Uniform Distribution Theory 2 (2007), no.2, 121–138

uniform
distribution
theory

CHARACTER SUMS WITH THE ALIQUOT DIVISORS FUNCTION

Sanka Balasuriya — Florian Luca

ABSTRACT. In this paper, we obtain nontrivial bounds for character sums involving the aliquot divisors function $s(n) = \sigma(n) - n$.

Communicated by Radhakrishnan Nair

1. Introduction

For every positive integer n, let $s(n)$ be the sum of the aliquot divisors of n given by

$$s(n) = \sum_{\substack{d \mid n \\ d \neq n}} d = \sigma(n) - n,$$

where $\sigma(n)$ is the sum of divisors function. Here, we consider some arithmetic properties of the aliquot sequence $(s(n))_{n \geqslant 1}$. In particular, for a fixed prime p, we obtain nontrivial estimates in certain ranges for character sums of the form

$$S_p(N) = \sum_{n=1}^{N} \chi(s(n)) \qquad \text{for } N \geqslant 1,$$

where χ is a nonprincipal multiplicative character modulo p. Note that if n is prime, it follows that $s(n) = 1$, therefore $\chi(s(n)) = 1$. Hence, we are left with giving a nontrivial upper bound on the expression $S_p(N) - \pi(N)$, where $\pi(N)$ stands, as usual, for the number of primes $q \leqslant N$.

Our results for the sum $S_p(N)$ depend on estimates for the cardinality of the sets

$$\mathcal{U}(p, N) = \{1 \leqslant n \leqslant N : \sigma(n) \equiv 0 \pmod p\},$$

and

$$\mathcal{T}(p, N) = \{1 < n \leqslant N : s(n) \equiv 0 \pmod p\}.$$

2000 Mathematics Subject Classification: 11N37.
Keywords: Character sums, divisors function, prime factor.

Upper bounds for exponential sums involving $s(n)$ have been obtained in [3], and analogous results for the Euler function $\varphi(n)$ have been obtained in [2, 4, 5]. We apply similar methods in the present paper although some modifications are needed not only since $s(n)$ is not a multiplicative function but also since it takes any prime number to 1.

Our main results are the following.

Theorem 1. *The following bound holds*

$$\#\mathcal{U}(p,N) \ll \frac{N}{p}\min\{(\log N)^{2/3}, (\log p)^2(\log\log N)^{5/3}\}$$

uniformly in the prime p and in N.

Theorem 2. *The inequality*

$$\#\mathcal{T}(p,N) \ll \frac{N(\log N)^2}{p^{1/12}}$$

holds uniformly in the prime p and in N.

Using the above results, we prove the following estimate.

Theorem 3. *The following bound holds*

$$\left|S_p(N) - \pi(N)\right| \ll \frac{N(\log N)^6}{p^{1/12}} + \frac{N}{v^{v/3+O(v(\log\log v)/\log v)}},$$

uniformly in the prime p and in N, where $v = (\log N)/(\log p)$.

Using Theorem 3, we get the following nontrivial bound on $S_p(N)$ in almost the entire range of p versus N.

Theorem 4. *The estimate*

$$|S_p(N)| \ll \frac{N}{\log\log\log N} + \frac{N}{v^{v/3+O(v(\log\log\log v)/\log v)}}$$

holds uniformly in the prime p and in N, where again $v = (\log N)/(\log p)$.

In particular, $|S_p(N)| = o(N)$ holds uniformly in the range $3 \leqslant p \leqslant N^{o(1)}$ as $N \to \infty$.

In the above statements, as well as throughout the paper, any implied constants in the symbols $\ll$, $\gg$ and O are *absolute* unless indicated otherwise. We recall that for positive functions F and G the notations $F = O(G)$, $F \ll G$ and $G \gg F$ are all equivalent to the assertion that the inequality $F \leqslant cG$ holds with some constant $c > 0$, whereas $F = o(G)$ means that $F/G \to 0$.

Throughout the paper, the letters p, q, r with or without subscripts are used to denote prime numbers, and k, ℓ, m, n denote positive integers.

Acknowledgements. We thank Professor Igor Shparlinski for his encouragement and invaluable suggestions. We also thank the anonymous referee for a careful reading of the paper and for pointing out some serious errors in a previous version of the manuscript. During the preparation of this paper, the first author was supported by an iMURS scholarship from the Macquarie University. The second author worked on this paper during Summer of 2007 when he visited the Tata Institute for Fundamental Research in Mumbai, India with a TWAS Associateship. He thanks the people of that institute for their hospitality and the TWAS for support.

2. Preliminaries

Let $P(n)$ be the largest prime factor of an integer $n \geqslant 1$, where we put $P(1) = 1$. An integer $n \geqslant 1$ is said to be *y-smooth* if $P(n) \leqslant y$. As usual, we define

$$\psi(x,y) = \#\{n \leqslant x \ : \ n \text{ is } y\text{-smooth}\} \qquad (x \geqslant y > 1).$$

The following bound is a relaxed and simplified version of [12, Corollary 1.3] (see also [6]).

Lemma 5. *The estimate*

$$\psi(x,y) \leqslant xu^{-u+O(u(\log\log u)/\log u)}, \qquad \text{where} \quad u = (\log x)/(\log y),$$

holds uniformly in the range $2 \leqslant y \leqslant x$ *and* $u \leqslant y^{1/2}$.

For given coprime integers $1 \leqslant a \leqslant b$, we let $\pi(x; b, a)$ be the number of primes $p \leqslant x$ such that $p \equiv a \pmod b$. The next statement is a simplified form of the Brun-Titchmarsh theorem (see, for example, [9, Section 2.3.1, Theorem 1] or [10, Chapter 3, Theorem 3.7]).

Lemma 6. *For any* $x \geqslant b$, *we have*

$$\pi(x; b, a) \ll \frac{x}{\varphi(b)\log(2x/b)}\,.$$

When b is bounded above by a power of the logarithm of x, then the Siegel-Walfiz theorem (see, for example, page 133 in [7]) gives us a much more precise estimate on $\pi(x; b, a)$. We record this as follows.

Lemma 7. *For every positive constant A, there is a constant $B = B(A)$ depending on A, such that the estimate*

$$\pi(x; b, a) = \frac{\pi(x)}{\phi(b)} + O\left(\frac{x}{\exp(B\sqrt{\log x})}\right)$$

holds uniformly in coprime integers $1 \leqslant a \leqslant b$ *with* $b \leqslant (\log x)^A$.

