}$ of the random element $f(\sigma,\omega)$ as $T\to\infty$.*

Note that weighted limit theorems for the Riemann zeta-function in various spaces were obtained in [7], [10] and [11]. In this case, the existence of the Euler product expansion over primes is applied.

Since all possible poles of the function $f(s)$ are included in a compact set, we have that there exists a number $V>0$ such that the function $f(s)$ is analytic in the region $\{s\in\mathbb{C}:\sigma>\sigma_1, t>V\}$.

Define

$$\widetilde{P}_{T,\sigma,w}(A)=\frac{1}{\widetilde{U}}\int\limits_{V}^{T} w(t)I_{\{t:f(\sigma+it)\in A\}}\mathrm{d}t,\quad A\in\mathcal{B}(\mathbb{C}),$$

where $\widetilde{U} = \widetilde{U}(T, w) = \int\limits_V^T w(t)\mathrm{d}t$. Clearly, $U = \widetilde{U} + \mathrm{O}(1)$. Then we have that

$$\begin{aligned} P_{T,\sigma,w}(A) &= \frac{1}{U}\left(\int\limits_V^T + \int\limits_{T_0}^V\right) w(t) I_{\{t: f(\sigma+it)\in A\}}\mathrm{d}t \\ &= \frac{1}{\widetilde{U}}\left(1 + \mathrm{O}\left(\widetilde{U}^{-1}\right)\right)\left(\int\limits_V^T + \int\limits_{T_0}^V\right) w(t) I_{\{t: f(\sigma+it)\in A\}}\mathrm{d}t \\ &= \frac{1}{\widetilde{U}}\int\limits_V^T w(t) I_{\{t: f(\sigma+it)\in A\}}\mathrm{d}t + \mathrm{o}(1) = \widetilde{P}_{T,\sigma,w}(A) + \mathrm{o}(1) \end{aligned}$$

uniformly in A, $A \in \mathcal{B}(\mathbb{C})$. Therefore, instead of the measure $P_{T,\sigma,w}$ we can consider the measure $\widetilde{P}_{T,\sigma,w}$. So, without loss of generality, we can suppose that the function $f(s)$ is analytic in the region $\{s \in \mathbb{C} : \sigma > \sigma_1, t > T_0\}$.

2. A limit theorem on Ω

In this section, we consider the weak convergence of the probability measure

$$Q_{T,w}(A) = \frac{1}{U}\int\limits_{T_0}^T w(t) I_{\{t: (\mathrm{e}^{-i\lambda_m t}: m\in\mathbb{N})\in A\}}\mathrm{d}t, \quad A \in \mathcal{B}(\Omega).$$

Lemma 5. *On $(\Omega, \mathcal{B}(\Omega))$ there exists a probability measure Q_w such that the measure $Q_{T,w}$ converges weakly to Q_w as $T \to \infty$.*

Proof. The dual group of Ω is isomorphic to

$$\bigoplus_{m=1}^{\infty} \mathbb{Z}_m,$$

where $\mathbb{Z}_m = \mathbb{Z} = \{\dots, -2, -1, 0, 1, 2, \dots\}$. An element $\mathbf{k} = (k_1, k_2, \dots) \in \bigoplus_{m=1}^{\infty} \mathbb{Z}_m$, where only a finite number of integers k_j, $j \in \mathbb{N}$, are distinct from zero, acts on Ω by

$$\omega \to \omega^{\mathbf{k}} = \prod_{m=1}^{\infty} \omega^{k_m}(m), \quad \omega \in \Omega.$$

Therefore, the Fourier transform $g_{T,w}(\mathbf{k})$ of the measure $Q_{T,w}$ is

$$g_{T,w}(\mathbf{k}) = \int\limits_{\Omega} \left(\prod_{m=1}^{\infty} \omega^{k_m}(m) \right) \mathrm{d}Q_{T,w} = \frac{1}{U} \int\limits_{T_0}^{T} w(t) \prod_{m=1}^{\infty} \mathrm{e}^{-itk_m\lambda_m} \mathrm{d}t. \tag{7}$$

