

REMARKS ON ONE TYPE OF UNIFORM DISTRIBUTION

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ABSTRACT. A sequence $\{x_n\}$ of numbers in $[0, 1]$ is called Buck's uniformly distributed if for every interval $I \subset [0, 1]$ the set $\{n \in \mathbb{N}, x_n \in I\}$ is Buck's measurable and its Buck's measure density is equal to the length of I . Some examples of sequences are given. An analogy of Weyl's criterion is proved.

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Introduction

The notion of uniformly distributed sequence was introduced in 1916 by H. Weyl in his paper [Wey]. Let $\{x_n\}$ be a sequence of real numbers from $[0, 1]$. This sequence is called uniformly distributed if and only if for every subinterval $I \subseteq [0, 1]$ it holds

$$\lim_{N \rightarrow \infty} \frac{1}{N} \text{card}\{n \leq N; x_n \in I\} = |I|, \quad (1)$$

where $|I|$ is the length of I . If \mathbb{N} is the set of positive integers and $A \subseteq \mathbb{N}$ then the value (if it exists):

$$d(A) := \lim_{N \rightarrow \infty} \frac{1}{N} \text{card}(A \cap [0, N])$$

is called the asymptotic density of the set A . Thus a sequence $\{x_n\}$, $x_n \in [0, 1]$, is uniformly distributed if and only if for every subinterval $I \subseteq [0, 1]$ the set $\{n \in \mathbb{N}; x_n \in I\}$ has the asymptotic density and

$$d(\{n \in \mathbb{N}; x_n \in I\}) = |I|. \quad (2)$$

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The aim of this paper is to study another type of uniform distribution motivated by (2). In the monograph [S-P, p. 6] there is considered the uniform distribution with respect to a given submeasure on a system of some subsets of the set of positive integers.

In 1946 R. C. Buck introduced and studied the notion of measure density, motivated from one side by asymptotic density and by measurability in sense of Carathéodory from other side. Let $m \in \mathbb{N}$, $a \in \mathbb{N} \cup \{0\}$. Denote by $m\mathbb{N} + a = \{a + k.m; k = 0, \dots\}$, the arithmetic progression with the difference m and prime element a . Clearly $m\mathbb{N} + a$ has the asymptotic density and $d(m\mathbb{N} + a) = \frac{1}{m}$. Let D_0 be the class of all sets of the form $m_1\mathbb{N} + a_1 \cup \dots \cup m_s\mathbb{N} + a_s$. Put for $S \subset \mathbb{N}$: $\mu^*(S) = \inf\{d(A); S \subset A, A \in D_0\}$. Then the value $\mu^*(S)$ is called the measure density of S . Put

$$D_\mu = \{S \subseteq \mathbb{N}; \mu^*(S) + \mu^*(\mathbb{N} \setminus S) = 1\}.$$

Then D_μ is an algebra of sets and the function $\mu = \mu^*/D_\mu$ is a finitely-additive probability measure on D_μ . For all above statements on measure density we refer to the Buck's paper [BUC].

The algebra D_μ has the following properties:

- (i) $D_0 \subset D_\mu$.
- (ii) $E \in D_\mu \iff \forall \varepsilon > 0 \exists E_1, E_2 \in D_\mu$ such that $E_1 \subset E \subset E_2$ and $\mu(E_2) - \mu(E_1) < \varepsilon$.
- (iii) Every set $E \in D_\mu$ has the asymptotic density and $d(E) = \mu(E)$.

For (i), (ii), (iii) we refer to [BUC] and [PAS]. Remark that in 1973 M. Parnes proved that the algebra D_μ is very small from the measure theoretical point of view, (see [PA]). Let $\{y_n\}$ be a sequence of numbers from the interval $[0, 1)$. Put for a set $L \subset [0, 1)$

$$A(\{y_n\}, L) = \{n \in \mathbb{N}; y_n \in L\}. \quad (3)$$

(Shortly we can write $A(L) = A(\{y_n\}, L)$, if $\{y_n\}$ is given.)

