Uniform Distribution Theory **2** (2007), no.1, 93–105

uniform distribution theory

DISCREPANCY BETWEEN QMC AND RQMC

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ABSTRACT. We introduce a class of functions in $d \geq 3$ dimensions which have arbitrary odd superposition effective dimensions between three and d inclusive. We prove that for the integration of any function in this class any Sobol' points of a fixed length have zero error, whereas Owen's scrambling of any Sobol' points of the same length has the same variance of error as simple Monte Carlo methods. Furthermore, for any function in the same class Owen's scrambling of high-discrepancy points, which consist of d copies of the van der Corput points in base two, gives zero-variance estimates for the integration.

Communicated by Reinhard Winkler

Dedicated to Professor Robert F. Tichy on the occasion of his 50th birthday

1. Introduction

Quasi-Monte Carlo (QMC) methods are the deterministic version of Monte Carlo (MC) methods [8]. Around 1993, they were first applied to finance problems successfully by researchers at Columbia University [26]. Paskov and Traub [16] reported that Sobol' sequences give much faster convergence than conventional Monte Carlo methods for the evaluation of Mortgage-Backed Securities (MBS). In 1996, Papageorgiou and Traub [15] applied generalized Faure (GFaure) sequences and Sobol' sequences to the evaluation of the most difficult tranche of a Collateralized Mortgage Obligation (CMO) with ten tranches, which they obtained as a real hard problem from Goldman-Sachs, and reported that both GFaure and Sobol' are much faster than Monte Carlo methods and that GFaure is several times faster than Sobol'. Since then, explanation of the success of quasi-Monte Carlo methods in finance has been a long open problem [14, 17, 20, 25].

2000 Mathematics Subject Classification: 65D30.
Keywords: Effective dimension, generalized Sobol' sequences, high dimensional integrals, Monte Carlo and quasi-Monte Carlo methods, Owen's scrambling.

Around 1995, Owen [9] proposed randomized quasi-Monte Carlo (RQMC) methods, for which he gave an exact formula of the variance of the so-called Owen's scrambling applied to any set of points in $[0,1]^d$. In his later papers [10, 11], he showed that scrambled $(0,m,d)$-nets provide much smaller variance than simple Monte Carlo methods when the integrands satisfy some smoothness condition. Caflisch et al. [2] applied RQMC successfully to finance problems including MBS evaluations, where they used scrambled Sobol' points. After that, many people (see, e.g., [5, 7]) have followed this direction of research. Today, for QMC as well as RQMC it is believed that Sobol' sequences with judiciously chosen direction numbers have very good convergence properties for many practical problems including problems in finance [4, 6].

In this paper, we study the difference between the integration errors of QMC and of RQMC. Denote a set of N points in $[0,1)^d$ by $X_n = (X_n^{(1)}, ..., X_n^{(d)}), n = 0, ..., N-1$. Let $(x_u, 1)$ denote the vector x from $[0,1]^d$ where all components whose indices are not in a subset $u \subseteq \{1, ..., d\}$ are replaced by 1. We denote the cardinality of u by $|u|$. Let $I(f) = \int_{[0,1]^d} f(x)dx$. For the integration error of QMC, we have the Hlawka-Zaremba identity [3]:

$$I(f) - \frac{1}{N}\sum_{n=0}^{N-1} f(X_n) = \sum_{\emptyset \neq u \subseteq \{1,\ldots,d\}} (-1)^{|u|} \int_{[0,1]^{|u|}} disc(x_u, 1) \frac{d^{|u|}}{dx_u} f(x_u, 1) dx_u,$$

where the local discrepancy is defined by

$$disc(x) = \prod_{i=1}^{d} x_i - \frac{|\{n; X_n \in [0,x_1) \times \cdots \times [0,x_d)\}|}{N}.$$