Let $\sum\limits_{m=1}^{\infty} k_m\lambda_m \neq 0$. Then integration by parts and properties of $w(t)$ give

$$\begin{aligned}
\int\limits_{T_0}^{T} w(t) \left(\prod_{m=1}^{\infty} \mathrm{e}^{-itk_m\lambda_m} \right) \mathrm{d}t &= \int\limits_{T_0}^{T} w(t) \exp\left\{ -it \sum_{m=1}^{\infty} k_m\lambda_m \right\} dt \\
&= \left(-i \sum_{m=1}^{\infty} k_m\lambda_m \right)^{-1} \int\limits_{T_0}^{T} w(t) \mathrm{d} \exp\left\{ -it \sum_{m=1}^{\infty} k_m\lambda_m \right\} \\
= \mathrm{O}\left(\left| \sum_{m=1}^{\infty} k_m\lambda_m \right|^{-1} \right) - &\left(-i \sum_{m=1}^{\infty} k_m\lambda_m \right)^{-1} \int\limits_{T_0}^{T} \exp\left\{ -it \sum_{m=1}^{\infty} k_m\lambda_m \right\} \mathrm{d}w(t) \\
&= \mathrm{O}\left(\left| \sum_{m=1}^{\infty} k_m\lambda_m \right|^{-1} \right).
\end{aligned} \tag{8}$$

We recall that here only a finite number of integers k_m are distinct from zero. Thus, (7) and (8) show that

$$g_{T,w}(\mathbf{k}) = \begin{cases} 1 & \text{if} \quad \sum\limits_{m=1}^{\infty} k_m\lambda_m = 0, \\ \mathrm{O}\left(U \left| \sum\limits_{m=1}^{\infty} k_m\lambda_m \right| \right)^{-1} & \text{if} \quad \sum\limits_{m=1}^{\infty} k_m\lambda_m \neq 0. \end{cases}$$

Hence, we obtain that

$$\lim_{T\to\infty} g_{T,w}(\mathbf{k}) = \begin{cases} 1 & \text{if} \quad \sum\limits_{m=1}^{\infty} k_m\lambda_m = 0, \\ 0 & \text{if} \quad \sum\limits_{m=1}^{\infty} k_m\lambda_m \neq 0, \end{cases}$$

and Theorem 1.4.2 of [5] shows that the measure $Q_{T,w}$ converges weakly to the measure defined by the Fourier transform

$$g_w(\mathbf{k}) = \begin{cases} 1 & \text{if} \quad \sum\limits_{m=1}^{\infty} k_m\lambda_m = 0, \\ \\ 0 & \text{if} \quad \sum\limits_{m=1}^{\infty} k_m\lambda_m \neq 0, \end{cases}$$

as $T \to \infty$. □

Corollary 6. *Suppose that the system $\{\lambda_m : m \in \mathbb{N}\}$ is linearly independent over the field of rational numbers. Then the measure $Q_{T,w}$ converges weakly to the Haar measure m_H as $T \to \infty$.*

Proof. If the system $\{\lambda_m : m \in \mathbb{N}\}$ is linearly independent over the field of rational numbers, then the limit Fourier transform

$$g_w(\mathbf{k}) = \begin{cases} 1 & \text{if} \quad \mathbf{k} = \mathbf{0}, \\ 0 & \text{if} \quad \mathbf{k} \neq \mathbf{0}, \end{cases}$$

corresponds the Haar measure m_H. □

3. Limit theorems for absolutely convergent series

Let

$$\sigma_2 > \sigma_a - \sigma_1, \qquad \text{and} \qquad v(m,n) = \exp\{-\mathrm{e}^{(\lambda_m - \lambda_n)\sigma_2}\}.$$

Define a series

$$f_n(s) = \sum_{m=1}^{\infty} a_m v(m,n) \mathrm{e}^{-\lambda_m s}. \tag{9}$$

Then in [4] it was proved that, under conditions (2) and (3), the series (9) converges absolutely for $\sigma > \sigma_1$. In this section, we consider the weak convergence of the probability measure

$$P_{T,n,\sigma,w}(A) = \frac{1}{U} \int_{T_0}^{T} w(t) I_{\{t : f_n(\sigma+it) \in A\}} \mathrm{d}t, \quad A \in \mathcal{B}(\mathbb{C}).$$

