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ON AN ARITHMETIC FUNCTION CONNECTED WITH THE DISTRIBUTION OF SUPERSINGULAR FERMAT VARIETIES

Werner Georg Nowak *

ABSTRACT. For an integer $m > 2$, let $\delta(m)$ denote the asymptotic density of primes for which some power is congruent to -1 modulo m. This arithmetic function has a deep algebraic background as indicated in the title. The task of this paper is to obtain a precise asymptotic formula for the sum $\sum\limits_{2<m\le x} \delta(m)$, where x is a large real variable. Furthermore, an analogous result for higher power moments of $\delta(m)$ is established.

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Dedicated to Professor Robert F. Tichy on the occasion of his 50th birthday

1. Introduction

To describe the problem considered in this article in purely number theoretic terms, let us call a prime p "m-admissible", where $m > 2$ is a given integer, if there exists a power of p which is congruent to -1 modulo m. For fixed m, let $\delta(m)$ denote the asymptotic density of the m-admissible primes, i.e.,

$$\delta(m) := \lim_{x\to\infty} \frac{\#\{p \in \mathbb{P},\ p \le x\ :\ p \text{ is } m\text{-admissible}\}}{x/\log x}\,. \tag{1.1}$$

This quantity has attracted interest because of its connection with so-called supersingular Fermat varieties: Is is known [6], [7], [9] that the Jacobian variety of a Fermat curve

$$X^m + Y^m = Z^m\,,$$

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considered in characteristic p, is supersingular (i.e., isogenous to a product of supersingular elliptic curves), if and only if p is m-admissible. See also [7] for some more equivalent formulations of this property, as well as for an extension to higher dimensional Fermat varieties.

An explicit formula for $\delta(m)$ has been established by N. Yui [9] and W.C. Waterhouse [7]. To state it, we define first arithmetic functions $\alpha(m), \gamma(m)$ by

$$2^{\alpha(m)} \parallel \phi(m)\,, \quad 2^{\gamma(m)} \parallel \gcd(\{p-1:\ p \mid m,\ p>2\ \}). \tag{1.2}$$

Here $\phi(\cdot)$ is the Euler totient function, p denotes primes throughout, and $2^a \parallel b$ means as usual that 2^a divides b but 2^{a+1} does not. Further, let

$$\mathcal{P}(m) := \{p \in \mathbb{P}:\ p \mid m\ \}\,, \quad \omega(m) := \#\mathcal{P}(m)\,, \quad \omega^*(m) := \#(\mathcal{P}(m) \setminus \{2\})\,.$$

Then, according to N. Yui [9] and W.C. Waterhouse [7],

$$\delta(m) = \begin{cases} 2^{-\alpha(m)} & \text{if } 4 \mid m\,, \\ 2^{-\alpha(m)}\,\dfrac{2^{\omega^*(m)\gamma(m)}-1}{2^{\omega^*(m)}-1} & \text{else.} \end{cases} \tag{1.3}$$

The average order of this arithmetic function is described by the sum $\sum_{2<m\le x} \delta(m)$, where x is a large real parameter. W.C. Waterhouse [7] showed that this is a $o(x)$. This permits the interpretation that supersingular Fermat varieties are overall a rare phenomenon. The result was subsequently sharpened and given a quantitative form by W. Schwarz and W.C. Waterhouse [5]: They proved that

$$\sum_{2<m\le x} \delta(m) \sim \frac{A_0\, x}{(\log x)^{2/3}}\,, \tag{1.4}$$

with a positive constant A_0 for which they gave an explicit (though quite involved) representation. Their argument is based on a deep Tauberian theorem of E. Wirsing [8].

The objective of the present paper is to establish a more precise asymptotic expansion for this sum.

Theorem 1. *For every fixed positive integer K there exist positive constants $A_0, A_1, \ldots, A_K$ such that, as $x \to \infty$,*

$$\sum_{2<m\le x} \delta(m) = x \sum_{k=0}^{K} A_k (\log x)^{2^{-k}/3-1} + O\left(x(\log x)^{2^{-K}/4-1}\right),$$

where the O-constant may depend on the integer K.

Remarks. 1. The error term can be readily improved to $O(x(\log x)^{2^{-K}/6-1})$, just by considering this asymptotics with K replaced by $K+1$. In fact, for any $\varepsilon > 0$, the remainder can be made less than $O\left(x(\log x)^{-1+\varepsilon}\right)$, by an appropriate choice of K.
2. The matter is related to the problem concerning the average p-adic valuation of the Euler totient function $\phi(n)$. The latter has been investigated in a recent paper by W.D. Banks, F. Luca and I.E. Shparlinski [1], Theorem 7.