Furthermore, writing $q_{b,a}$ for the least prime in the arithmetic progression $a \pmod b$, the following Linnik type bound on $q_{b,a}$ is due to Heath-Brown (see [11]).

Lemma 8. *The inequality* $q_{b,a} \ll b^{5.5}$ *holds uniformly in coprime integers* $1 \leqslant a \leqslant b$.

We will also need the following lower bound for a linear form in logarithms. For a rational number $\alpha = r/s$ with coprime integers r and $s > 0$, we write $H(\alpha) = \max\{|r|, s\}$. The following result is a particular case of Baker's theorem on lower bounds for linear forms in logarithms (see [1], for example).

Lemma 9. *Let* $\alpha_1, \alpha_2, \alpha_3$ *be rational numbers different from* 0, ± 1, *and let* b_1, b_2, b_3 *be positive integers. Put* $B = \max\{b_1, b_2, b_3, 3\}$. *Assume that*

$$\Gamma = \alpha_1^{b_1} \alpha_2^{b_2} \alpha_3^{b_3} - 1 \neq 0.$$

Then

$$-\log |\Gamma| \ll (\log B) \prod_{i=1}^{3} \log(H(\alpha_i)).$$

The above lemmas are enough for the proof of Theorems 1 and 2. For the proof of Theorem 3 however, we also need the following bound for character sums over prime numbers, which is a relaxed version of [15, Theorem 1].

Lemma 10. *For any nonprincipal multiplicative character* χ *modulo* $p \geqslant 2$, *any integer* a *coprime to* p, *and any real* $z \geqslant 2$,

$$\sum_{q \leqslant z} \chi(q + a) \leqslant z p^{o(1)} (\log z)^5 \left(\frac{1}{p^{1/2}} + \frac{p^{1/2}}{z^{1/2}} + \frac{1}{z^{1/6}} \right),$$

as $p \to \infty$.

3. Proof of Theorem 1

By the multiplicativity of the sum of divisors function, it follows that whenever $n \in \mathcal{U}(p, N)$, there is a prime power q^k such that $p \mid \sigma(q^k)$ and $q^k \| n$. Recall that $q^k \| n$ means that $q^k \mid n$ and $q^{k+1} \nmid n$. Note that since $\sigma(q^k) < 2q^k \leqslant 2N$, it follows that unless $p \leqslant 2N$, the set $\mathcal{U}(p, N)$ is empty. From now on, we assume that $p \leqslant 2N$ and for simplicity we write $\mathcal{U} = \mathcal{U}(p, N)$ (i.e., we omit the dependence on either p or N).

Let $\mathcal{U}_0$ be the subset of those $n \in \mathcal{U}$ for which there exists a prime $q \| n$ such that $p \mid \sigma(q)$. Then $p \mid q+1$. Since $q \mid n$, it follows that $q \leqslant N$. Let $q \equiv -1 \pmod p$ be fixed. The number of positive integers $n \leqslant N$ such that $q \mid n$ does not exceed N/q. Hence, summing up over all the possibilities for q, we get

$$\#\mathcal{U}_0 \leqslant \sum_{\substack{q \leqslant N \\ q \equiv -1 \pmod p}} \frac{N}{q} = N \sum_{\substack{q \leqslant 2N \\ q \equiv -1 \pmod p}} \frac{1}{q} \ll \frac{N \log\log N}{p}. \tag{1}$$

In the above estimates, we used the familiar fact that if $1 \leqslant a \leqslant b$ are coprime then

$$\sum_{\substack{q \leqslant N \\ q \equiv a \pmod b}} \frac{1}{q} \ll \frac{1}{q_{b,a}} + \frac{\log\log N}{\phi(b)}$$

holds uniformly for $1 \leqslant a \leqslant b \leqslant N$, where we recall that $q_{b,a}$ denotes the smallest prime number $q \equiv a \pmod b$. Indeed, the above estimate follows by Abel's summation formula from Lemma 6.

From now on, we assume that $n \in \mathcal{U}\backslash\mathcal{U}_0$. Every such n has the property that $p \mid \sigma(q^k)$ for some prime power $q^k \| n$, where $k \geqslant 2$. Note that $2^k \leqslant q^k \leqslant N$, therefore $k \leqslant c \log N$, where $c = (\log 2)^{-1}$. For fixed p and k, the congruence

$$\sigma(q^k) = q^k + q^{k-1} + \cdots + 1 \equiv 0 \pmod p$$

puts q into $s_k \leqslant k$ arithmetical progressions $q \equiv q_i \pmod p$. Here, we write q_i for the smallest positive integer in the above progression and we assume that $1 \leqslant q_1 < q_2 < \cdots < q_s \leqslant p-1$. Note that

$$2q_i^k \geqslant q_i^k \left(1 + \frac{1}{q_i} + \cdots\right) \geqslant p,$$

therefore $q_i \gg p^{1/k}$. Put $\lambda_{i,k} = (q_{p,q_i}^{k+1} - 1)/(p(q_{p,q_i} - 1))$. Note that $\lambda_{i,k} < 2q_{p,q_i}^k/p$, and that $1 \leqslant \lambda_{1,k} < \cdots < \lambda_{s_k,k}$. Fix $i \in \{1, \ldots, s_k\}$ and write $\mathcal{U}_{i,k}$ for the subset of $\mathcal{U}\backslash\mathcal{U}_0$ formed by those integers $n \leqslant N$ such that $q^k \mid n$ for some $q \equiv q_i \pmod p$. Given such a prime q, the number of such integers $n \leqslant N$ does not exceed N/q^k. Summing up the contributions to $\mathcal{U}\backslash\mathcal{U}_0$ over all such primes q, we get

$$\begin{aligned} \#\mathcal{U}_{i,k} &\leqslant \sum_{\substack{q \leqslant N \\ q \equiv q_i \pmod p}} \frac{N}{q^k} \leqslant \frac{N}{q_{p,q_i}^k} + \sum_{\ell \geqslant 1} \frac{N}{(q_i + \ell p)^k} \leqslant \frac{N}{q_{p,q_i}^k} + \frac{N}{p^k}\zeta(2) \\ &\ll \frac{N}{p\lambda_{i,k}} + \frac{N}{p^k}. \end{aligned}$$

Summing up the above inequality over all the possibilities for $i \in \{1, \dots, s_k\}$, we get that

$$\sum_{i=1}^{s_k} \#\mathcal{U}_{i,k} \ll \frac{N}{p} \sum_{i=1}^{s_k} \frac{1}{\lambda_{i,k}} + \frac{Nk}{p^k}.$$