Theorem 7. *Let $\sigma > \sigma_1$. Then on $(\mathbb{C}, \mathcal{B}(\mathbb{C}))$ there exists a probability measure $P_{n,\sigma}$ such that the measure $P_{T,n,\sigma,w}$ converges weakly to $P_{n,\sigma}$ as $T \to \infty$.*

P r o o f. Define the function $h_{n,\sigma} : \Omega \to \mathbb{C}$ by the formula

$$h_{n,\sigma}(\omega) = \sum_{m=1}^{\infty} a_m v(m,n)\omega(m)\mathrm{e}^{-\lambda_m \sigma}, \quad \omega \in \Omega.$$

Since the series (9) converges absolutely for $\sigma > \sigma_1$, we have that the series defining $h_{n,\sigma}(\omega)$ converges uniformly in ω. Therefore, the function $h_{n,\sigma}$ is continuous. Moreover,

$$h_{n,\sigma}\big((\mathrm{e}^{-i\lambda_m t} : m \in \mathbb{N})\big) = f_n(\sigma + it).$$

Hence, $P_{T,n,\sigma,w} = Q_{T,w} h_{n,\sigma}^{-1}$, and by Lemma 5 and Theorem 5.1 of [2] we obtain that the measure $P_{T,n,\sigma,w}$ converges weakly to $Q_w h_{n,\sigma}^{-1}$ as $T \to \infty$. □

Now let, for $\omega \in \Omega$,

$$f_n(s,\omega) = \sum_{m=1}^{\infty} a_m v(m,n)\omega(m)\mathrm{e}^{-\lambda_m s}.$$

The latter series also converges absolutely for $\sigma > \sigma_1$. Let $\hat{\omega}$ be a fixed element of Ω. Define one more probability measure

$$\hat{P}_{T,n,\sigma,w}(A) = \frac{1}{U}\int_{T_0}^{T} w(t) I_{\{t : f_n(\sigma+it,\hat{\omega}) \in A\}} \mathrm{d}t, \quad A \in \mathcal{B}(\mathbb{C}).$$

THEOREM 8. *Let $\sigma > \sigma_1$. Suppose that the system $\{\lambda_m : m \in \mathbb{N}\}$ is linearly independent over the field of rational numbers. Then on $\big(\mathbb{C}, \mathcal{B}(\mathbb{C})\big)$ there exists a probability measure $P_{n,\sigma}$, such that both measures $P_{T,n,\sigma,w}$ and $\hat{P}_{T,n,\sigma,w}$ converge weakly to $P_{n,\sigma}$ as $T \to \infty$.*

P r o o f. Since the system $\{\lambda_m : m \in \mathbb{N}\}$ is linearly independent over the field of rational numbers, Corollary 6 and Theorem 7 show that the measure $P_{T,n,\sigma,w}$ converges weakly to $m_H h_{n,\sigma}^{-1}$ as $T \to \infty$. Define $\hat{h}_{n,\sigma} : \Omega \to \mathbb{C}$ by the formula

$$\hat{h}(\omega) = \sum_{m=1}^{\infty} a_m v(m,n)\omega(m)\hat{\omega}(m)\mathrm{e}^{-\lambda_m \sigma}, \quad \omega \in \Omega.$$

Then, similarly to the proof of Theorem 7, we obtain that the measure $P_{T,n,\sigma,w}$ converges weakly to $m_H \hat{h}_{n,\sigma}^{-1}$ as $T \to \infty$. Now let $h_1 : \Omega \to \Omega$ be given by $h_1(\omega) = \omega\hat{\omega}$, $\omega \in \Omega$. Then we have that $\hat{h}_{n,\sigma}(\omega) = h_{n,\sigma}\big(h_1(\omega)\big)$, and the invariance of the Haar measure m_H yields

$$m_H \hat{h}_{n,\sigma}^{-1} = m_H\left(h_{n,\sigma} h_1\right)^{-1} = (m_H h_1^{-1}) h_{n,\sigma}^{-1} = m_H h_{n,\sigma}^{-1}.$$

Hence, the measure $\hat{P}_{T,n,\sigma,w}$ also converges weakly to $m_H h_{n,\sigma}^{-1}$ as $T \to \infty$. □

4. Approximation in the mean

To pass from the function $f_n(s)$ to $f(s)$, we need an approximation in the mean for $f(s)$ by $f_n(s)$.