On the other hand, for the variance of the estimate based on Owen's scrambling in base b [10, 11, 12], we have

$$\mathrm{V}(f; N; X_n) \equiv \mathrm{E}\left[\left(I(f) - \frac{1}{N}\sum_{n=0}^{N-1} f(X_n)\right)^2\right] = \frac{1}{N} \sum_{\emptyset \neq u \subseteq \{1,\ldots,d\}} \sum_{\kappa} \Gamma_{u,\kappa} \sigma_{u,\kappa}^2, \tag{1}$$

where the expectation is with respect to Owen's scrambling, $b \geq 2$ is an integer, and κ is a vector of $|u|$ nonnegative integers k_i for $i \in u$. The gain coefficient is defined by

$$\Gamma_{u,\kappa} = \frac{1}{N(b-1)^{|u|}} \sum_{n=0}^{N-1} \sum_{n'=0}^{N-1} \prod_{i \in u} (b 1_{\lfloor b^{k_i+1} X_n^{(i)} \rfloor = \lfloor b^{k_i+1} X_{n'}^{(i)} \rfloor} - 1_{\lfloor b^{k_i} X_n^{(i)} \rfloor = \lfloor b^{k_i} X_{n'}^{(i)} \rfloor}),$$

and $\sigma^2_{u,\kappa}$ is the variance of a step function determined from the b-ary Haar wavelet expansion of f. The definition of the step function is given later in Section 2.3.

We can see that the gain coefficient is to the function variance as the local discrepancy is to the function variation, and that both the gain coefficient and the local discrepancy depend only on the point set, whereas the variance and the variation depend only on the function.

The main focus of this study is on the difference between the gain coefficient and the local discrepancy, i.e., that the gain coefficient is based on the product of one-dimensional quantities (hereafter we call this *the product structure* of the gain coefficient), whereas the d-dimensional local discrepancy is not the product of the one-dimensional local discrepancies. The organization of the paper is as follows: First, we introduce a class of functions in $d \geq 3$ dimensions which have arbitrary odd superposition effective dimensions between three and d inclusive. Then, we prove that for the integration of any function in this class any Sobol' points of a fixed length yield zero error, whereas Owen's scrambling of any Sobol' points of the same length has the same variance of error as simple Monte Carlo methods. Furthermore, for any function in the same class Owen's scrambling of high-discrepancy points, which consist of d copies of the van der Corput points in base two, gives zero-variance estimates for the integration. The key to the proofs of these results is the product structure of the gain coefficient. In the last section, we discuss the significance of this result and future research directions.

2. Main Results

2.1. Definitions and notations

The notion of effective dimension is important for studying the efficiency of RQMC [13]. It is defined based on the ANOVA (Analysis of Variance) decomposition of a function. So, we start with the definition of ANOVA [10]: Let $u \subseteq \{1, 2, ..., d\}$ and let $\bar{u} = \{1, 2, ..., d\} - u$ be its complement. Also let $X = (x_1, ..., x_d) \in [0, 1)^d$ and X^u denote those coordinates of X indexed by elements of u. Then, the ANOVA decomposition of $f(x_1, ..., x_d)$ is defined by

$$f(x_1, ..., x_d) = \sum_{u \subseteq \{1,...,d\}} \alpha_u(x_1, ..., x_d),$$

where the sum is over all 2^d subsets of coordinates of $[0,1)^d$. The terms $\alpha_u(x_1,...,x_d)$ are defined recursively starting with

$$\alpha_\emptyset(x_1,...,x_d) := I(f) \equiv \int_{[0,1)^d} f(z_1,...,z_d)dz_1...dz_d,$$

and

$$\alpha_u(x_1,...,x_d) := \int_{Z^u=X^u, Z^{\bar{u}}\in[0,1)^{\bar{u}}} (f(z_1,...,z_d) - \sum_{v\subset u} \alpha_v(z_1,...,z_d)) \prod_{j\in\bar{u}} dz_j,$$

where the sum is over proper subsets $v \neq u$. When $u = \{1,...,d\}$,

$$\alpha_{\{1,...,d\}}(x_1,...,x_d) = f(x_1,...,x_d) - \sum_{v\subset\{1,...,d\}} \alpha_v(x_1,...,x_d).$$

The meaning of $\alpha_u(x_1,...,x_d)$ is the effect of the subset X^u on $f(x_1,...,x_d)$ minus the effect of its proper subset X^v with $v \subset u$. The $\alpha_u(x_1,...,x_d)$ have the following orthogonal properties:

- Let $i \in u$. If we fix all the $x_j, j \neq i$, then

$$\int_0^1 \alpha_u(x_1,...,x_d)dx_i = 0.$$

  Thus, when $\emptyset \neq u \subseteq \{1,...,d\}$,

$$\int_{[0,1)^d} \alpha_u(x_1,...,x_d)dx_1...dx_d = 0.$$

- When $u \neq v$,

$$\int_{[0,1)^d} \alpha_u(x_1,...,x_d)\alpha_v(x_1,...,x_d)dx_1...dx_d = 0.$$

  Hence, the variance of $f(x_1,...,x_d)$ is given by

$$\sigma^2 = \int_{[0,1)^d} (f(x_1,...,x_d) - \alpha_\emptyset(x_1,...,x_d))^2 dx_1...dx_d = \sum_{u\neq\emptyset} \sigma_u^2,$$

  where

$$\sigma_u^2 := \sigma^2(\alpha_u) = \begin{cases} 0 & \textit{if } u = \emptyset, \\ \int_{[0,1)^d} \alpha_u(x_1,...,x_d)^2 dx_1...dx_d & \textit{otherwise}. \end{cases}$$

The definition of superposition effective dimension is given as follows:

$$D_{super} := \min\{i : 1 \leq i \leq d \text{ such that } \sum_{|u|\leq i} \sigma_u^2 \geq (1-\epsilon)\sigma^2\},$$

where ϵ can be set to any value in $[0,1)$, and is often chosen to be 0.01 in practice.

Next, we recall the definition of Walsh functions:

$$\mathrm{wal}(0,x) = 1 \text{ for } x \in [0,1),$$

and for a nonnegative integer $m \geq 1$,

$$\mathrm{wal}(m,x) = (-1)^{\sum_{j=1}^{\infty} m_{j-1}a_j} = (-1)^{(\mathbf{m},X)} \text{ for } x \in [0,1),$$

where $m = m_0 + m_1 2 + \cdots$, and $x = a_1 2^{-1} + a_2 2^{-2} + \cdots$ in its canonical base 2 representation, and $\mathbf{m} = (m_0, m_1, ...)$ and $X = (a_1, a_2, ...)$ are the binary vector representation of m and x, respectively. The Rademacher functions are the subclass of the Walsh functions for which m is a power of 2.

Let t_k be an integer with $2^{k-1} \leq t_k < 2^k$ for $k = 1, 2, ...$, and denote its binary representation by $t_k = t_{k,1} + t_{k,2}2 + \cdots + t_{k,k}2^{k-1}$ with $t_{k,k} = 1$. We define a nonsingular lower triangular infinite matrix T, where its (k,j)-element for $j \leq k$ is equal to $t_{k,j}$. Hereafter, we denote

$$r_0^{(T)}(x) = 1, \text{ for } x \in [0,1),$$

and for $k = 1, 2, ...$,

$$r_k^{(T)}(x) = \mathrm{wal}(t_k, x).$$

Note that the matrix T specifies uniquely a subclass of the Walsh functions, and that the identity matrix I corresponds to the Rademacher functions.

2.2. Functions with arbitrary effective dimension

From now on, we fix d matrices $T_1, ..., T_d$ which specify d subclasses of the Walsh functions. First, we introduce the following functions in d dimensions:

Definition 1. We define for $2 \leq s \leq d$ and $k \geq 1$,

$$\phi_{s,k}(x_1, ..., x_d) = \sum_{\substack{\{1,2\} \subseteq u \subseteq \{1,...,d\} \\ |u|=s}} c_{u,k} \prod_{i \in u} r_k^{(T_i)}(x_i),$$

where $c_{u,k}$ are constants.

The following lemma is straightforward from Definition 1.

Lemma 1. *Let $\{1,2\} \subseteq u \subseteq \{1,...,d\}$. For any $k \geq 1$ and for any $w \subset \{1,...,d\}$ with $|w| \leq |u|$ and $w \neq u$, we have*

$$\int_{Z^w = X^w, Z^{\bar{w}} \in [0,1)^{\bar{w}}} \prod_{i \in u} r_k^{(T_i)}(z_i) \prod_{j \in \bar{w}} dz_j = 0.$$

Proof. The proof follows from the orthogonal property of the Walsh functions and that $u \cap \bar{w} \neq \emptyset$ for any $w \neq u$. $\square$

Here, we define a class $\mathfrak{F}_{s,m}$ of functions in d dimensions.

