Uniform Distribution Theory **2** (2007), no.1, 107–113

uniform distribution theory

ON THE CONVERGENCE OF A SERIES OF BUNDSCHUH

Johannes Schoissengeier

ABSTRACT. If $(c_n)_{n\geq 1}$ is a non-increasing sequence of positive real numbers tending to 0 in P. Bundschuh [Arch. Math. **29** (1977), 518–523] the question was asked for which real numbers α the series $\sum_{n=1}^{\infty}(-1)^{[2n\alpha]}c_n$ is convergent. For series of the form $g(\alpha)=\sum_{n=1}^{\infty}\frac{(-1)^{[2n\alpha]}}{n}$ it has been shown in Bundschuh [ibid] that it converges for numbers α with bounded continued fraction expansion and also for numbers like e, thereby solving a problem posed in H.D. Ruderman [Amer. Math. Monthly **83** (1977), 573]. Whether the series is convergent for $\alpha=\pi$ has remained open. We could also deal with the more general series $\sum_{n=1}^{\infty}(-1)^{[2n\alpha]}c_n$, but then the technique would obscure the ideas of the proof.

The author and S. Tričković [J. Math. Anal. Appl. **342** (2006), 238–247] have proved that $\sum_{n=1}^{\infty}\frac{(-1)^{[2n\alpha]}}{n}$ is convergent almost everywhere, that the function $g(\alpha)$ is odd, has period 1 and represents a function in $\mathrm{L}_2[0,1]$. Furthermore every open non empty interval contains two subsets P and N of the power of the continuum such that for $\alpha\in P$ we have $g(\alpha)=+\infty$ and for $\alpha\in N$ we have $g(\alpha)=-\infty$. If $\alpha=\frac{p}{q}$ is rational and p and q are coprime then the series is convergent if and only if q is even.

In this paper we determine all real numbers α for which the series $g(\alpha)$ is convergent.

Communicated by Werner Georg Nowak

Dedicated to Professor Robert F. Tichy on the occasion of his 50th birthday

Statement and proof of the result

In the sequel we may assume that α is an irrational number and that $0<\alpha<\frac{1}{2}$. Let $\alpha=[0;a_1,a_2,...]$ be the continued fraction expansion of α with convergents

2000 Mathematics Subject Classification: 11J70, 11J71, 11K38, 11K60.
Keywords: Continued fractions, Ostrowski-expansions, series.

$\frac{p_n}{q_n} = [0; a_1, ..., a_n]$, where $p_{-2} = 0,\ p_{-1} = 1,\ q_{-2} = 1,\ q_{-1} = 0$. We define $s_{i,j} = q_{\min\{i,j\}}(q_{\max\{i,j\}}\alpha - p_{\max\{i,j\}})$. By $\{t\}$ we denote the fractional part of the real number t.

If $N \geq 1$ is a positive integer there are uniquely determined non-negative integers $b_0, b_1, ...$ such that $b_0 < a_1, b_i \leq a_{i+1}, b_i = a_{i+1} \implies b_{i-1} = 0$, and such that $N = \sum\limits_{i=0}^{\infty} b_i q_i$. This representation is called the *Ostrowski-expansion of* N *to base* α. There is clearly a largest $m \geq 0$ such that $b_m \neq 0$.

For $0 < x < 1$ there are uniquely determined non-negative integers $c_0, c_1, ...$ and $\delta \in \{0, 1\}$ such that $c_0 < a_1, c_i \leq a_{i+1}, c_i = a_{i+1} \implies c_{i-1} = 0$, and $x = \sum\limits_{i=0}^{\infty} c_i(q_i\alpha - p_i) + \delta$. We call this representation of x again the *Ostrowski–expansion of* x *to base* α. By i_x we denote the smallest i such that $c_i \neq 0$. It can easily be shown that $\delta = 1 \iff i_x$ is odd $\iff x > 1 - \alpha$.

By $c_{[0,x)}$ we denote the characteristic function of the interval $[0, x)$. The following Theorem can be found in [3] or [6, p.361].