The sequence $\{y_n\}$ is called *Buck's uniformly distributed* if and only if for every subinterval $I \subseteq [0, 1)$ the set $A(\{y_n\}, I)$ belongs to D_μ and

$$\mu(A(\{y_n\}, I)) = |I|, \quad (4)$$

where $|I|$ is the length of I . Instead of Buck's uniformly distributed we shall write only B.u.d.

Remark that (iii) yields that every B.u.d. sequence is uniformly distributed in the sense (2).

1. Some elementary statements

PROPOSITION 1. *Let $\{y_n\}$ be a sequence of numbers from $[0, 1)$. Then $\{y_n\}$ is B.u.d. if and only if*

- a) $\forall x \in (0, 1); A([0, x]) \in D_\mu \wedge \mu(A([0, x])) = x.$
- b) $\forall x \in (0, 1); A([0, x]) \in D_\mu \wedge \mu(A([0, x])) = x.$

Proof. Suppose that a) holds. For $x \in [0, 1)$ and $\varepsilon > 0$ we have $A([0, x]) \subset A([0, x]) \subset A([0, x + \varepsilon])$. Thus (ii) yields $A([0, x]) \in D_\mu$ and $x \leq \mu(A[0, x]) \leq x + \varepsilon$ and so for $\varepsilon \rightarrow 0^+$ we have $\mu(A[0, x]) = x$. Now for an arbitrary interval I (suppose for instance that $I = [a, b]$) we have

$$A(I) = A([0, b]) \setminus A([0, a]) \in D_\mu$$

and $\mu(A(I)) = b - a$. Analogously we can prove b). If we consider that for $x \in [0, 1)$ we have $A(\{x\}) = A([0, x]) - A([0, x])$ we obtain it immediately. \square

COROLLARY 1. *If y_n is B.u.d. then, $\forall x \in [0, 1)$, we have $\mu(A(\{x\})) = 0$ (and $A(\{x\}) \in D_\mu$).*

PROPOSITION 2. *Let $\{y_n\}$ be a sequence of numbers from $[0, 1)$. Then $\{y_n\}$ is B.u.d. if and only if $\mu^*(A(I)) = |I|$ for every subinterval $I \subset [0, 1)$.*

Proof. One implication is trivial. Implication \Leftarrow . Consider $I = [0, x)$, $J = [x, 1)$. Then $\mu^*(A([0, x])) = x$, $\mu^*(A([x, 1))) = 1 - x$ and so $A([0, x]) \in D_\mu$ and the assertion follows now from Proposition 1 a). \square

PROPOSITION 3. *Let $E \subset [0, 1)$ be a dense set and $\{y_n\}$ a sequence of numbers from $[0, 1)$. The sequence $\{y_n\}$ is B.u.d. if and only if*

- a) $\forall x \in E; A([0, x]) \in D_\mu \wedge \mu(A([0, x])) = x.$
- b) $\forall x \in E; A([0, x]) \in D_\mu \wedge \mu(A([0, x])) = x.$
- c) *For every interval $I \subset [0, 1)$, whose endpoints belong to E , we have $A(I) \in D_\mu$ and $\mu(A(I)) = |I|$.*

Proof. Let x_0 be an arbitrary number from $[0, 1)$. Then there exist $x_1, x_2 \in E$ such that $x_1 < x_0 < x_2$ and $x_2 - x_1 < \varepsilon$ (for $\varepsilon > 0$). And so $A([0, x_1]) \subseteq A([0, x_0]) \subseteq A([0, x_2])$ and $A([0, x_1]) \subseteq A([0, x_0]) \subseteq A([0, x_2])$. Thus from (ii) we have $A([0, x_0]) \in D_\mu$ and the assertion follows in the cases a) and b). c) is trivial. \square