Theorem 9. *Let $\sigma > \sigma_1$. Then*

$$\lim_{n\to\infty} \limsup_{T\to\infty} \frac{1}{U} \int_{T_0}^{T} w(t)\big|f(\sigma+it) - f_n(\sigma+it)\big|\mathrm{d}t = 0.$$

P r o o f. Let σ_2 be the same as in Section 3. For $\sigma_2 > \sigma_a - \sigma_1$, define

$$l_n(s) = \frac{s}{\sigma_2}\Gamma\left(\frac{s}{\sigma_2}\right)\mathrm{e}^{\lambda_n s},$$

where, as usual, $\Gamma(s)$ denotes the Euler gamma-function. Then we have from [9], Section 4, that

$$f_n(s) = \frac{1}{2\pi i} \int_{\sigma_2-i\infty}^{\sigma_2+i\infty} f(s+z) l_n(z) \frac{\mathrm{d}z}{z}.$$

We move the contour of integration in this formula to the left. Let $\sigma_3 > \sigma_1$ and $\sigma_3 < \sigma$. Then the residue theorem yields

$$f_n(s) = \frac{1}{2\pi i} \int_{\sigma_3-\sigma-i\infty}^{\sigma_3-\sigma+i\infty} f(s+z) l_n(z) \frac{\mathrm{d}z}{z} + f(s) + R_n(s),$$

where

$$R_n(s) = \sum_j \operatorname*{Res}_{z=z_j-s} f(s+z) \frac{l_n(z_j-s)}{z_j-s},$$

and z_j runs over possible poles of the function $f(s)$ in the region $\sigma > \sigma_1$. Hence

$$\begin{aligned} &\frac{1}{U}\int_{T_0}^{T} w(t)\,|f(\sigma+it)-f_n(\sigma+it)|\,\mathrm{d}t \\ &\ll \int_{-\infty}^{\infty} \big|l_n(\sigma_3-\sigma+iv)\big| \left(\frac{1}{U}\int_{v+T_0}^{v+T} \big|w(t-v)f(\sigma_3+it)\big|\mathrm{d}t\right)\mathrm{d}v \\ &\quad + \frac{1}{U}\int_{T_0}^{T} w(t)\big|R_n(\sigma+it)\big|\mathrm{d}t. \end{aligned} \tag{10}$$

We can choose σ_3 so that the line $\sigma = \sigma_3$ does not contain poles of $f(s)$. The properties of the gamma-function show that

$$\frac{1}{U}\int_{T_0}^{T} w(t)\big|R_n(\sigma+it)\big|\mathrm{d}t \ll \frac{1}{U}\int_{T_0}^{T}\big|R_n(\sigma+it)\big|\mathrm{d}t = \mathrm{o}(1) \tag{11}$$

as $T \to \infty$. Moreover, in view of (5) we find that

$$\begin{aligned} &\frac{1}{U}\int_{v+T_0}^{v+T} w(t-v)\big|f(\sigma_3+it)\big|\mathrm{d}t \\ &\ll \frac{1}{U}\left(\int_{v+T_0}^{v+T} w(t-v)\mathrm{d}t \int_{v+T_0}^{v+T} w(t-v)\big|f(\sigma+it)\big|^2\mathrm{d}t\right)^{\frac{1}{2}} \\ &\ll \big(1+|v|\big)^{\frac{1}{2}} \ll 1+|v|. \end{aligned}$$

From this, (10) and (11) we get

$$\begin{aligned} &\limsup_{T\to\infty}\frac{1}{U}\int_{T_0}^{T} w(t)\big|f(\sigma+it)-f_n(\sigma+it)\big|\mathrm{d}t \\ &\qquad\ll \int_{-\infty}^{\infty}\big|l_n(\sigma_3-\sigma+iv)\big|\big(1+|v|\big)\mathrm{d}v. \end{aligned} \tag{12}$$

However, since $\sigma_3 < \sigma$, the definition of $l_n(s)$ shows that

$$\lim_{n\to\infty} \int\limits_{-\infty}^{\infty} \big|l_n(\sigma_3 - \sigma + iv)\big|\big(1 + |v|\big)\mathrm{d}v = 0.$$

From this and (12) the theorem follows. □

5. Proof of Theorem 3

By Theorem 7 we have that the measure $P_{T,n,\sigma,w}$ converges weakly to some measure $P_{n,\sigma}$, as $T \to \infty$.