Finally, summing the above inequality up over all $k \geqslant 2$, we get

$$\sum_{2 \leqslant k \leqslant c \log N} \sum_{i=1}^{s_k} \#\mathcal{U}_{i,k} \ll \frac{N}{p} \sum_{2 \leqslant k \leqslant c \log N} \sum_{i=1}^{s_k} \frac{1}{\lambda_{i,k}} + N \sum_{k \geqslant 2} \frac{k}{p^k}. \tag{2}$$

It is clear that the second sum is $O(N/p)$. Thus, it remains to deal with the first double sum. Let $\Lambda = \{\lambda_{i,k} : 2 \leqslant k \leqslant c \log N,\ i = 1, \dots, s_k\}$. As (i, k) is a pair such that $2 \leqslant k \leqslant c \log N$ and $1 \leqslant i \leqslant s_k$, we have that $\lambda_{i,k} \in \Lambda$, but each number from Λ might appear from more that one pair (i, k) (incidentally, a well-known conjecture about the Goormaghtigh Diophantine equation asserts that in fact the only such case is $31 = (5^3 - 1)/(5 - 1) = (2^5 - 1)/(2 - 1)$; see [16]). Fixing $\lambda \in \Lambda$, the multiplicity with which is appears as $\lambda_{i,k}$ is bounded by the number of solutions (q, k) with q a prime of the Diophantine equation

$$p\lambda = \frac{q^{k+1} - 1}{q - 1}. \tag{3}$$

It remains to count, for a fixed $M = p\lambda$, the number $A(M)$ of solutions (q, k) with q prime to the equation

$$M = \frac{q^{k+1} - 1}{q - 1}.$$

A result of Loxton [14], says that $A(M) \leqslant (\log M)^{1/2+o(1)}$ as $M \to \infty$. Since for us $M \leqslant 2N$, we get that

$$A(p\lambda) \leqslant (\log N)^{1/2+o(1)} \qquad \text{as } N \to \infty. \tag{4}$$

When p is sufficiently small, one can do better as follows. Assume that $A(p\lambda) \geqslant 2$ and let $(q_1, k_1) \neq (q_2, k_2)$ be two solutions to equation (3). If $k_1 = k_2$, then $q_1 = q_2$, which is impossible. So, let us assume that $k_1 > k_2$. Then, $q_1 < q_2$. Equation

$$\frac{q_1^{k_1+1} - 1}{q_1 - 1} = \frac{q_2^{k_2+1} - 1}{q_2 - 1},$$

can be rewritten as

$$q_1^{k_1+1}(q_2 - 1) - q_2^{k_2+1}(q_1 - 1) = (q_2 - q_1) > 0,$$

therefore

$$q_1^{k_1+1} q_2^{-(k_2+1)} \left(\frac{q_2 - 1}{q_1 - 1} \right) - 1 = \frac{q_2 - q_1}{q_2^{k_2+1}(q_1 - 1)}.$$

The expression appearing above is positive and obviously $< q_2^{-k_2}$. Applying Lemma 9 with Γ being the expression appearing in the left hand side of the above equation, where

$$\alpha_1 = q_1,\ \alpha_2 = q_2^{-1},\ \alpha_3 = \frac{q_2-1}{q_1-1},\ b_1 = k_1+1,\ b_2 = k_2+1,\ b_3 = 1$$

(note that $H(\alpha_1) = q_1,\ H(\alpha_2) = q_2,\ H(\alpha_3) < q_2$), yields

$$k_2 \log q_2 \ll \log q_1 (\log q_2)^2 \log k_1,$$

leading to $k_2 \ll (\log q_2)^2 \log k_1$. Obviously $k_1 \ll \log N$. Furthermore, since the numbers q_1 and q_2 are the first primes in certain arithmetic progression modulo p, we get, by Lemma 8, that $q_2 \ll p^{5.5}$. Thus, $k_2 \ll (\log p)^2 \log\log N$. This shows that if $A(p\lambda) \geqslant 2$, then

$$p\lambda \leqslant 2q_2^{k_2} \ll \exp(k_2 \log q_2) = \exp(O((\log p)^3 \log\log N)),$$

and now Loxton's result shows that

$$A(p\lambda) \leqslant (\log p\lambda)^{1/2+o(1)} \leqslant (\log\log N)^{1/2+o(1)} (\log p)^{3/2+o(1)} \qquad \text{as } N \to \infty. \tag{5}$$

Thus, putting

$$A = \max\{A(p\lambda_{i,k}) : 2 \leqslant k \leqslant c \log N,\ 1 \leqslant i \leqslant s_k\},$$

we get that

$$A \leqslant \min\{(\log N)^{1/2+o(1)}, (\log p)^2 (\log\log N)^{1/2+o(1)}\}$$

uniformly in both p and N as $N \to \infty$. In particular, the inequality

$$A \ll \min\{(\log N)^{3/5}, (\log p)^2 (\log\log N)^{2/3}\} \tag{6}$$

holds uniformly in N and p. Thus,

$$\sum_{2\leqslant k\leqslant c(\log N)} \sum_{i=1}^{s_k} \frac{1}{\lambda_{i,k}} \ll A \sum_{\lambda\in\Lambda} \frac{1}{\lambda} \ll A \log \#\Lambda. \tag{7}$$

Since

$$\#\Lambda \leqslant \sum_{2\leqslant k\leqslant c\log N} s_k \leqslant \sum_{2\leqslant k\leqslant c\log N} k \ll (\log N)^2,$$

we get that $\log \#\Lambda \ll \log\log N$, which together with estimates (1), (2), (6), (7), and the fact that $(\log N)^{3/5} \log\log N \ll (\log N)^{2/3}$ completes the proof of Theorem 1.

4. Proof of Theorem 2

We may assume that p is sufficiently large otherwise there is nothing to prove. We also assume that $\mathcal{T}(p,N)$ is nonempty. Since there exists a number $n_0 \leqslant N$ such that $p \mid s(n_0)$ and $s(n_0) \neq 0$ (because $n_0 > 1$), it follows that

$$p \leqslant s(n_0) < \sigma(n_0) \ll N \log\log N. \tag{8}$$

We let $\alpha \in (0, 1/10)$ be a number to be determined later. Recall that a number m is squarefull if $q^2 \mid m$ whenever q is a prime factor of m. The set $\mathcal{T}_1$ of numbers $n \leqslant N$ such that $d \mid n$ for some squarefull $d \geqslant p^{2\alpha}$ has cardinality at most

$$\begin{aligned} \#\mathcal{T}_1 \quad &\leqslant \quad \sum_{\substack{p^{2\alpha} \leqslant d \leqslant N \\ d \text{ squarefull}}} \sum_{\substack{1<n\leqslant N \\ d \mid n}} 1 \leqslant \sum_{\substack{p^{2\alpha}<d \\ d \text{ squarefull}}} \left\lfloor \frac{N}{d} \right\rfloor \\ &\ll \quad N \sum_{\substack{p^{2\alpha}<d \\ d \text{ squarefull}}} \frac{1}{d} \ll \frac{N}{p^{\alpha}}. \end{aligned} \tag{9}$$

For the last inequality above, we used the fact that the estimate

$$\sum_{\substack{t \leqslant d \\ d \text{ squarefull}}} \frac{1}{d} \ll \frac{1}{t^{1/2}}$$

holds uniformly in t (with the particular value $t = p^{2\alpha}$), which in turn follows via the Abel summation formula from the fact that the counting function of the set of squarefull integers $d \leqslant t$ is $O(t^{1/2})$ (see, for example, Theorem 14.4 in [13]).