Theorem A. *Let* N *be a positive integer with Ostrowski-expansion* $\sum\limits_{n=1}^{m} b_i q_i$ *(with* $b_m \neq 0$*), and let* x *be in* $(0, 1)$ *with Ostrowski-expansion*

$$x = \sum_{i=0}^{\infty} c_i(q_i\alpha - p_i) + \delta,$$

both to base α*. Assume that* $x \notin \big\{\{\alpha\}, \{2\alpha\}, ..., \{N\alpha\}\big\}$*. For* $k \geq -1$ *let*

$$\delta_k = \begin{cases} 1 & \sum\limits_{i=0}^{k} c_i q_i > \sum\limits_{i=0}^{k} b_i q_i \\ 0 & else \end{cases}$$

and let t *be the first integer* $\geq m$ *with* $\delta_t = 1$*.*
Then

$$\sum_{n=1}^{N} c_{[0,x)}\big(\{n\alpha\}\big) - Nx = \sum_{i=0}^{\infty}(-1)^i \min\{c_i, b_i\} - \sum_{i=0}^{m}(-1)^i \delta_i \delta_{i-1}$$
$$- \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} b_i c_j s_{i,j} - \frac{1}{2}\big((-1)^{i_x} - (-1)^t\big).$$

Lemma 1. *For* $i \geq 0$ *let* $c_i = \frac{1}{4}\big(a_{i+1} + \frac{1}{2}(-1)^{q_{i-1}} - \frac{1}{2}(-1)^{q_{i+1}}\big)\big(1 - (-1)^{q_i}\big)$*. Then for* $j \geq 0$ *we have* $\sum\limits_{i=0}^{\infty} c_i s_{i,j} = \frac{1}{4}(-1)^j\big(1 - (-1)^{q_j}\big)$*.*

Proof. We note that $\big((-1)^{q_{i+1}} - (-1)^{q_{i-1}}\big)\big(1 + (-1)^{q_i}\big) = 0$. Therefore as $a_{i+1}s_{i,j} = s_{i+1,j} - s_{i-1,j} + (-1)^j\delta_{i,j}$ we have

$$\begin{aligned}
\sum_{i=0}^{\infty} c_i s_{i,j} =& \frac{1}{4}\sum_{i=0}^{\infty}\big(s_{i+1,j} - s_{i-1,j} + (-1)^j\delta_{i,j}\big)\big(1-(-1)^{q_i}\big)\\
&+ \frac{1}{8}\sum_{i=0}^{\infty} s_{i,j}\big((-1)^{q_{i-1}} - (-1)^{q_{i+1}}\big)\big(1-(-1)^{q_i}\big)\\
=&\frac{1}{4}\sum_{i=1}^{\infty} s_{i,j}\big(1-(-1)^{q_{i-1}}\big) - \frac{1}{4}\sum_{i=-1}^{\infty} s_{i,j}\big(1-(-1)^{q_{i+1}}\big)\\
&+ \frac{1}{4}(-1)^j\big(1-(-1)^{q_j}\big) + \frac{1}{8}\sum_{i=0}^{\infty} s_{i,j}\big((-1)^{q_{i-1}} - (-1)^{q_{i+1}}\big)\big(1-(-1)^{q_i}\big)\\
=&\frac{1}{4}\sum_{i=0}^{\infty} s_{i,j}\big(1-(-1)^{q_{i-1}}\big) - \frac{1}{4}\sum_{i=0}^{\infty} s_{i,j}\big(1-(-1)^{q_{i+1}}\big)\\
&+ \frac{1}{4}(-1)^j\big(1-(-1)^{q_j}\big) + \frac{1}{8}\sum_{i=0}^{\infty} s_{i,j}\big((-1)^{q_{i+1}} - (-1)^{q_{i-1}}\big)\big((-1)^{q_i}-1\big)\\
=&\frac{1}{4}(-1)^j\big(1-(-1)^{q_j}\big) + \frac{1}{8}\sum_{i=0}^{\infty} s_{i,j}\big((-1)^{q_{i+1}} - (-1)^{q_{i-1}}\big)\big(1+(-1)^{q_i}\big)\\
=&\frac{1}{4}(-1)^j\big(1-(-1)^{q_j}\big).
\end{aligned}$$