Definition 2. For any integer $m \geq 2$ and $2 \leq s \leq d$, we define a class $\mathfrak{F}_{s,m}$ which consists of functions

$$f(x_1, ..., x_d) = c_0 + \sum_{k=\lfloor (m-1)/2 \rfloor + 2}^{m} \phi_{s,k}(x_1, ..., x_d),$$

where c_0 is constant.

Let's consider the ANOVA decomposition of $f \in \mathfrak{F}_{s,m}$. First, we have

$$\alpha_\emptyset(x_1, ..., x_d) = \int_{[0,1)^d} f(x_1, ..., x_d) dx_1 ... dx_d = c_0.$$

The orthogonal property of the Walsh functions leads to

$$\sigma^2(f) = \int_{[0,1)^d} (f(x_1, ..., x_d) - c_0)^2 dx_1 ... dx_d = \sum_{k=\lfloor (m-1)/2 \rfloor + 2}^{m} \sum_{\substack{\{1,2\} \subseteq u \subseteq \{1,...,d\} \\ |u|=s}} c_{u,k}^2.$$

Let $\mathfrak{U} = \{u \mid \{1,2\} \subseteq u \subseteq \{1, ..., d\}$ with $|u| = s\}$. From Lemma 1, when $w \notin \mathfrak{U}$ and $\emptyset \neq w \subset \{1, ..., d\}$ with $|w| \leq s$, we have

$$\begin{aligned}
&\alpha_w(x_1, ..., x_d) = \\
&= \int_{Z^w = X^w, Z^{\bar{w}} \in [0,1)^{\bar{w}}} (f(z_1, ..., z_d) - \sum_{v \subset w} \alpha_v(z_1, ..., z_d)) \prod_{j \in \bar{w}} dz_j \\
&= \int_{Z^w = X^w, Z^{\bar{w}} \in [0,1)^{\bar{w}}} (f(z_1, ..., z_d) - c_0) \prod_{j \in \bar{w}} dz_j \\
&= \int_{Z^w = X^w, Z^{\bar{w}} \in [0,1)^{\bar{w}}} \sum_{k=\lfloor (m-1)/2 \rfloor + 2}^{m} \phi_{s,k}(z_1, ..., z_d) \prod_{j \in \bar{w}} dz_j \\
&= \sum_{k=\lfloor (m-1)/2 \rfloor + 2}^{m} \sum_{\substack{\{1,2\} \subseteq u \subseteq \{1,...,d\} \\ |u|=s}} c_{u,k} \int_{Z^w = X^w, Z^{\bar{w}} \in [0,1)^{\bar{w}}} \prod_{i \in u} r_k^{(T_i)}(z_i) \prod_{j \in \bar{w}} dz_j \\
&= 0.
\end{aligned}$$

When $w \in \mathfrak{U}$, also from Lemma 1 we have

$$\alpha_w(x_1, ..., x_d) = \sum_{k=\lfloor (m-1)/2 \rfloor + 2}^{m} c_{w,k} \prod_{i \in w} r_k^{(T_i)}(x_i).$$

Thus, $\sigma_w^2 = \sigma^2(\alpha_w) = \sum_{k=\lfloor (m-1)/2 \rfloor + 2}^{m} c_{w,k}^2$. We now arrive at the following theorem:

Theorem 1. *For any function $f \in \mathfrak{F}_{s,m}$, its superposition effective dimension is equal to s.*

Note that the above theorem holds for every $\epsilon \in [0,1)$ in the definition of superposition effective dimension.

2.3. Integration errors of QMC and of RQMC

Here, we consider a class of (t,d)-sequences in base $b=2$ whose generator matrices are written as $(T_i)^{-1}U_i, i=1,...,d$, where $T_i, i=1,...,d$, are matrices specifying subclasses of the Walsh functions, and $U_i, i=1,...,d$, are arbitrary nonsingular upper-triangular matrices. Hereafter, we denote this class by $\mathfrak{S}_d$. We prove the following theorem:

Theorem 2. *If s is an odd integer between three and d inclusive, then for any function f in $\mathfrak{F}_{s,m}$, we have*

$$e_{2^m}(f;X_n) \equiv \left| I(f) - \frac{1}{2^m}\sum_{n=0}^{2^m-1} f(X_{n+\ell}) \right| = 0,$$

where $X_n, n=0,1,...,$ is any sequence in $\mathfrak{S}_d$, and ℓ is any multiple of 2^m.