2. The sequence of van der Corput

Let $q_0 = 1, q_1, q_2, \dots$ be a sequence of positive integers such that $q_j > 1$ for $j > 1$. Put

$$Q_n = q_0 \dots q_n, n = 0, 1, 2, \dots \quad (5)$$

It is well known that every positive integer a can be uniquely expressed in the form

$$a = c_0 Q_0 + c_1 Q_1 + \dots + c_n Q_n, \quad (6)$$

where $0 \leq c_j < q_{j+1}$, $j = 0, \dots, n-1$. Similarly every $\alpha \in [0, 1)$ can be uniquely expressed in the form

$$\alpha = \sum_{n=0}^{\infty} \frac{c_n}{Q_{n+1}}, \quad 0 \leq c_n < q_{n+1} \quad (7)$$

and $c_n < q_{n+1} - 1$ for infinitely many n . If $a \in \mathbb{N}$ has the form (6), put

$$\gamma(a) = \frac{c_0}{Q_1} + \dots + \frac{c_n}{Q_{n+1}}. \quad (8)$$

If $q_1 = q_2 = \dots = b$, b for some $b \in \mathbb{N}$, $b > 1$, we obtain the classical van der Corput sequence with the base b (see [D-T, pp. 41, 368]; [S-P, p. 2-102]). Remark that this sequence plays an important role in the theory of discrepancy.

THEOREM 1. *If $\{\gamma(k)\}$ is given by (8), then $\{\gamma(k)\}$ is B.u.d. sequence.*

PROOF. For the proof we shall use some elementary properties of Cantor's expansions. Consider the intervals

$$I_{k+1}^j = \left[\frac{j}{Q_{k+1}}, \frac{j+1}{Q_{k+1}} \right), \quad j = 0, \dots, Q_{k+1} - 1, k = 0, \dots$$

Clearly the endpoints of these intervals form a dense set in $[0, 1)$. Thus Proposition 3 c) guaranties that it suffices to prove

$$\mu^*(A(I_{k+1}^j)) = \frac{1}{Q_{k+1}}. \quad (9)$$

To interval I_{k+1}^j we can uniquely associate a finite sequence $\bar{c}_0, \dots, \bar{c}_k$, $0 \leq \bar{c}_s < q_{s+1}$, $s = 0, \dots, k$ such that $\alpha \in I_{k+1}^j$ if and only if $\alpha = \sum_{n=0}^{\infty} \frac{c_n}{Q_{n+1}}$, where $c_0 = \bar{c}_0, \dots, c_k = \bar{c}_k$. Similarly there exists an arithmetic progression $r + Q_{k+1}\mathbb{N}$ such that $a \in r + Q_{k+1}\mathbb{N}$ if and only if $a = c_0 Q_0 + \dots + c_k Q_k + \dots$, where $c_0 = \bar{c}_0, \dots, c_k = \bar{c}_k$. Thus it follows

$$\gamma(a) \in I_{k+1}^j \iff a \in r + Q_{k+1}\mathbb{N}$$

and so $A(I_{k+1}^j) = r + Q_{k+1}\mathbb{N}$ which implies (9). \square

Let α be an irrational number. It is a well known statement that the sequence of fractional parts $\{\{n\alpha\}\}$ is uniformly distributed. We prove that this sequence is not B.u.d.

THEOREM 2. *Let α be an irrational number. Put $y_n = \{n\alpha\}$, $n = 1, 2, \dots$. Then for every $x \in (0, 1)$ we have*

$$\mu^*(A([0, x])) = 1 = \mu^*(A([x, 1])).$$

PROOF. In 1984 O. Strauch [S] proved that for every proper subinterval I the set $A(I)$ does not contain an infinite arithmetic progression. Therefore the complement $\mathbb{N} \setminus A(I)$ has a nonempty intersection with every arithmetic progression. Thus $\mu^*(\mathbb{N} \setminus A(I)) = 1$ (see [PAS]). For $I = [0, x]$, $I = [x, 1]$ we obtain the assertion. \square

3. Buck's measurable sequences

Suppose that $\{x_n\}$ is a sequence of numbers from $[0, 1)$. This sequence is called *Buck's measurable* if and only if for every subinterval $I \subseteq [0, 1)$ the set $A(\{x_n\}, I)$ belongs to the algebra D_μ .

Remark that similar generalization of Weyl's u.d. sequence was made by I. Schoenberg [Sch] in 1929. Thus $\{x_n\}$ is B.u.d. if it is Buck's measurable and $\mu(A(\{x_n\}, I)) = |I|$.