□

By a similar computation one can show that

$$\sum_{i=0}^{m} c_i q_i = \frac{1}{4}q_{m+1}\big(1-(-1)^{q_m}\big) + \frac{1}{4}q_m\big(1-(-1)^{q_{m+1}}\big).$$

Lemma 2. *For $i \ge 0$ let $c_i := \frac{1}{4}\big(a_{i+1} + \frac{1}{2}(-1)^{q_{i-1}} - \frac{1}{2}(-1)^{q_{i+1}}\big)\big(1-(-1)^{q_i}\big)$. Then $\sum\limits_{i=0}^{\infty} c_i(q_i\alpha - p_i)$ is the Ostrowski-expansion of $\frac{1}{2}$ to base α.*

Proof. First we prove that c_i is a non-negative integer. This is clear if $2|q_i$. Otherwise

$$c_i = \frac{1}{2}\big(a_{i+1} + \frac{1}{2}(-1)^{q_{i-1}} - \frac{1}{2}(-1)^{q_{i+1}}\big).$$

If a_{i+1} is even, then $q_{i-1} \equiv q_{i+1} \pmod 2$ and we are done again. Otherwise $q_{i-1} \not\equiv q_{i+1} \pmod 2$, and therefore again $0 \le c_i$, and $c_i \in \mathbb{Z}$.

As $\alpha < \frac{1}{2}$ we have $a_1 > 1$. Note that therefore $c_0 = \frac{1}{2}\big(a_1 + \frac{1}{2} - \frac{1}{2}(-1)^{a_1}\big) < a_1$. If $c_i = a_{i+1}$, then $2 \nmid q_i$ and $a_{i+1} = \frac{1}{2}\big(a_{i+1} + \frac{1}{2}(-1)^{q_{i-1}} - \frac{1}{2}(-1)^{q_{i+1}}\big)$, which implies $a_{i+1} = 1$ and that q_{i-1} is even. But then $c_{i-1} = 0$.

The representation $\sum\limits_{i=0}^{\infty} c_i(q_i\alpha - p_i) = \frac{1}{2}$ is a special case of Lemma 1 (put $j = 0$). □

Corollary 1. *Let N be a positive integer with Ostrowski-expansion $\sum\limits_{n=1}^{m} b_i q_i$ (with $b_m \neq 0$) to base α. For $k \geq 0$ let*

$$c_k = \frac{1}{4}\big(a_{k+1} + \frac{1}{2}(-1)^{q_{k-1}} - \frac{1}{2}(-1)^{q_{k+1}}\big)\big(1 - (-1)^{q_k}\big),$$

for $k \geq -1$ let

$$\delta_k = \begin{cases} 1 & \textit{if } \sum\limits_{i=0}^{k} c_i q_i > \sum\limits_{i=0}^{k} b_i q_i, \\ 0 & \textit{else}, \end{cases}$$

and let t be the first integer $\geq m$ with $\delta_t = 1$. Then for $\alpha < 1/2$

$$\sum_{n=1}^{N} c_{[0,\frac{1}{2})}\big(\{n\alpha\}\big) - N/2 = \sum_{i=0}^{m}(-1)^i\bigg(\min\{c_i, b_i\} - \frac{1}{4}b_i\big(1 - (-1)^{q_i}\big)\bigg)$$
$$- \sum_{i=0}^{m}(-1)^i\delta_i\delta_{i-1} - \frac{1}{2}\big(1 - (-1)^t\big) = \frac{1}{2}\sum_{\substack{i=0 \\ 2\nmid q_i}}^{m}(-1)^i \min\{b_i, a_{i+1} - b_i\} + O(m).$$

Proof. This follows from Theorem A, Lemma 1 and Lemma 2 by noting that $i_{1/2} = 0$. □