For the remaining n, let $n = qm$, where $q = P(n)$. Assume first that $q > p^{\alpha}$. Then $q > P(m)$. In this case, $s(n) = qs(m) + \sigma(m)$. Note that $m \neq 1$ since when $m = 1$, we get that $s(n) = 1$, but this is not a multiple of p. Let $\mathcal{T}_2$ be the subset formed by these n such that $p \mid qs(m)$. Then $p \mid \sigma(m)$, so $m \in \mathcal{U}(p, N/p^{\alpha})$. Since for a fixed m the number of values of $q \leqslant N/m$ is $\leqslant N/m$ (in this last count we ignore the fact that q is a prime), we get that

$$\#\mathcal{T}_2 \leqslant N \sum_{m \in U(p,N/p^{\alpha})} \frac{1}{m}. \tag{10}$$

Theorem 1 tells us that the inequality

$$\#\mathcal{U}(p,t) \ll \frac{t \log N}{p} \tag{11}$$

holds uniformly for $t \leqslant N$, and now by the Abel summation formula we get that

$$\#\mathcal{T}_2 \ll \frac{N(\log N)^2}{p}.$$

Let $\mathcal{T}_3$ be the set of such numbers $n \leqslant N$ such that $p \nmid qs(m)$ and still $q > p^\alpha$. If $n = qm \in \mathcal{T}_3$, we then have that $p \nmid qs(m)\sigma(m)$, and $q \equiv -\sigma(m)s(m)^{-1} \pmod p$, where $s(m)^{-1}$ stands for the inverse of $s(m)$ modulo p. Let $a_m \in \{1, \ldots, p-1\}$ be such that $a_m \equiv -\sigma(m)s(m)^{-1} \pmod p$. Thus, if m is fixed, then q is in the fixed arithmetic progression $a_m \pmod p$, and the number of such numbers is $\leqslant N/(pm) + 1 \leqslant 2N/p^\alpha m$ (here, we ignore again the fact that q is prime, and we use the fact that $mp^\alpha < mq \leqslant N$). Thus,

$$\#\mathcal{T}_3 \leqslant \frac{2N}{p^\alpha} \sum_{m \leqslant N} \frac{1}{m} \ll \frac{N \log N}{p^\alpha}. \tag{12}$$

Let us now consider the set $\mathcal{T}_4$ of the remaining n in $\mathcal{T}(p, N)$. Then $P(n) \leqslant p^\alpha$. Since $\sigma(n) \ll n \log(P(n)) \ll n \log p$, $p \mid s(n)$, and $s(n) \neq 0$, we get that $p \leqslant s(n) < \sigma(n) \ll n \log p$, therefore $n \gg p/\log p$. Let λ be the smallest divisor of n such that $n/\lambda < p^{1/2}/\log p$. Write $n = \lambda d$. Thus, $d < p^{1/2}/\log p$. Note that $d \geqslant p^{1/2-\alpha}/(\log p)$ because n is p^α-smooth.

We now fix λ. We count the number of $n = \lambda d \leqslant N$ which are multiples of λ. Write $d = d_1 d_2$, where d_1 and d_2 are coprime, d_1 is divisible only by primes dividing λ, and d_2 is coprime to λ. Note that $d_1 < p^{2\alpha}$. Indeed, for if not, then $d_1 \geqslant p^{2\alpha}$, so n is divisible by $d_1 \prod_{r \mid d_1} r$, where, as stated in the Introduction, we use r to denote a prime number. The last number above is squarefull and exceeds $p^{2\alpha}$. However, positive integers n with such a divisor have already been accounted for in $\mathcal{T}_1$. Thus, $d_1 < p^{2\alpha}$. Write $d_2 = d_3 d_4$, where d_3 and d_4 are coprime, d_3 is squarefull, and d_4 is squarefree. Since $n \notin \mathcal{T}_1$, we get that $d_3 < p^{2\alpha}$. Fix λ, d_1 and d_3, and write $\ell = \lambda d_1 d_3$. Then $d_4 \geqslant d/(d_1 d_3) \geqslant p^{1/2-5\alpha}/(\log p)$. Note that $\beta = 1/2 - 5\alpha > 0$ since we are assuming that $\alpha \in (0, 1/10)$. Thus, $\ell \leqslant N/d_4 \leqslant N(\log p)/p^\beta$. We now show that the number d_4 is uniquely determined. Indeed, assume that there exist two distinct values d_4 and d_4' both $< p^{1/2}/\log p$ and coprime to ℓ, such that both numbers $n = \ell d_4$ and $n' = \ell d_4'$ have the property that $p \mid s(n)$ and $p \mid s(n')$. Since d_4 and ℓ are coprime, we get $s(n) = \sigma(\ell)\sigma(d_4) - \ell d_4 \equiv 0 \pmod p$. Since $P(n) \leqslant p^\alpha < p$, we get that p does not divide n. Thus, p does not divide $\sigma(\ell)$ either, and so $\sigma(d_4)d_4^{-1} \equiv \ell\sigma(\ell)^{-1} \pmod p$. The same congruence holds when d_4 is replaced by d_4'. Thus, $\sigma(d_4)d_4^{-1} \equiv \sigma(d_4')d_4'^{-1} \pmod p$, leading to