In this part we derive some analogies of the well known Weyl's criterium. We shall use one statement from [PAS]. Let $S \subset \mathbb{N}$, $B \in \mathbb{N}$. Denote

$$S \pmod{B} = \{s \pmod{B}; s \in S\}$$

and

$$R(S : B) = |S \pmod{B}|.$$

Then it holds.

THEOREM A. *Let $\{B_n\}$ be a sequence of positive integers fulfilling the condition*

$$\forall d \in \mathbb{N} \exists n_0 \forall n \geq n_0; d|B_n. \quad (c)$$

Then for every subset $S \subset \mathbb{N}$ we have

$$\mu^*(S) = \lim_{n \rightarrow \infty} \frac{R(S : B_n)}{B_n}$$

(see [PAS]).

PROPOSITION 4. *Let $\{x_n\}$ be a sequence of numbers from $[0, 1)$. Then $\{x_n\}$ is Buck's measurable if and only if for every interval $I \subseteq [0, 1)$ there exists a value $C(I)$ such that*

$$\lim_{n \rightarrow \infty} \frac{1}{B_n} \sum_{j=1}^{B_n} \chi_I(x_{k_j^{(n)}}) = C(I) \quad (10)$$

holds for arbitrary system of complete remainders systems $R_n = \{k_1^{(n)}, \dots, k_{B_n}^{(n)}\}$ modulo $B_n, n = 1, 2, \dots$. Here $\{B_n\}$ is a sequence from Theorem A and $\chi_I(\cdot)$ is the indicator function of I . In addition in this case there holds $C(I) = \mu(A(I))$.

PROOF. The value $R(A(I) : B_n)$ is the maximal number of elements from $A(I)$ incongruent modulo B_n . Thus

$$\frac{1}{B_n} \sum_{j=1}^{B_n} \chi_I(x_{k_j^{(n)}}) \leq \frac{R(A(I) : B_n)}{B_n}$$

and so

$$\limsup_{n \rightarrow \infty} \frac{1}{B_n} \sum_{j=1}^{B_n} \chi_I(x_{k_j^{(n)}}) \leq \mu^*(A(I)). \quad (11)$$

Suppose now that $A(I) \in D_\mu$ and

$$\liminf_{n \rightarrow \infty} \frac{1}{B_n} \sum_{j=1}^{B_n} \chi_I(x_{k_j^{(n)}}) < \mu^*(A(I)).$$

Then there exists $\alpha < \mu^*(A(I))$ such that

$$\frac{1}{B_{n_i}} \sum_{j=1}^{B_{n_i}} \chi_I(x_{k_j^{(n_i)}}) \rightarrow \alpha, \quad i \rightarrow \infty$$

for suitable subsequence $\{n_i\}$. But

$$1 = \frac{1}{B_{n_i}} \sum_{j=1}^{B_{n_i}} \chi_I(x_{k_j^{(n_i)}}) + \frac{1}{B_{n_i}} \sum_{j=1}^{B_{n_i}} \chi_{\neg I}(x_{k_j^{(n_i)}}),$$

where $\neg I$ denotes the complement of I . For $i \rightarrow \infty$ we obtain from (11)

$$1 \leq \alpha + \mu^*(A(\neg I)) < \mu^*(A(I)) + \mu^*(A(\neg I)) = 1$$

– a contradiction.

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Suppose now that (10) holds. Denote $q(n) = R(A(I) : B_n)$. Then $A(I)$ contains the numbers $k_1^{(n)}, \dots, k_{q(n)}^{(n)}$ incongruent modulo B_n . These numbers we can complete to a complete remainder system modulo B_n .

$$R_n = \{k_1^{(n)}, \dots, k_{q(n)}^{(n)}, k_{q(n)+1}^{(n)}, \dots, k_{B_n}^{(n)}\}.$$

Nearly (10) yields

$$\frac{q(n)}{B_n} = \frac{1}{B_n} \sum_{j=1}^{B_n} \chi_I(k_j^{(n)}) \rightarrow C(I)$$

and so $C(I) = \mu^*(A(I))$. If we consider that $1 = \chi_I + \chi_{-I}$, then we obtain

$$1 = C(I) + C(-I) = \mu^*(A(I)) + \mu^*(A(-I))$$

and the assertion follows. □

As an immediate consequence we have

COROLLARY 2. *The sequence $\{x_n\}$, $x_n \in [0, 1)$ is B.u.d. if and only if for every subinterval $I \subset [0, 1)$ there holds*

$$\lim_{n \rightarrow \infty} \frac{1}{B_n} \sum_{j=1}^{B_n} \chi_I(x_{k_j^{(n)}}) = |I|,$$

using the same notation as in Proposition 4.