P r o o f. The proof is essentially the same as Theorem 2 of [22]. Based on Lemmata 2, 3, and 4 of [22], we have

$$e_{2^m}(\phi_{s,k};X_n) = 0$$

for $1 \le k \le m$. Since f is a linear combination of $\phi_{s,k}$, the proof follows immediately. □

By definition [19, 21], the generalized Sobol' sequences with lower triangular matrices $(T_i)^{-1}, i=1,...,d$, are a subset of $\mathfrak{S}_d$, where $U_i, i=1,...,d$, are constructed based on irreducible polynomials and the so-called direction numbers. Therefore, the above theorem also holds for any generalized Sobol' sequence in $\mathfrak{S}_d$. Note that Sobol' sequences are the special case of generalized Sobol' sequences with $T_i = I, i=1,...,d$.

Now, we consider Owen's scrambling of generalized Sobol' points. The following lemma is useful in the succeeding development.

Lemma 2. *Let $\emptyset \ne u \subseteq \{1,...,d\}$ and $k_i \ge 0$ for $i \in u$. If each subinterval defined by*

$$\prod_{i\in u}\left[\frac{j_i}{b^{k_i+1}}, \frac{j_i+1}{b^{k_i+1}}\right) \times \prod_{i\notin u}[0,1), \tag{2}$$

where $0 \le j_i < b^{k_i+1}$ for $i \in u$, contains at most one point of $X_n, n=0,...,N-1$, then we have $\Gamma_{u,\bar\kappa} = 1$, where $\bar\kappa = \kappa + (1,...,1)$.

P r o o f. If the assumption holds, then for any pair (n, n') with $0 \le n \ne n' \le N-1$, there exists some $i \in u$ such that

$$b1_{\lfloor b^{k_i+2}X_n^{(i)}\rfloor = \lfloor b^{k_i+2}X_{n'}^{(i)}\rfloor} - 1_{\lfloor b^{k_i+1}X_n^{(i)}\rfloor = \lfloor b^{k_i+1}X_{n'}^{(i)}\rfloor} = 0,$$

thereby

$$\prod_{i\in u}(b1_{\lfloor b^{k_i+2}X_n^{(i)}\rfloor = \lfloor b^{k_i+2}X_{n'}^{(i)}\rfloor} - 1_{\lfloor b^{k_i+1}X_n^{(i)}\rfloor = \lfloor b^{k_i+1}X_{n'}^{(i)}\rfloor}) = 0.$$

For the case of $n = n'$ with $0 \le n \le N-1$, we have

$$\prod_{i\in u}(b1_{\lfloor b^{k_i+2}X_n^{(i)}\rfloor = \lfloor b^{k_i+2}X_{n'}^{(i)}\rfloor} - 1_{\lfloor b^{k_i+1}X_n^{(i)}\rfloor = \lfloor b^{k_i+1}X_{n'}^{(i)}\rfloor}) = (b-1)^{|u|}.$$

Thus, the proof is complete. □

Owen [10, 11, 12] analyzed the variance of the integration error for his scrambling. His formula for the variance of the integration error was given in (1). Here, we give the definition of the step function, which we denote by $\nu_{u,\kappa}$. First, we need recall the b-ary Haar wavelets on $[0, 1)$:

$$\psi_{k,t,c}(x) = b^{\frac{k}{2}}(b1_{\lfloor b^{k+1}x\rfloor = bt+c} - 1_{\lfloor b^k x\rfloor = t}),$$

where k, t, c are integers with $k \ge 0$, $0 \le t < b^k$, and $0 \le c < b$. A function $f \in L_2[0, 1)^d$ has the b-ary Haar wavelet expansion:

$$f(x_1, ..., x_d) = I(f) + \sum_{\emptyset \ne u \subseteq (1,...,d)} \sum_\kappa \sum_\tau \sum_\gamma h_{u,\kappa,\tau,\gamma} \prod_{i\in u} \psi_{k_i,t_i,c_i}(x_i),$$

where κ, τ, and γ are vectors of $|u|$ elements k_i, t_i, and c_i for $i \in u$, respectively, and the coefficients are given by

$$h_{u,\kappa,\tau,\gamma} = \int_{[0,1)^d} f(x_1, ..., x_d) \prod_{i\in u} \psi_{k_i,t_i,c_i}(x_i) dx_1, ..., dx_d.$$