$$p \mid \sigma(d_4)d_4' - \sigma(d_4')d_4. \tag{13}$$

Since $\max\{d_4, d_4'\} < p^{1/2}/\log p$, it follows that

$$\max\{\sigma(d_4), \sigma(d_4')\} \ll \frac{p^{1/2}\log\log p}{\log p}.$$

In particular, for large p, we have

$$|\sigma(d_4)d_4' - \sigma(d_4')d_4| < p. \tag{14}$$

Inequality (14) and congruence (13) yield $\sigma(d_4)/d_4 = \sigma(d_4')/d_4'$. It is however easy to see that the function $s \mapsto \sigma(s)/s$ is injective when restricted to squarefree numbers s (hint: observe that $\sigma(s)/s$ determines uniquely $P(s)$ when s is squarefree). Since d_4 and d_4' are squarefree, we get that $d_4 = d_4'$, which is a contradiction. Hence, we see that n is uniquely determined by the triple (λ, d_1, d_3), where $\ell = \lambda d_1 d_3 \leqslant N(\log p)/p^\beta$. Given ℓ, the divisor λ can be chosen in at most $\tau(\ell)$ ways, where for a positive integer k we define $\tau(k)$ to be the number of divisors of k. Given both ℓ and λ, we have that $d_1d_3 = \ell/\lambda$ is such that d_1 is divisible only by primes dividing λ, and d_3 is coprime to λ, therefore the pair (d_1, d_3) is uniquely determined. Thus, we just showed that

$$\#\mathcal{T}_4 \leqslant \sum_{\ell \leqslant N(\log p)/p^\beta} \tau(\ell) \ll \frac{N(\log p)(\log N)}{p^\beta} \ll \frac{N(\log N)^2}{p^\beta}. \tag{15}$$

In the last estimate (15) above, we used the known fact that the estimate

$$\sum_{m \leqslant t} \tau(m) \ll t \log t$$

holds uniformly in t, together with the bound (8) which implies that $\log p \ll \log N$. Note that $\mathcal{T}_1$, $\mathcal{T}_2$, $\mathcal{T}_3$ and $\mathcal{T}_4$ cover $\mathcal{T}(p, N)$. Now the desired result follows from estimates (9), (10), (12) and (15) by noticing that if one chooses $\alpha = 1/12$, then $\alpha = \beta = 1/12$. □

Remark. We point out that a different bound on $\mathcal{T}(p, N)$ appears also in [3].

5. Proof of Theorem 3

We may clearly assume that N is arbitrarily large, that $p \geqslant (\log N)^{72}$, and that $v \geqslant 100$, since the result is trivial otherwise. Put $K = p^3$ and let $\mathcal{E}_1$ be the set of integers $n \leqslant N$ such that n is K-smooth. Clearly, $\#\mathcal{E}_1 = \Psi(N, K)$. Note that

$$u = \frac{\log N}{\log K} = \frac{\log N}{3\log p} = \frac{v}{3}. \tag{16}$$

Since $p > (\log N)^{72}$, we have that the inequality $u \leqslant p^{1/2}$ holds uniformly in our range for p and N once N is sufficiently large. Now Lemma 5 yields

$$\#\mathcal{E}_1 = \psi(N,K) \leqslant \frac{N}{u^{u+O(u(\log\log u)/\log u)}} \leqslant \frac{N}{v^{v/3+O(v(\log\log v)/\log v)}} \tag{17}$$

uniformly in p and N.

Next, let $\mathcal{E}_2$ be the set of integers $n \leqslant N$ not in $\mathcal{E}_1$ such that $P^2(n) \mid n$. For each such integer n there is a prime $q > p$ such that $q^2 \mid n$. Thus,

$$\#\mathcal{E}_2 \leqslant \sum_{q>p} \sum_{\substack{n \leqslant N \\ q^2 \mid n}} 1 \leqslant \sum_{q>p} \frac{N}{q^2} \ll \frac{N}{p}. \tag{18}$$

Let $\mathcal{E}_3$ be the set of primes $n \leqslant N$. Clearly, $s(n) = 1$ for all $n \in \mathcal{E}_3$.

Now let $\mathcal{N} = \{1, \ldots, N\} \setminus (\mathcal{E}_1 \cup \mathcal{E}_2 \cup \mathcal{E}_3)$. From the above bounds (17)-(18), it follows that

$$\begin{aligned} S_p(N) &= \sum_{n \in \mathcal{N}} \chi(s(n)) + \pi(N) + O(\#\mathcal{E}_1 + \#\mathcal{E}_2) \\ &= \pi(N) + \sum_{n \in \mathcal{N}} \chi(s(n)) + O\left(\frac{N}{v^{v/3+O(v(\log\log v)/\log v)}} + \frac{N}{p} \right). \end{aligned} \tag{19}$$

Every integer $n \in \mathcal{N}$ can be uniquely represented in the form $n = mq$, where

$$2 \leqslant m < N/K \qquad \text{and} \qquad L_m = \max\{K, P(m)\} < q \leqslant N/m.$$

Conversely, if the numbers m and q satisfy the above inequalities, then $n = mq$ lies in $\mathcal{N}$. Observing that $s(mq) = s(m)q + \sigma(m)$, we arrive at

$$\begin{aligned} \sum_{n \in \mathcal{N}} \chi(s(n)) &= \sum_{2 \leqslant m < N/K} \sum_{L_m < q \leqslant N/m} \chi(s(m)q + \sigma(m)) \\ &= \sum_{\substack{2 \leqslant m < N/K \\ p \nmid \sigma(m)s(m)}} \sum_{L_m < q \leqslant N/m} \chi(s(m)q + \sigma(m)) \\ &+ O\left(\sum_{m \in \mathcal{U}(p,N/K) \cup \mathcal{T}(p,N/K)} \frac{N}{m} \right). \end{aligned} \tag{20}$$

Let us deal first with the "errors" in (20). For the sum over the first subset, we get

$$\sum_{m \in \mathcal{U}(p,N/K)} \frac{N}{m} \leqslant N \sum_{m \in \mathcal{U}(p,N/K)} \frac{1}{m} \ll \frac{N(\log N)^2}{p}, \tag{21}$$

where the last estimate above follows like in the proof of Theorem 2 based on the estimate (11) and on Abel's summation formula. As for the sum over the second one, we get

$$\sum_{m\in\mathcal{T}(p,N/K)} \frac{N}{m} \ll N \sum_{m\in\mathcal{T}(p,N/K)} \frac{1}{m} \ll \frac{N(\log N)^3}{p^{1/12}}. \tag{22}$$

The last estimate above follows also from Abel's summation formula via the fact that, by Theorem 2, the estimate

$$\#\mathcal{T}(p,t) \ll \frac{t(\log N)^2}{p^{1/12}}$$

holds uniformly in $t \leqslant N$.