Corollary 2 implies that $\{x_n\}$ is B.u.d. if and only if

$$\lim_{n \rightarrow \infty} \frac{1}{B_n} \sum_{j=1}^{B_n} g(x_{k_j^{(n)}}) = \int_0^1 g(x) dx$$

holds for every step function $g = \sum_i c_i \chi_{I_i}$.

For every Riemannian integrable function $f : [0, 1] \rightarrow (\infty, \infty)$ and $\varepsilon > 0$ there exist two steps functions g_1, g_2 that

$$g_1 < f < g_2, \quad \int_0^1 (g_2(x) - g_1(x)) dx < \varepsilon. \quad (12)$$

Thus we get

THEOREM 3. *Suppose the notation from Proposition 4. Then $\{x_n\}$ is B.u.d. if and only if*

$$\lim_{n \rightarrow \infty} \frac{1}{B_n} \sum_{j=1}^{B_n} f(x_{k_j^{(n)}}) = \int_0^1 f(x) dx \quad (13)$$

for arbitrary Riemannian integrable function $f : [0, 1] \rightarrow (-\infty, \infty)$.

Every Riemannian integrable function can be approximated in the sense (12) by two continuous functions g_1, g_2 and so we get

THEOREM 3'. *Suppose the notation from Proposition 4. Then $\{x_n\}$ is B.u.d. if and only if*

$$\lim_{n \rightarrow \infty} \frac{1}{B_n} \sum_{j=1}^{B_n} f(x_{k_j^{(n)}}) = \int_0^1 f(x) \, dx$$

for every continuous function $f : [0, 1] \rightarrow (-\infty, \infty)$.

Every continuous function can be uniformly approximated by trigonometric polynomials and so we have analogy to Weyl's criterion:

THEOREM 4. *Suppose the notation from Proposition 4. Then $\{x_n\}$ is B.u.d. if and only if*

$$\lim_{n \rightarrow \infty} \frac{1}{B_n} \sum_{j=0}^{B_n} \exp 2\pi i h x_{k_j^{(n)}} = 0$$

for every $h \in \mathbb{Z}$, $h \neq 0$.

This yields

COROLLARY 3. *If $\{x_n\}$ is B.u.d., then the sequence of fractional parts $\{\{kx_n\}\}$ is B.u.d. for $k \in \mathbb{Z}$, $k \neq 0$.*

A sequence $\{x_n\}$ is called *polyadically continuous* if and only if $\forall \varepsilon > 0$ there exists $B \in \mathbb{N}$ such that

$$n \equiv m \pmod{B} \implies |x_n - x_m| < \varepsilon. \quad (14)$$

THEOREM 5. *Every u.d. sequence which is polyadically continuous is B.u.d.*

Proof. Let $\{x_n\}$ be a u.d. sequence. Then for every continuous function f there holds

$$\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=1}^N f(x_n) = \int_0^1 f(x) \, dx.$$

Suppose that a sequence $\{B_n\}$ fulfils the condition (c) of Theorem A. Clearly

$$\int_0^1 f(x) \, dx = \lim_{n \rightarrow \infty} \frac{1}{B_n} \sum_{j=1}^{B_n} f(x_j).$$

If $R_n = \{k_1^{(n)}, \dots, k_{B_n}^{(n)}\}$ is a complete remainder system modulo B_n , we can suppose that $j \equiv k_j^{(n)} \pmod{B_n}$, $j = 1, \dots, B_n$. If $\{x_n\}$ is polyadically continuous,