Lemma 10. *The family of probability measures $\{P_{n,\sigma} : n \in \mathbb{N}\}$ is tight.*

Proof. Clearly, for $M > 0$,

$$\frac{1}{U} \int\limits_{T_0}^{T} w(t) I_{\{t : |f_n(\sigma+it)| > M\}} \mathrm{d}t \le \frac{1}{MU} \int\limits_{T_0}^{T} w(t)\big|f_n(\sigma + it)\big|\mathrm{d}t. \tag{13}$$

Moreover, in view of Theorem 9, Cauchy - Schwarz's inequality, the definition of U, and (5) with $v = 0$

$$\begin{aligned}
&\sup_{n\in\mathbb{N}} \limsup_{T\to\infty} \frac{1}{U} \int\limits_{T_0}^{T} w(t)\big|f_n(\sigma + it)\big|\mathrm{d}t \\
&\le \sup_{n\in\mathbb{N}} \limsup_{T\to\infty} \frac{1}{U} \int\limits_{T_0}^{T} w(t)\big|f(\sigma + it) - f_n(\sigma + it)\big|\mathrm{d}t \\
&\qquad\qquad + \limsup_{T\to\infty} \frac{1}{U} \int\limits_{T_0}^{T} w(t)\big|f(\sigma + it)\big|\mathrm{d}t \\
&\ll 1 + \limsup_{T\to\infty} \frac{1}{U} \left(\int\limits_{T_0}^{T} w(t)\mathrm{d}t \int\limits_{T_0}^{T} w(t)\big|f(\sigma + it)\big|^2\mathrm{d}t \right)^{\frac{1}{2}} \le R < \infty.
\end{aligned} \tag{14}$$

We take $M = R\varepsilon^{-1}$, where ε is arbitrary positive number. Then Theorem 2.1 of [2] and (13), (14) imply, for all $n \in \mathbb{N}$,

$$\begin{aligned} P_{n,\sigma}\big(\{s \in \mathbb{C} : |s| > M\}\big) & \\ &\le \liminf_{T\to\infty} P_{T,n,\sigma,w}\big(\{s \in \mathbb{C} : |s| > M\}\big) \\ &= \liminf_{T\to\infty} \frac{1}{U}\int_{T_0}^{T} w(t) I_{\{t:|f_n(\sigma+it)|>M\}}\mathrm{d}t \\ &\le \limsup_{T\to\infty} \frac{1}{U}\int_{T_0}^{T} w(t) I_{\{t:|f_n(\sigma+it)|>M\}}\mathrm{d}t \le \varepsilon. \end{aligned}$$

Hence, it follows that, for all $n \in \mathbb{N}$,

$$P_{n,\sigma}(K_\varepsilon) > 1 - \varepsilon,$$

where $K_\varepsilon = \{s \in \mathbb{C} : |s| \le M\}$ is a compact set in $\mathbb{C}$. This means that the family $\{P_{n,\sigma} : n \in \mathbb{N}\}$ is tight. $\square$

Proof of Theorem 3. On a certain probability space $\big(\hat{\Omega}, \mathcal{B}(\hat{\Omega}), \mathbb{P}\big)$, define a random variable $\eta = \eta_T$ having the distribution

$$\mathbb{P}(\eta \in A) = \frac{1}{U}\int_{T_0}^{T} w(t) I_A \mathrm{d}t, \quad A \in \mathcal{B}(\mathbb{C}).$$

We set

$$X_{T,n}(\sigma) = f_n(\sigma + i\eta),$$

and denote by $\xrightarrow{\mathcal{D}}$ the convergence in distribution. Then the assertion of Theorem 7 can be rewritten in the form

$$X_{T,n}(\sigma) \xrightarrow[T\to\infty]{\mathcal{D}} X_n(\sigma), \tag{15}$$

where $X_n(\sigma)$ is a complex-valued random element with the distribution $P_{n,\sigma}$.