Theorem 1. *Let α be irrational with continued fraction expansion $\alpha = [0; a_1, a_2, ...]$ and convergents $\frac{p_n}{q_n}$. The following assertions are equivalent:*

(1) $\sum\limits_{n=0}^{\infty} \frac{(-1)^{[2n\alpha]}}{n}$ *is convergent (resp. $= \infty$, resp. $= -\infty$).*

(2) $\sum\limits_{\substack{t=0 \\ 2\nmid q_t}}^{\infty} \frac{(-1)^t \log a_{t+1}}{q_t}$ *is convergent (resp. $= \infty$, resp. $= -\infty$).*

P r o o f. First we restrict ourselves to the case $\alpha < 1/2$. Note that $[2n\alpha]$ is even if and only if $\{n\alpha\} < 1/2$. Hence

$$\begin{aligned}\sum_{i=1}^{n}(-1)^{[2i\alpha]} &= \sum_{i=1}^{n}\Big(c_{[0,1/2)}\big(\{i\alpha\}\big) - c_{[1/2,1)}\big(\{i\alpha\}\big)\Big)\\ &= 2\Big(\sum_{i=1}^{n} c_{[0,1/2)}\big(\{i\alpha\}\big) - n/2\Big)\\ &= \sum_{\substack{j=0\\ 2\nmid q_j}}^{\infty}(-1)^j \min\big\{b_j(n),\ a_{j+1} - b_j(n)\big\} + O(\log n),\end{aligned}$$

by Corollary 1, as $q_m \le n$ implies $m = O(\log n)$; here and in the sequel $b_i(n)$ denotes the i-th digit of n in the Ostrowski-expansion of n to base α.

Therefore by summation by parts

$$\sum_{n=1}^{N}\frac{(-1)^{[2n\alpha]}}{n} = \sum_{n=1}^{N-1}\frac{1}{n(n+1)}\Big(\sum_{\substack{i=0\\ 2\nmid q_i}}^{\infty}(-1)^i \min\big\{b_i(n),\ a_{i+1}-b_i(n)\big\}+O(\log n)\Big) + \frac{2}{N}\Big(\sum_{n=1}^{N} c_{[0,1/2)}\big(\{n\alpha\}\big) - N/2\Big).$$

The last term tends to 0 as $(n\alpha)_{n\ge 1}$ is uniformly distributed mod 1.

In order to simplify notation and to prove all three statements simultaneously we write for two sequences $(u_n)_{n\ge1}$, and $(v_n)_{n\ge1}$ $u_n \sim v_n$ if $u_n - v_n$ is convergent. Let m be chosen such that $q_m \le N < q_{m+1}$.

Then

$$\sum_{n=1}^{N}\frac{(-1)^{[2n\alpha]}}{n} \sim \sum_{n=1}^{N}\frac{1}{n(n+1)}\sum_{\substack{i=0\\ 2\nmid q_i}}^{\infty}(-1)^i \min\big\{b_i(n)\,, a_{i+1} - b_i(n)\big\}.$$

The inner sum terminates. If t_n is the largest index t with $b_t(n) \ne 0$ (that is $q_t \le n < q_{t+1}$) we have

$$\sum_{\substack{i<t_n\\ 2\nmid q_i}}(-1)^i \min\{b_i(n), a_{i+1} - b_i(n)\} = O\Big(\sum_{i<t_n} a_{i+1}\Big)$$

and hence

$$\sum_{n=1}^{N}\frac{1}{n(n+1)}\sum_{\substack{i<t_n\\2\nmid q_i}}(-1)^i\min\left\{b_i(n),\ a_{i+1}-b_i(n)\right\}=O\left(\sum_{t=0}^{m}\sum_{n=q_t}^{q_{t+1}-1}\frac{1}{n^2}\sum_{i<t}a_{i+1}\right)$$

$$=O\left(\sum_{t=0}^{m}\frac{1}{q_t}\sum_{i<t}a_{i+1}\right)=O\left(\sum_{i=0}^{\infty}a_{i+1}\sum_{t>i}\frac{1}{q_t}\right)=O\left(\sum_{i=0}^{\infty}\frac{a_{i+1}}{q_{i+1}}\right)=O(1).$$