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then $\{f(x_n)\}$ is also polyadically continuous, because f is uniformly continuous. Let $\varepsilon > 0$ and B be from (14). Then from (c) we have $B|B_n, n \geq n_0$ for suitable n_0 , hence $|f(x_j) - f(x_{k_j^{(n)}})| < \varepsilon$. And so

$$\left| \frac{1}{B_n} \sum_{j=1}^{B_n} f(x_j) - \frac{1}{B_n} \sum_{j=1}^{B_n} f(x_{k_j^{(n)}}) \right| < \varepsilon$$

for $n \geq n_0$. For $n \rightarrow \infty$ we have

$$\left| \int_0^1 f(x) \, dx - \limsup_{n \rightarrow \infty} \frac{1}{B_n} \sum_{j=1}^{B_n} f(x_{k_j^{(n)}}) \right| \leq \varepsilon$$

$$\left| \int_0^1 f(x) \, dx - \liminf_{n \rightarrow \infty} \frac{1}{B_n} \sum_{j=1}^{B_n} f(x_{k_j^{(n)}}) \right| \leq \varepsilon$$

and the assertion follows. \square

If we consider the sequence of van der Corput $\{\gamma(a)\}$ given by (8), then for $a_1 \equiv a_2 \pmod{Q_n}$ we have that a_1, a_2 have the same first n digits in (6) and so $|\gamma(a_1) - \gamma(a_2)| \leq Q_n^{-1}$. Therefore $\{\gamma(a)\}$ is an example of polyadically continuous sequence.

If $\{x_n\}$ is a Buck's measurable sequence, then the function

$$g(x) = \mu(A(\{x_n\}, [0, x])) \tag{15}$$

is called the *Buck's distribution function* of $\{x_n\}$. Remark that for the asymptotic density in 1929 Schoenberg introduced the notion of *asymptotic distribution function* as $g(x) = d(A(\{x_n\}, [0, x]))$ (see [Sch], [K-N], [D-T], [S-P]).

Let $g : [0, 1] \rightarrow [0, 1], g(0) = 0, g(1) = 1$ be an increasing and continuous function. Then $g^{-1} : [0, 1] \rightarrow [0, 1]$ is also increasing and continuous. Consider the van der Corput sequence $\{\gamma(n)\}$. Put $x_n = g^{-1}(\gamma(n)), n = 1, 2, \dots$. Then $\{x_n\}$ is polyadically continuous, because $\{\gamma(n)\}$ is polyadically continuous. Moreover, for $x \in [0, 1)$ we have

$$x_n < x \iff \gamma(n) < g(x),$$

thus g and $\{x_n\}$ fulfil (15). We proved

PROPOSITION 5. *If $g : [0, 1] \rightarrow [0, 1], g(0) = 0, g(1) = 1$ is an increasing and continuous function, then $\{x_n\}$, where $x_n = g^{-1}(\gamma(n))$, is a polyadically continuous sequence with Buck's distribution function g .*

For the case of the asymptotic density there hold stronger results (see [K-N], [D-T]).

Proposition 4 yields that if $\{x_n\}$ is Buck's measurable, then

$$\lim_{n \rightarrow \infty} \frac{1}{B_n} \sum_{j=1}^{B_n} h(x_{k_j^{(n)}}) = \int_0^1 h(x) dg(x)$$

for every step function h , where $g(x) = \mu(A[0, x])$. As every continuous function f can be approximated in the sense (12) (instead of Riemann integral we consider Riemann-Stieltjes integral with respect to $g(x) = \mu(A([0, x]))$) we obtain

THEOREM 6. *Suppose the notation from Proposition 4. a) If $\{x_n\}$ is Buck's measurable then, for every continuous function $f : [0, 1] \rightarrow (-\infty, \infty)$ we have*

$$\lim_{n \rightarrow \infty} \frac{1}{B_n} \sum_{j=1}^{B_n} f(x_{k_j^{(n)}}) = \int_0^1 f(x) dg(x), \quad (e)$$

where $g(x)$ is the Buck's distribution function of $\{x_n\}$.

b) Let $g(x)$ be a continuous, non decreasing function defined on $[0, 1]$ and $g(0) = 0, g(1) = 1$. Then $g(x)$ is the Buck's distribution function of $\{x_n\}$ if and only if (e) holds.

Part b) follows from the fact that in the case that $g(x)$ is continuous, every step function can be approximated by two continuous functions in the sense (12) (See [K-N, p. 54], proof of the criterium for asymptotic distribution functions).