We now deal with the "main term" in (20). Assuming that $p \nmid s(m)\sigma(m)$, let $a_m \in \{1,\dots,p-1\}$ be the least positive integer in the arithmetic progression $\sigma(m)s(m)^{-1} \pmod p$. We have

$$\begin{aligned}
\sum_{\substack{2\leqslant m<N/K\\ p\nmid\sigma(m)s(m)}} \sum_{L_m<q\leqslant N/m} \chi(s(m)q+\sigma(m)) \\
&= \sum_{\substack{2\leqslant m<N/K\\ p\nmid\sigma(m)s(m)}} \sum_{L_m<q\leqslant N/m} \chi(s(m)(q+a_m)) \\
&= \sum_{\substack{2\leqslant m<N/K\\ p\nmid\sigma(m)s(m)}} \chi(s(m)) \sum_{L_m<q\leqslant N/m} \chi(q+a_m). \qquad (23)
\end{aligned}$$

Since a_m is not a multiple of p, by Lemma 10, we have that

$$\sum_{L_m<q\leqslant N/m} \chi(q+a_m) \leqslant \frac{N}{m}\left(p^{-1/2}+N^{-1/2}m^{1/2}p^{1/2}+N^{-1/6}m^{1/6}\right)p^{o(1)}(\log N)^5,$$

as $p\to\infty$. For $m<N/K$, the first term inside the above parentheses dominates the other terms, and we thus obtain

$$\sum_{L_m<q\leqslant N/m} \chi(q+a_m) \leqslant \frac{Np^{o(1)}(\log N)^5}{mp^{1/2}} \qquad \text{as } p\to\infty.$$

Substituting the above inequality in equation (23), we get,

$$\begin{aligned}\sum_{\substack{2\leqslant m<N/K\\ p\nmid\sigma(m)s(m)}}\ \sum_{L_m<q\leqslant N/m}\chi(s(m)q+\sigma(m)) &\leqslant \sum_{\substack{2\leqslant m<N/K\\ p\nmid\sigma(m)s(m)}}\frac{Np^{o(1)}(\log N)^5}{mp^{1/2}}\\ &\leq \frac{Np^{o(1)}(\log N)^5}{p^{1/2}}\sum_{1\leqslant m<N/K}\frac{1}{m}\\ &\ll \frac{N(\log N)^6}{p^{1/2+o(1)}}\end{aligned}$$

as $p\to\infty$. In particular, the estimate

$$\sum_{\substack{2\leqslant m<N/K\\ p\nmid\sigma(m)s(m)}}\ \sum_{L_m<q\leqslant N/m}\chi(s(m)q+\sigma(m))\ll\frac{N(\log N)^6}{p^{1/3}} \tag{24}$$

holds uniformly in N and p. The desired estimate follows now from estimates (19), (20), (21), (22), and (24).

6. Proof of Theorem 4

For a set $\mathcal{A}$ of positive integers and a positive real number t we put $\mathcal{A}(t)=\mathcal{A}\cap[1,t]$.

We may assume that $v\geqslant 100$, otherwise there is nothing to prove. By Theorem 3 and the Prime Number Theorem, it follows that

$$|S_p(N)|\ll\frac{N}{\log N}+\frac{N}{v^{v/3+O(v(\log\log v)/\log v)}}$$

provided that $p\geqslant(\log N)^{100}$. So, from now on we may assume that $p<M=(\log N)^{100}$. Then $v\geqslant(\log N)/(\log M)=(\log N)/(100\log\log N)$, therefore

$$\frac{N}{v^{v/3+O(v(\log\log v)/\log v)}}\leqslant N^{1-1/300+o(1)},\qquad (N\to\infty),$$

therefore certainly the above expression is $\ll N/(\log\log\log N)$. Thus, it remains to show that

$$|S_p(N)|\ll\frac{N}{\log\log\log N}\qquad\text{for } p\leqslant M. \tag{25}$$

Lemma 4 in [8], shows that there exists a constant $c_1>0$ such that $\sigma(n)$ is a multiple of all primes $q\leqslant c_1\log\log N/\log\log\log N$ for all $n\leqslant N$ with $O(N/(\log\log\log N)^2)$ exceptions. Set $L=c_1\log\log N/(\log\log\log N)$, assume that $p \leqslant L$, and suppose that $p \mid \sigma(n)$. Then $s(n) \equiv -n \pmod p$, therefore $\chi(s(n)) = \chi(-n) = \chi(-1)\chi(n)$. Thus,

$$\begin{aligned} S_p(N) &= \chi(-1)\sum_{n\leqslant N}\chi(n) + O\left(\frac{N}{\log\log\log N}\right) = O\left(p + \frac{N}{\log\log\log N}\right) \\ &= O\left(\frac{N}{\log\log\log N}\right). \end{aligned}$$

Thus, we may assume that $p \in (L, M]$. Put $U = N^{1/\log\log\log N}$ and $V = N^{1/(\log\log\log N)^2}$. Write $n = PQ\ell$, where $P = P(n)$ and $Q = P(n/P)$. The set $\mathcal{E}_1$ of $n \leqslant N$ such that $P \leqslant U$ is, by Lemma 5, of cardinality $O(N/\log\log N)$ as $N \to \infty$. Thus, we may restrict our attention to the positive integers $n \leqslant N$ not in $\mathcal{E}_1$. The set $\mathcal{E}_2$ of such integers such that $Q = P$ is of cardinality $O(N/U) = O(N/\log\log\log N)$ (see, for example, the estimate for $\#\mathcal{E}_2$ in the proof of Theorem 3). We claim that the set $\mathcal{E}_3$ of $n \leqslant N$ such that $Q \leqslant V$ is also of cardinality $O(N/\log\log\log N)$. To see this, assume that $m = Q\ell < N/U$ is some number with $P(m) = Q \leqslant V$. Fixing m, the number of possibilities for $P \leqslant N/m$ is at most

$$\pi(N/m) \leqslant \frac{N}{m\log(N/m)} \ll \frac{N}{m\log U} = \frac{N\log\log\log N}{m\log N}.$$

Summing up over all m with $P(m) \leqslant V$, we get that the number of possibilities for such $n \leqslant N$ is

$$\begin{aligned} \#\mathcal{E}_3 &\ll \frac{N\log\log\log N}{\log N}\sum_{\substack{1\leqslant m\\ P(m)\leqslant V}}\frac{1}{m} \ll \frac{N\log\log\log N}{\log N}\prod_{p\leqslant V}\left(1-\frac{1}{p}\right)^{-1} \\ &\ll \frac{N(\log\log\log N)\log V}{\log N} = \frac{N}{\log\log\log N}. \end{aligned}$$

Here, we used Mertens's estimate for the product $\prod_{p\leqslant t}(1-1/p)$ with the choice $t = V$.