Then, the step function is defined by

$$\nu_{u,\kappa}(x_1, ..., x_d) = \sum_\tau \sum_\gamma h_{u,\kappa,\tau,\gamma} \prod_{i\in u} \psi_{k_i,t_i,c_i}(x_i).$$

Thus, $\nu_{u,\kappa}$ is constant within the subintervals defined in (2). The following lemma is useful:

Lemma 3. *Let $\emptyset \ne u \subseteq \{1, ..., d\}$. For any $k \ge 1$, a function $\prod_{i\in u} r_k^{(T_i)}(x_i)$ has the following step functions in its binary ($b = 2$) Haar wavelet expansion:*

$$\nu_{u,\kappa}(x_1, ..., x_d) = \begin{cases} \prod_{i\in u} r_k^{(T_i)}(x_i) & \text{if } \kappa = (k-1, ..., k-1), \\ 0 & \text{otherwise.} \end{cases}$$

P r o o f. The proof is immediate from the definitions of $\nu_{u,\kappa}$ and $r_k^{(T)}$. □

Note that we have the variance

$$\frac{\sigma^2(f)}{N} = \frac{1}{N} \sum_{\emptyset \neq u \subseteq \{1,\dots,d\}} \sum_{\kappa} \sigma^2_{u,\kappa}$$

for simple Monte Carlo methods with N samples. Now, we will prove the following theorem:

Theorem 3. *Let s be an integer between two and d. Then, for any function f in $\mathfrak{F}_{s,m}$, we have*

$$\mathrm{V}(f;2^m;X_{n+\ell}) = \mathrm{E}\left[\left(I(f) - \frac{1}{2^m}\sum_{n=0}^{2^m-1} f(X_{n+\ell})\right)^2\right] = \frac{\sigma^2(f)}{2^m},$$

where $X_n, n = 0, 1, \dots,$ is a generalized Sobol' sequence with any direction numbers, and ℓ is any multiple of 2^m.

P r o o f. Since the first two coordinates of any generalized Sobol' sequence constitute a $(0,2)$-sequence, there exists at most one point of $X_{n+\ell}, n = 0, \dots, 2^m - 1$, in each subinterval defined in (2) with $k_i = k$ for all $i \in u$, if $k \geq \lfloor (m-1)/2 \rfloor$. Therefore, according to Lemma 2, we have $\Gamma_{u,\bar{\kappa}} = 1$ for all u satisfying $\{1,2\} \subseteq u$, if $k \geq \lfloor (m-1)/2 \rfloor$, where $\bar{\kappa} = (k+1, \dots, k+1)$. From Lemma 3, we have the step function

$$\nu_{u,\bar{\kappa}}(x_1, \dots, x_d) = c_{u,k+2} \prod_{i \in u} r_{k+2}^{(T_i)}(x_i),$$

thereby

$$\mathrm{V}(f;2^m;X_{n+\ell}) = \frac{1}{2^m} \sum_{k=\lfloor (m-1)/2 \rfloor}^{m-2} \sum_{\substack{\{1,2\} \subseteq u \subseteq \{1,\dots,d\} \\ |u| = s}} c^2_{u,k+2} \Gamma_{u,\bar{\kappa}} = \frac{\sigma^2(f)}{2^m}.$$

Thus, the proof is complete. □

2.4. Owen's scrambling of high-discrepancy points

First, we need to point out that for some class of functions "low-discrepancy" is not a necessary condition for the speed-up of their numerical integration. For more details on this point and the definition of high-discrepancy sequences, see Tezuka [23]. Here, we consider a d-dimensional high-discrepancy sequence consisting of d copies of the van der Corput sequence in base two, that is to say, all d coordinates of the sequence are identical to the van der Corput sequence in base two.