Analogously to Theorem 5, we can prove

COROLLARY 4. *If $\{x_n\}$ is a polyadically continuous sequence with continuous asymptotic distribution function, then $\{x_n\}$ is Buck's measurable.*

Now we can construct Buck's measurable sequences, using some results about multiplicative functions. Suppose that to a prime number p a positive value a_p is associated such that the series $\sum a_p$ converges. Then the sequence $x_n = \prod_{p|n} (1 - a_p)$ is a sequence with a continuous asymptotic distribution function (see [E, p. 185]), and it can be easily proved that $\{x_n\}$ is polyadically continuous. Thus Corollary 4 yields that $\{x_n\}$ is Buck's measurable.

4. Some metric properties

Denote by B the set of all B.u.d. sequences and by U the set of all u.d. sequences. The property (iii) yields that $B \subset U$.

THEOREM 7. *The set B is closed with respect to the supremum metric.*

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Proof. Let $\{x_n\} \in \overline{B}$ - closure of B in the supremum metric. Then for $\varepsilon > 0$ there exists $\{y_n\} \in B$ such that $|x_n - y_n| < \varepsilon$, $n = 1, 2, \dots$. If R_n is the system from Proposition 4, we have

$$\begin{aligned} & \left| \frac{1}{B_n} \sum_{j=1}^{B_n} \exp 2\pi i h x_{k_j^{(n)}} - \frac{1}{B_n} \sum_{j=1}^{B_n} \exp 2\pi i h y_{k_j^{(n)}} \right| \\ & \leq \frac{1}{B_n} \sum_{j=1}^{B_n} |\dots| \leq 2\pi h |x_{k_j^{(n)}} - y_{k_j^{(n)}}| \leq 2\pi h \varepsilon. \end{aligned}$$

For $\varepsilon \rightarrow 0^+$ we get, using Theorem 4, that

$$\frac{1}{B_n} \sum_{j=1}^{B_n} \exp 2\pi i h x_{k_j^{(n)}} = 0, \quad h \neq 0, h \in Z,$$

and the assertion follows. □

It is a well known fact that $\lambda^\infty(U) = 1$, where λ^∞ is a product measure on $[0, 1)^\infty$ (see [K-N]).

Let $A \subset \mathbb{N}$. If there exists $\lim_{N \rightarrow \infty} \frac{1}{N} \text{card}(A \cap [k, k + N])$ uniformly with respect to $k \in \mathbb{N}$ then its value is called the *uniform density* of A ($= u(A)$). (cf. [GLS, Theorem 1.1, Theorem 1.2]). A sequence $\{y_n\}, y_n \in [0, 1)$ is called *well distributed* if and only if $u(A(\{y_n\}, I)) = |I|$ for every interval $I \subset [0, 1)$ (cf. [K-N, p. 51]). Every set $A \in D_\mu$ has a uniform density $u(A)$ and $u(A) = \mu(A)$. Thus $\lambda^\infty(B) = 0$. Thus every B.u.d. sequence is well distributed. The product measure of the set of well distributed sequences is 0 (see [K-N, p. 201, Theorem 3.8]). Theorem 2 gives also the example of sequence which is well distributed but no B.u.d. (cf. [K-N, p. 42, Example 5.2]).

5. The theorem of John von Neumann:

John von Neumann proved that every dense sequence in $(0, 1)$ can be rearranged to a u.d. sequence. We prove this result for B.u.d. sequence.

Directly from Theorem 4 we obtain:

PROPOSITION 6. *Let $\{y_n\}$ be a B.u.d. sequence and $|x_n - y_n| \rightarrow 0$ as $n \rightarrow \infty$. Then $\{x_n\}$ is B.u.d.*

PROPOSITION 7. *Let $\{y_n\}$ be a B.u.d. sequence and $\{x_n\}$ be such that $x_n = y_n$, for $n \in A$, where $A \in D_\mu$ and $\mu(A) = 1$. Then $\{x_n\}$ is B.u.d.*