This implies

$$\sum_{n=1}^{N}\frac{(-1)^{[2n\alpha]}}{n}\sim\sum_{\substack{n=1\\2\nmid q_{t_n}}}^{N}\frac{1}{n(n+1)}(-1)^{t_n}\left(\min\left\{b_{t_n}(n),a_{t_n+1}-b_{t_n}(n)\right\}\right)$$

$$=\sum_{\substack{t=0\\2\nmid q_t}}^{m}(-1)^t\sum_{n=q_t}^{q_{t+1}-1}\frac{1}{n(n+1)}\min\left\{b_t(n),a_{t+1}-b_t(n)\right\}$$

$$+\frac{1}{2}\big(1-(-1)^{q_m}\big)\sum_{n=q_m}^{N}(-1)^m\frac{1}{n(n+1)}\min\left\{b_m(n),\ a_{m+1}-b_m(n)\right\}.$$

For $q_t\le n<q_{t+1}$ we have $b_t(n)=[\frac{n}{q_t}]$. Therefore the first inner sum is equal to

$$\sum_{q_t\le n\le a_{t+1}q_t/2}\frac{1}{n(n+1)}\frac{n}{q_t}+\sum_{a_{t+1}q_t/2<n\le a_{t+1}q_t}\frac{1}{n(n+1)}\left(a_{t+1}-\frac{n}{q_t}\right)+O\left(\frac{1}{q_t}\right)$$

$$=\frac{1}{q_t}\log\frac{a_{t+1}}{2}+O\left(a_{t+1}\sum_{n>a_{t+1}q_t/2}\frac{1}{n^2}\right)+O\left(\frac{1}{q_t}\right)$$

$$=\frac{1}{q_t}\log a_{t+1}+O\left(\frac{1}{q_t}\right).$$

In the same way one sees that the last sum $=(-1)^m\frac{1}{q_m}\log\frac{N}{q_m}+O(\frac{1}{q_m})$. Collecting everything we get the result in this case.

If $\alpha>1/2$, let $1-\alpha=[0;a_1',a_2',...]$, and let $\frac{p_n'}{q_n'}$ be the convergents of $1-\alpha$. Then $a_1'=a_2+1,\ a_i'=a_{i+1}$ for $i>1,\ q_i'=q_{i+1}$. From $g(\alpha)=-g(1-\alpha)$ we get the result in this case. □

It is now clear that the series is convergent for $\alpha=\pi$ as we know that then $a_{t+1}=O(q_t^c)$ for some absolute constant $c>0$ (see [2]).

REFERENCES

[1] BUNDSCHUH, P.: *Konvergenz unendlicher Reihen und Gleichverteilung mod 1*, Arch. Math. **29** (1977), 518–523.
[2] MAHLER, K.: *On the approximation of* π, Nederl. Akad. Wet. Proc., Ser. A **56** (1953), 30–42.
[3] ROÇADAS, L.: *New aspects on the distribution of* $(n\alpha)$*-sequences and Ostrowski expansions*, Thesis, Vila Real, 2006.
[4] RUDERMAN, H.D.: *Advanced problem 6105*, Amer. Math. Monthly **83** (1977), 573.
[5] SCHOISSENGEIER, J.–TRIČKOVIĆ, S.: *On the divergence of a certain series*, J. Math. Anal. Appl. **342** (2006), 238–247.
[6] SÓS, V.T.: *On the Discrepancy of the Sequence* $\{n\alpha\}$, in: *Topics in Number Theory* (Proc. Colloq., Debrecen, 1974), Colloq. Math. János Bolyai **13**, North-Holland, Amsterdam, 1976, pp. 359–367

Received May 15, 2007
Accepted July 19, 2007

Johannes Schoißengeier
Fakultät für Mathematik
Universität Wien
Nordbergstraße 15
A-1090 Wien
ÖSTERREICH
E-mail: johannes.schoissengeier@univie.ac.at