We can prove the following theorem:

Theorem 4. *Let s be an odd integer between three and d inclusive. Then, for any function f in $\mathfrak{F}_{s,m}$, we have*

$$\mathrm{V}(f;2^m;X_{n+\ell}) = \mathrm{E}\left[\left(I(f)-\frac{1}{2^m}\sum_{n=0}^{2^m-1} f(X_{n+\ell})\right)^2\right]=0,$$

where $X_n, n=0,1,...,$ is a sequence consisting of d copies of the van der Corput sequence in base two, and ℓ is any multiple of 2^m.

P r o o f. We denote the van der Corput sequence in base two by $V_n, n=0,1,...$ Since $|u|=s$ is odd and $k_i=k$ for all $i\in u$, we have, for any pair (n,n') with $0\le n,n'\le N-1$,

$$\prod_{i\in u}(b1_{\lfloor b^{k+1}X_n^{(i)}\rfloor=\lfloor b^{k+1}X_{n'}^{(i)}\rfloor}-1_{\lfloor b^{k}X_n^{(i)}\rfloor=\lfloor b^{k}X_{n'}^{(i)}\rfloor})=$$
$$=b1_{\lfloor b^{k+1}V_n\rfloor=\lfloor b^{k+1}V_{n'}\rfloor}-1_{\lfloor b^{k}V_n\rfloor=\lfloor b^{k}V_{n'}\rfloor}$$

if $b=2$. For the van der Corput sequence in base 2, we have

$$\sum_{n=0}^{2^m-1}\sum_{n'=0}^{2^m-1}(b1_{\lfloor b^{k+1}V_n\rfloor=\lfloor b^{k+1}V_{n'}\rfloor}-1_{\lfloor b^{k}V_n\rfloor=\lfloor b^{k}V_{n'}\rfloor})=0,$$

where $k<m$. Thus, in the case of d copies of the van der Corput sequence, we have

$$\Gamma_{u,\kappa}=0,$$

for $\kappa=(k,...,k)$ with $k<m$, if $|u|=s$ is odd. From Lemma 3, we have the step function

$$\nu_{u,\kappa}(x_1,...,x_d)=c_{u,k+1}\prod_{i\in u}r_{k+1}^{(T_i)}(x_i),$$

where $\kappa=(k,...,k)$, thereby

$$\mathrm{V}(f;2^m;X_{n+\ell})=\frac{1}{2^m}\sum_{k=\lfloor(m-1)/2\rfloor+1}^{m-1}\sum_{\substack{\{1,2\}\subseteq u\subseteq\{1,\dots,d\}\\|u|=s}}c_{u,k+1}^2\Gamma_{u,\kappa}=0.$$

Thus, the proof is complete. $\square$

3. Discussion

Recently, it has been shown [1, 22, 24] that the notion of effective dimension is not so important for studying the efficiency of QMC, although it is essential for RQMC. In this paper, we made a further investigation on the difference between the integration errors of QMC and of RQMC. Our focus was on the product structure of the gain coefficient. As shown in the proof of Lemma 2, if all the points are separated from each other by the elementary interval, the gain coefficient becomes equal to one no matter how uniformly distributed are the points. This result is due to the product structure. As for the scrambling of high-discrepancy points, the product structure also plays a key role in the proof of Theorem 4 to make the variance become zero. Therefore, the gain coefficient is said to be not necessarily a measure of high dimensional uniformity of points. This is because it is based on the product of one-dimensional quantities and thereby one-dimensional uniformity is enough to make it small in some cases.

We stress that the function class $\mathfrak{F}_{s,m}$ introduced in this paper is artificial, but very simple. For example, in two dimensions, it contains chess-board functions and their linear combinations. Even for such simple functions, the integration errors of QMC and RQMC are totally different. Maybe, there exist other classes which have similar conclusions. It would be interesting to know if our results are true for some integrands encountered in finance.

Acknowledgements. The author is grateful to an anonymous referee for his careful reading and valuable comments. Thanks also go to Harald Niederreiter, Art Owen, Anargyros Papageorgiou, and Henryk Woźniakowski for their comments.

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Received March 21, 2007
Revised June 21, 2007

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