Proof. We have

$$\begin{aligned} & \left| \frac{1}{B_n} \sum_{j=1}^{B_n} \exp 2\pi i h y_{k_j^{(n)}} - \frac{1}{B_n} \sum_{j=1}^{B_n} \exp 2\pi h i x_{k_j^{(n)}} \right| \\ & \leq \frac{1}{B_n} \sum_{\substack{j \leq B_n \\ k_j^{(n)} \in \neg A}} |\exp 2\pi i h x_{k_j^{(n)}} - \exp 2\pi i h y_{k_j^{(n)}}| \\ & \leq \frac{2}{B_n} R(\neg A : B_n). \end{aligned}$$

But $\mu(\neg A) = 1 - \mu(A) = 0$, so we have

$$\lim_{n \rightarrow \infty} \frac{2}{B_n} R(\neg A : B_n) = 0$$

and the assertion follows from Theorem 4. \square

THEOREM 8. *Every sequence $\{x_n\}$ dense in $[0, 1)$ can be rearranged to a B.u.d. sequence.*

Proof. Consider the sequence $\{\gamma(n)\}$ of van der Corput. This is a B.u.d. sequence (see Theorem 1). As $\{x_n\}$ is dense, for $n = 1, 2, \dots$ there exists k_n that $|x_{k_n} - \gamma(n)| < \frac{1}{n}$. And so Proposition 5 yields that $\{x_{k_n}\}$ is a B.u.d. Let $\{x_{k'_n}\}$ be the sequence of elements of $\{x_n\}$ which do not occur in $\{x_{k_n}\}$. If $\{x_{k'_n}\}$ is finite, the proof is finished. Let $\{x_{k'_n}\}$ be infinite. Let $A \subseteq \mathbb{N}$ be such infinite set that $\mu(A) = 0$. For instance $A = \{n!; n = 1, 2, \dots\}$. Then $B = \mathbb{N} \setminus A \in D_\mu$ and $\mu(B) = 1$. Define the sequence $\{z_n\}$ as follows:

$$z_n = x_{k_n}; \quad n \in B.$$

Let $A = \{s_1 < s_2 < \dots\}$, then

$$\begin{aligned} z_{s_{2j}} &= x_{k'_j}, & j &= 1, 2, \dots \\ z_{s_{2j+1}} &= x_{k_{s_j}}, & j &= 1, 2, \dots \end{aligned}$$

Then $\{z_n\}$ is a rearrangement of $\{x_n\}$ and Proposition 6 implies that $\{z_n\}$ is B.u.d. \square

6. The B.u.d. preserving mapping

In the paper [PSSa] the mapping $g : [0, 1] \rightarrow [0, 1]$, such that $\{g(y_n)\}$ is u.d., if $\{y_n\}$ u.d. is studied. Such mapping is called *u.d. preserving*. In this paper

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it is proved, that if g is Riemannian integrable, then g is u.d. preserving if and only if

$$\int_0^1 h(g(x)) \, dx = \int_0^1 h(x) \, dx \quad (16)$$

holds for every continuous function $h : [0, 1] \rightarrow (-\infty, \infty)$.

A mapping $g : [0, 1] \rightarrow [0, 1]$ is called *B.u.d. preserving* if for every B.u.d. sequence $\{y_n\}$ is $\{g(y_n)\}$ B.u.d.

THEOREM 9. *A Riemannian integrable function $g : [0, 1] \rightarrow [0, 1]$ is B.u.d. preserving if and only if (16) holds for every continuous function $h : [0, 1] \rightarrow (-\infty, \infty)$.*

PROOF. In the third part we proved that $\{y_n\}$ is B.u.d. if and only if

$$\lim_{n \rightarrow \infty} \sum_{j=1}^{B_n} h(y_{k_j^{(n)}}) = \int_0^1 h(x) \, dx$$

(see the notation of Proposition 4). If we consider that both sequences $\{y_n\}$, $\{g(y_n)\}$ are B.u.d., then using the standard procedure from [PSSa] we obtain the assertion. \square

COROLLARY 5. *If $g : [0, 1] \rightarrow [0, 1]$ is a continuous function, fulfilling (16) then the sequence $\{g(\gamma(n))\}$ is a polyadically continuous B.u.d. sequence.*

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