Since, by Lemma 10, the family $\{P_{n,\sigma} : n \in \mathbb{N}\}$ is tight, by the Prokhorov theorem, see [2], it is relatively compact. Therefore, there exists a sequence $\{P_{n_1,\sigma}\} \subset \{P_{n,\sigma}\}$ such that $P_{n_1,\sigma}$ converges weakly to a certain probability measure P_σ on $(\mathbb{C}, \mathcal{B}(\mathbb{C}))$. Hence, we have that

$$X_{n_1}(\sigma) \xrightarrow[n_1\to\infty]{\mathcal{D}} P_\sigma. \tag{16}$$

Here and in the sequal, as in [2], Chapter 1, Section 4, we use a mixted terminology of convergence in distribution and weak convergence of distributions.

Let

$$X_T(\sigma) = f(\sigma + i\eta).$$

Then, by Theorem 9, for every $\varepsilon > 0$,

$$\begin{aligned}
&\lim_{n\to\infty} \limsup_{T\to\infty} \mathbb{P}(|X_T(\sigma) - X_{T,n}(\sigma)| \geq \varepsilon) \\
&= \lim_{n\to\infty} \limsup_{T\to\infty} \frac{1}{U} \int_{T_0}^{T} w(t) I_{\{t:|f(\sigma+it)-f_n(\sigma+it)|\geq\varepsilon\}} \mathrm{d}t \\
&\leq \lim_{n\to\infty} \limsup_{T\to\infty} \frac{1}{U\varepsilon} \int_{T_0}^{T} w(t)|f(\sigma+it) - f_n(\sigma+it)| \mathrm{d}t = 0.
\end{aligned}$$

Now this, (15) and (16) together with Theorem 4.2 of [2] show that

$$X_T(\sigma) \xrightarrow[T\to\infty]{\mathcal{D}} P_\sigma, \tag{17}$$

and this is equivalent to the assertion of the theorem. □

6. Proof of Theorem 4

To prove Theorem 4, we have to obtain the weak convergence of the probability measure

$$\hat{P}_{T,\sigma,w}(A) = \frac{1}{U} \int_{T_0}^{T} w(t) I_{\{t\in f(\sigma+it,\omega)\in A\}} \mathrm{d}t, \quad A \in \mathcal{B}(\mathbb{C}), \quad \omega \in \Omega.$$

For this, we need an analogue of Theorem 9 for $f(\sigma + it, \omega)$.

We start with some elements of ergodic theory. Let $a_t = \{\mathrm{e}^{-it\lambda_m} : m \in \mathbb{N}\}$, $t \in \mathbb{R}$, and define $\varphi_t(\omega) = a_t\omega$ for $\omega \in \Omega$. Then $\{\varphi_t : t \in \mathbb{R}\}$ is a one-parameter group of measurable measure preserving transformations on Ω. We recall that $A \in \mathcal{B}(\Omega)$ is called an invariant set with respect to the group $\{\varphi_t : t \in \mathbb{R}\}$ if, for each $t \in \mathbb{R}$, the sets A and $A_t = \varphi_t(A)$ differ at most by a set of m_H-measure zero. All invariant sets form a sub-σ-field of $\mathcal{B}(\Omega)$. The one-parameter group $\{\varphi_t : t \in \mathbb{R}\}$ is ergodic if its σ-field of invariant sets consists only of sets having m_H-measure equal to 0 or 1.

Lemma 11. *Suppose that the system $\{\lambda_m : m \in \mathbb{N}\}$ is linearly independent over the field of rational numbers. Then the one-parametric group $\{\varphi_t : t \in \mathbb{R}\}$ is ergodic.*

Proof. This is Lemma 5 from [8]. □

We apply Lemma 11 for the proof of the following statement.

Lemma 12. *Suppose that the system $\{\lambda_m : m \in \mathbb{N}\}$ is linearly independent over the field of rational numbers, and let $T \to \infty$ and $\sigma > \sigma_1$. Then*

$$\int\limits_{T_0+v}^{T+v} w(t-v)\big|f(\sigma+it,\omega)\big|^2 \mathrm{d}t \ll U\big(1+|v|\big)^\alpha$$

for almost all $\omega \in \Omega$ and all $v \in \mathbb{R}$.