We also discard the set $\mathcal{E}_4$ of positive integers $n \leqslant N$ divisible by the square of a prime $> V$ since the number of such $n \leqslant N$ is $O(N/V) = O(N/\log\log\log N)$. On the remaining set of $n \leqslant N$, we have that $P > Q > P(\ell)$, $Q > V$ and $P > U$. Let $\mathcal{M}$ be the set of positive integers $m < N/U$, $mP(m) \leqslant N$, $P(m) = Q > V$ and $P(m)\|m$. Put, as in the proof of Theorem 3, $L_m = \max\{P(m), U\}$. Then all but $O(N/\log\log\log N)$ of the positive integers $n \leqslant N$ are of the form mP, for some $m \in \mathcal{M}$ and $P \in (L_m, N/m]$, and distinct pairs (m, P) of this form give rise to distinct n's. Thus,

$$N = \sum_{m \in \mathcal{M}} (\pi(N/m) - \pi(L_m)) + O\left(\frac{N}{\log\log\log N}\right).$$

Furthermore, as in the proof of Theorem 3, we also have

$$S_p(N) = \sum_{m \in \mathcal{M}} \sum_{L_m < P \leqslant N/m} \chi(s(m)P + \sigma(m)) + O\left(\frac{N}{\log\log\log N}\right).$$

We next need to understand the subset $\mathcal{M}_1$ of $\mathcal{M}$ formed by numbers m such that $p \mid s(m)$. Let $m = Q\ell$ be such a number. Then $s(m) = Qs(\ell) + \sigma(\ell) \equiv 0 \pmod p$. Let $\mathcal{M}_2$ be the subset of $\mathcal{M}_1$ such that $p \mid s(\ell)$. Then $p \mid \sigma(\ell)$, therefore $p \mid \ell$. Furthermore, $p \mid \sigma(\ell) \mid \sigma(m)$. Write $m = pm_1$. Let $t \leqslant N$ and let us bound the cardinality of $\mathcal{M}_2(t)$. If $p \mid m_1$, then $p^2 \mid m$ and the number of such m is $\leqslant t/p^2$. If $p \nmid m_1$, then $\sigma(m) = (p+1)\sigma(m_1)$, therefore $m_1 \leqslant t/p$ is a number such that $p \mid \sigma(m_1)$. Theorem 1 shows that the number of such m_1 is

$$\#\mathcal{U}(p, t/p) \ll \frac{t(\log\log t)^{5/3}(\log p)^2}{p^2} \ll \frac{t(\log\log N)^{5/3}(\log p)^2}{p^2}.$$

Thus, we just proved that the estimate

$$\#\mathcal{M}_2(t) \ll \frac{t(\log\log N)^{5/3}(\log p)^2}{p^2} \tag{26}$$

holds uniformly in $t \leqslant N$.

Next, let us look at $\mathcal{M}_3 = \mathcal{M}_1 \backslash \mathcal{M}_2$. In this case, $Qs(\ell) \equiv -\sigma(\ell) \pmod p$. Furthermore, $p \nmid s(\ell)$, and $Q > V > M > p$, therefore $p \nmid \sigma(\ell)$. Thus, $-\sigma(\ell)s(\ell)^{-1} \pmod p$ is a nonzero congruence class modulo p and we denote its residue by $a_\ell \in \{1, \ldots, p-1\}$. This shows that the congruence class of Q modulo p is determined by ℓ whenever $m \in \mathcal{M}_2$. Fixing ℓ with $p \nmid s(\ell)$ and t such that $t > \ell V$, the number of possible values of $Q \leqslant t/\ell$ such that $Q \equiv -\sigma(\ell)s(\ell)^{-1} \pmod p$ is, by Lemma 6,

$$\pi(t/\ell; p, a_\ell) \ll \frac{t}{p\ell\log(2t/(\ell p))} \ll \frac{t}{p\ell\log(V^{1/2})} \ll \frac{t(\log\log\log N)^2}{p\ell\log N}.$$

Here, we used the fact that the inequalities

$$\frac{t}{\ell p} \geqslant \frac{V}{p} \geqslant \frac{V}{M} \geqslant V^{1/2}$$

hold whenever N is sufficiently large. The above bound is uniform in our range for t and ℓ. Summing up over all the possibilities for $\ell \leqslant t/V$ once t is fixed, we get that

$$\#\mathcal{M}_3(t) \ll \frac{t(\log\log\log N)^2}{p\log N}\sum_{\ell\leqslant N}\frac{1}{\ell} \ll \frac{t(\log\log\log N)^2}{p}. \tag{27}$$

Estimates (26), (27), and the fact that $\#\mathcal{M}_1(t) \leqslant \#\mathcal{M}_2(t) + \#\mathcal{M}_3(t)$, show that

$$\#\mathcal{M}_1(t) \ll t\left(\frac{(\log\log N)^{5/3}(\log p)^2}{p^2} + \frac{(\log\log\log N)^2}{p}\right).$$

By Abel's summation formula, we get that

$$\sum_{m\in\mathcal{M}_1}\frac{1}{m} \ll \left(\frac{(\log\log N)^{5/3}(\log p)^2}{p^2} + \frac{(\log\log\log N)^2}{p}\right)\log N.$$

Thus,

$$\begin{aligned}\sum_{m\in\mathcal{M}_1} & (\pi(N/m)-\pi(L_m)) \leqslant \sum_{m\in\mathcal{M}_1}\frac{N}{m\log(N/m)} \\ \leqslant & \sum_{m\in\mathcal{M}_1}\frac{N}{m\log U} \ll \frac{N\log\log\log N}{\log N}\sum_{m\in\mathcal{M}_1}\frac{1}{m} \\ \ll & N\left(\frac{(\log\log N)^{5/3}(\log\log\log N)(\log p)^2}{p^2} + \frac{(\log\log\log N)^3}{p}\right).\end{aligned}$$

Since $p \geqslant L$, it follows that the last bound above is

$$\ll \frac{N(\log\log\log N)^5}{(\log\log N)^{1/3}} \ll \frac{N}{\log\log\log N}.$$