Proof. It is not difficult to see that $\big\{\omega(m) : m \in \mathbb{N}\big\}$ is a sequence of pairwise orthogonal random variables on the probability space $\big(\Omega, \mathcal{B}(\Omega), m_H\big)$. Then from this we have that

$$\mathbb{E}\big|f(\sigma+it,\omega)\big|^2 = \sum_{m=1}^{\infty} \mathbb{E}\big|a_m\omega(m)\mathrm{e}^{-\lambda_m(\sigma+it)}\big|^2 = \sum_{m=1}^{\infty} |a_m|^2\mathrm{e}^{-2\lambda_m\sigma} < \infty \tag{18}$$

for every $t \in \mathbb{R}$ by Lemma 2 of [12]. Moreover, by the definition of $\varphi_t(\omega)$ we have that

$$\big|f\big(\sigma+iv,\varphi_t(\omega)\big)\big|^2 = \big|f(\sigma+iv,a_t\omega)\big|^2 = \big|f(\sigma+it+iv,\omega)\big|^2. \tag{19}$$

Lemma 11 implies the ergodicity of the random process $\big|f\big(\sigma+iv,\varphi_t(\omega)\big)\big|^2$. Therefore, in virtue of (6) and (19)

$$\begin{aligned}\frac{1}{U}\int\limits_{T_0}^{T} w(t)\,\big|f\big(\sigma+iv,\varphi_t(\omega)\big)\big|^2\,\mathrm{d}t =&\frac{1}{U}\int\limits_{T_0}^{T} w(t)\big|f(\sigma+it+iv,\omega)\big|^2\mathrm{d}t\\ =&\mathbb{E}\big|f(\sigma,\omega)\big|^2 + \mathrm{o}\big(1+|v|\big)^\alpha\end{aligned}$$

for almost all $\omega \in \Omega$ and all $v \in \mathbb{R}$ as $T \to \infty$. This and (18) prove the lemma. □

Theorem 13. *Suppose that $\sigma > \sigma_1$, and that the system $\{\lambda_m : m \in \mathbb{N}\}$ is linearly independent over the field of rational numbers. Then*

$$\lim_{n\to\infty}\limsup_{T\to\infty}\frac{1}{U}\int\limits_{T_0}^{T}\big|f(\sigma+it,\omega) - f_n(\sigma+it,\omega)\big|\mathrm{d}t = 0$$

for almost all $\omega \in \Omega$.

Proof. We use the same arguments as in the proof of Theorem 9, and apply Lemma 12. □

Theorem 14. *Suppose that all hypotheses of Theorem 4 are satisfied. Then on* $(\mathbb{C}, \mathcal{B}(\mathbb{C}))$ *there exists a probability measure* P_σ *such that the measures* $P_{T,\sigma,w}$ *and* $\hat{P}_{T,\sigma,w}$ *both converge weakly to* P_σ *as* $T \to \infty$.

Proof. In view of Theorem 3, it remains to prove that the measure $\hat{P}_{T,\sigma,w}$ also converges to the measure P_σ.

By Theorem 8 the measures $P_{T,n,\sigma,w}$ and $\hat{P}_{T,n,\sigma,w}$ both converge weakly to the same measure $P_{n,\sigma}$ as $T \to \infty$. Let

$$\hat{X}_{T,n}(\sigma) = f_n(\sigma + i\eta, \omega).$$

Then we have that

$$\hat{X}_{T,n}(\sigma) \xrightarrow[T\to\infty]{\mathcal{D}} X_n(\sigma), \tag{20}$$

where $X_n(\sigma)$ is a complex-valued random element with the distribution $P_{n,\sigma}$. The relation (17) shows that the measure P_σ in (16) is independent of the choice of the sequence $\{P_{n_1,\sigma}\}$. Since the sequence $\{P_{n,\sigma}\}$ is relatively compact, hence we have that

$$X_n(\sigma) \xrightarrow[n\to\infty]{\mathcal{D}} P_\sigma. \tag{21}$$

Define

$$\hat{X}_T(\sigma) = f(\sigma + i\eta, \omega).$$

Then the application of Theorem 13 shows that, for every $\varepsilon > 0$ and almost all $\omega \in \Omega$,