Thus, putting $\mathcal{M}_0 = \mathcal{M}\backslash\mathcal{M}_1$, we have just proved that both formulae

$$N = \sum_{m\in\mathcal{M}_0}(\pi(N/m)-\pi(L_m)) + O\left(\frac{N}{\log\log\log N}\right), \tag{28}$$

and

$$S_p(N) = \sum_{m\in\mathcal{M}_0}\sum_{L_m<P<N/m}\chi(s(m)P+\sigma(m)) + O\left(\frac{N}{\log\log\log N}\right) \tag{29}$$

hold uniformly in our range for p and N. Now let $m \in \mathcal{M}_0$, and let λ be any congruence class modulo p different from $\sigma(m)$. Primes $P \in (L_m, N/m]$ such that $s(m)P+\sigma(m) \equiv \lambda \pmod p$ are precisely those such that $P \equiv (\lambda - \sigma(m))s(m)^{-1} \pmod p$. Let $a := a(\lambda, m) \in \{1,\ldots,p-1\}$ be such that $a \equiv (\lambda-\sigma(m))s(m)^{-1}$

(mod p). Since $p \leqslant (\log N)^{100} \leqslant (\log U)^{200} \leqslant (\log L_m)^{200}$ uniformly for large N, we get, by Lemma 7 with $A = 200$, that the number of such primes is

$$\pi(N/m; p, a) - \pi(L_m; p, a) = \frac{\pi(N/m) - \pi(L_m)}{\phi(p)} + O\left(\frac{N}{m \exp(c_2\sqrt{\log U})}\right)$$

with some positive constant c_2. This shows that, for a fixed $m \in \mathcal{M}_0$, we have that the inner sums in (29) can be estimated as

$$\begin{aligned}\sum_{L_m < P \leqslant N/m} \chi(s(m)P + \sigma(m)) &= \frac{\pi(N/m) - \pi(L_m)}{p-1} \sum_{\substack{\lambda \pmod p \\ \lambda \not\equiv \sigma(m) \pmod p}} \chi(\lambda) \\ &+ O\left(\frac{pN}{m \exp(c_2\sqrt{\log U})}\right).\end{aligned}$$

Since

$$\sum_{\substack{\lambda \pmod p \\ \lambda \not\equiv \sigma(m) \pmod p}} \chi(\lambda) = O(1),$$

we get, summing up over all the possible values of $m \in \mathcal{M}_0$, and using estimates (29) and (28), that

$$\begin{aligned} S_p(N) &= \sum_{m \in \mathcal{M}_0} \sum_{L_m < P \leqslant N/m} \chi(s(m)P + \sigma(m)) + O\left(\frac{N}{\log\log\log N}\right) \\ &\ll \sum_{m \in \mathcal{M}_0} \left(\frac{\pi(N/m) - \pi(L_m)}{p-1} + \frac{Np}{m \exp(c_2\sqrt{\log U})}\right) + \frac{N}{\log\log\log N} \\ &= \frac{1}{p-1}\left(\sum_{m \in \mathcal{M}_0} (\pi(N/m) - \pi(L_m))\right) + \frac{Np}{\exp(c_2\sqrt{\log U})} \sum_{m \leqslant N} \frac{1}{m} \\ &+ \frac{N}{\log\log\log N} \ll \frac{N}{p-1} + \frac{Np\log N}{\exp(c_2\sqrt{\log U})} + \frac{N}{\log\log\log N} \\ &\ll \frac{N}{L} + \frac{N(\log N)^{101}}{\exp((\log N)^{1/3})} + \frac{N}{\log\log\log N} \ll \frac{N}{\log\log\log N}, \end{aligned}$$

which completes the proof of this theorem.

REFERENCES

[1] BAKER, A. – WÜSTHOLZ, G.: *Logarithmic forms and group varieties*, J. Reine Angew. Math. **442** (1993), 19–62.

[2] BALASURIYA, S. – SHPARLINSKI, I. E. – SUTANTYO, D.: *Multiplicative character sums with the Euler function*, Studia Sci. Math. Hungar. (to appear).
[3] BALASURIYA, S. – BANKS, W. – SHPARLINSKI, I. E.: *Congruences and exponential sums with the sum of aliquot divisors function*, Int. J. Number Theory (to appear).
[4] BANKS, W. – SHPARLINSKI, I. E.: *Congruences and exponential sums with the Euler function*, in: *High Primes and Misdemeanours*, Lectures in Honour of the 60th Birthday of Hugh Cowie Williams, Amer. Math. Soc., 2004, pp. 49–60.
[5] BANKS, W. – SHPARLINSKI, I. E.: *Congruences and rational exponential sums with the Euler function*, Rocky Mountain J. Math. **36** (2006), 1415–1426.
[6] CANFIELD, E. R. – ERDŐS, P.–POMERANCE, C.: *On a problem of Oppenheim concerning "Factorisatio Numerorum"*, J. Number Theory **17** (1983), 1–28.
[7] DAVENPORT, H.: *Multiplicative Number Theory*, Second edition, Grad. Text in Math. Vol. **74**, Springer-Verlag, Berlin, 1980.
[8] DE KONINCK, J.-M. – LUCA, F.: *On the composition of the Euler's function and the sum of divisors function*, Colloq. Math. **108** (2007), 31–51.
[9] GREAVES, G.: *Sieves in Number Theory*, Springer-Verlag, Berlin, 2001.
[10] HALBERSTAM, H. – RICHERT, H.–E.: *Sieve Methods*, Academic Press, London, 1974.
[11] HEATH-BROWN, D. R.: *Zero-free regions for Dirchlet L-functions and the least prime in an arithmetic progression*, Proc. London Math. Soc. **64** (1991), 265–338.
[12] HILDEBRAND, A. – TENENBAUM, G.: *Integers without large prime factors*, J. Théor. Nombres Bordeaux **5** (1993), 411–484.
[13] IVIĆ, A.: *The Riemann Zeta-Function*, Wiley-Interscience, New York, 1985.
[14] LOXTON, J. H.: *Some problems involving powers of integers*, Acta Arith. **46** (1986), no. 2, 113–123.
[15] Z. KH. RAKHMONOV: *On the distribution of values of Dirichlet characters and their applications*, Proc. Steklov Inst. Math. **207** (1995), 263–272.
[16] SHOREY, T. N.: *An equation of Goormaghtigh and diophantine approximations*, in: *Current Trends in Number Theory*, (S. D. Adhikari, S. A. Katre and B. Ramakrishnan, eds), Hindustan Book Agency, New Delhi, 2002, pp. 185–197.

Received July 18, 2007
Accepted January 24, 2008

Sanka Balasuriya
Department of Computing
Macquarie University
Sydney, NSW 2109,
AUSTRALIA
E-mail: sanka@ics.mq.edu.au

Florian Luca
Instituto de Matemáticas
Universidad Nacional Autonoma de México
C.P. 58089, Morelia, Michoacán,
MÉXICO
E-mail: fluca@matmor.unam.mx