$$\begin{aligned}
&\lim_{n\to\infty} \limsup_{T\to\infty} \mathbb{P}\left(\left|\hat{X}_T(\sigma) - \hat{X}_{T,n}(\sigma)\right| \ge \varepsilon\right) \\
&= \lim_{n\to\infty} \limsup_{T\to\infty} \frac{1}{U} \int_{T_0}^{T} w(t) I_{\left\{t:|f(\sigma+it,\omega)-f_n(\sigma+it,\omega)|\ge\varepsilon\right\}} \mathrm{d}t \\
&\le \lim_{n\to\infty} \limsup_{T\to\infty} \frac{1}{U\varepsilon} \int_{T_0}^{T} w(t) \left|f(\sigma+it,\omega) - f_n(\sigma+it,\omega)\right| \mathrm{d}t = 0.
\end{aligned}$$

This, (20), (21) and Theorem 4.2 of [2] imply the weak convergence of the measure $\hat{P}_{T,\sigma,w}$ to P_σ as $T \to \infty$. □

P r o o f o f T h e o r e m 4. Let $A \in \mathcal{B}(\mathbb{C})$ be a fixed continuity set of the measure P_σ in Theorem 13. Then we have by Theorem 13 and Theorem 2.1 of [2] that

$$\lim_{T\to\infty} \frac{1}{U} \int_{T_0}^{T} w(t) I_{\{t: f(\sigma+it,\omega)\in A\}} \mathrm{d}t = P_\sigma(A). \tag{22}$$

Now define on $(\Omega, \mathcal{B}(\Omega), m_H)$ the random variable θ by

$$\theta(\omega) = \begin{cases} 1 & \text{if} \quad f(\sigma,\omega) \in A, \\ 0 & \text{if} \quad f(\sigma,\omega) \notin A. \end{cases}$$

Then, clearly, we have that

$$\mathbb{E}\theta = \int_{\Omega} \theta(\omega)\mathrm{d}m_H = m_H\big(\omega \in \Omega : f(\sigma,\omega) \in A\big) = P_{f,\sigma}(A). \tag{23}$$

Lemma 1 implies the ergodicity of the random process $\theta(\varphi_t(\omega))$. This and the relation (6) with $v = 0$ yield

$$\lim_{T\to\infty} \frac{1}{U} \int_{T_0}^{T} w(t)\theta\big(\varphi_t(\omega)\big)\mathrm{d}t = \mathbb{E}\theta \tag{24}$$

for almost all $\omega \in \Omega$. Moreover, the definitions of θ and $\varphi_t(\omega)$ show that

$$\begin{aligned} \frac{1}{U} \int_{T_0}^{T} w(t)\theta\big(\varphi_t(\omega)\big)\mathrm{d}t &= \frac{1}{U} \int_{T_0}^{T} w(t) I_{\{t: f(\sigma,\varphi_t(\omega))\in A\}} \mathrm{d}t \\ &= \frac{1}{U} \int_{T_0}^{T} w(t) I_{\{t: f(\sigma+it,\omega)\in A\}} \mathrm{d}t. \end{aligned}$$

Now this, (23) and (24) give

$$\lim_{T\to\infty} \frac{1}{U} \int_{T_0}^{T} w(t) I_{\{t: f(\sigma+it,\omega)\in A\}} \mathrm{d}t = P_f(A)$$

for almost all $\omega \in \Omega$. Hence, in view of (22)

$$P_\sigma(A) = P_f(A) \tag{25}$$

for any continuity set A of P_σ. However, all continuity sets constitute the determining class, therefore (25) holds for all $A \in \mathcal{B}(\mathbb{C})$. The theorem is proved. $\square$

Remark. As it was observed in [5], a weight function $w(t)$ for which

$$w(T)\mu^{-1}(T) \ll 1,$$

where $\mu(T) = \inf\limits_{t\in[T_0;T]} w(t)$, satisfies the hypothesis (6) with $\alpha = 1$. So, the class of functions $w(t)$ in Theorem 4 is rather wide.

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