2. Auxiliary results

The progress accomplished is primarily due to the application of a more accurate Tauberian theorem, with an error term, which has been established by R.W.K. Odoni [3].

Odoni's Lemma. *Let f be a nonnegative, multiplicative arithmetic function. Assume that $f(p^k)$ is uniformly bounded for all primes p and integers $k \geq 1$. Suppose further that there exists a constant $c > 0$ so that*

$$\sum_{p \leq x} f(p) = \frac{cx}{\log x} + O\left(\frac{x}{(\log x)^2}\right)$$

as $x \to \infty$. Then there exists a positive constant A such that, for every fixed $\varepsilon > 0$,

$$\sum_{1 \leq n \leq x} f(n) = Ax(\log x)^{c-1} + O\left(x(\log x)^{c-3/2+\varepsilon}\right).$$

P r o o f. This is a special case of Theorem III in Odoni [3]. As Odoni remarks at the end of that paper, the error term could be improved slightly by a method exhibited in the monograph of H. Halberstam and H.-E. Richert [2]. However, this is unimportant for our purpose. □

An important idea in the approach of Schwarz and Waterhouse [5] is to approximate $\delta(m)$ by

$$g(m) := 2^{-\alpha(m)}, \tag{2.1}$$

where the latter function has the advantage to be multiplicative. Furthermore, Schwarz and Waterhouse in fact proved that

$$\sum_{p \leq x} g(p) = \frac{1}{3}\frac{x}{\log x} + O\left(\frac{x}{(\log x)^2}\right).$$

Applying Odoni's Lemma readily yields the following.

Corollary. For every fixed $\varepsilon > 0$ and large x,

$$\sum_{2<m\le x} g(m) = \frac{A_0\, x}{(\log x)^{2/3}} + O\left(\frac{x}{(\log x)^{7/6-\varepsilon}}\right) , \qquad (2.2)$$

where the constant A_0 is the same as in (1.4).

3. Proof of Theorem 1

Our strategy is to construct a more precise approximation to $\delta(m)$, by a sum of multiplicative arithmetic functions. To this end, for every integer $k \ge 1$, we define sets of primes

$$\mathbb{P}_k := \{p \in \mathbb{P} :\ 2^k \parallel (p-1)\ \} , \qquad \mathbb{P}_k^* := \bigcup_{j=1}^{k} \mathbb{P}_j .$$

Lemma 1. *For each integer $k \ge 1$, define nonnegative multiplicative arithmetic functions β_k and θ_k as follows:*

$$\beta_k(m) = \begin{cases} 2^{k\omega^*(m)-\alpha(m)} & \text{if } 4 \nmid m \text{ and } \mathcal{P}(m) \cap \mathbb{P}_k^* = \emptyset, \\ 0 & \text{else,} \end{cases}$$

resp.,

$$\theta_k(m) = \begin{cases} 2^{-\omega(m)} & \text{if } 4 \nmid m \text{ and } \mathcal{P}(m) \cap \mathbb{P}_{k+1}^* = \emptyset, \\ 0 & \text{else.} \end{cases}$$

Put finally, for $m > 2$ and $K \ge 1$,

$$\Delta_K(m) := \delta(m) - \left(g(m) + \sum_{k=1}^{K} \beta_k(m)\right) .$$

Then

$$0 \le \Delta_K(m) \le 4\,\theta_K(m)$$

for arbitrary integers $m > 2$ and $K \ge 1$.

P r o o f. If $4 \mid m$, then $\delta(m) = g(m)$, hence the assertion is obvious. Thus we assume that $4 \nmid m$ in what follows.

For given $m > 2$ and $K \ge 1$, let L denote the least positive integer for which $\mathcal{P}(m) \cap \mathbb{P}_L^* \ne \emptyset$. I.e., there exists $p \in \mathcal{P}(m)$ for which $2^L \parallel (p-1)$, and $2^L \mid (p'-1)$ for all odd $p' \in \mathcal{P}(m)$. Thus, by (1.2), $\gamma(m) = L$.

Case 1. $L \le K+1$. By definition, $\theta_K(m) = 0$. In view of (1.3) and (2.1),

$$\delta(m) = 2^{-\alpha(m)} \sum_{k=0}^{L-1} 2^{k\omega*(m)} = g(m) + \sum_{k=1}^{L-1} \beta_k(m),$$

and the last sum can readily be extended to $1 \le k \le K$, since $\beta_k(m) = 0$ for $L \le k \le K$.

Case 2. $L > K+1$. Now $\theta_K(m) = 2^{-\omega(m)}$. Again by (1.3),

$$\delta(m) \ge 2^{-\alpha(m)} \frac{2^{(K+1)\omega^*(m)} - 1}{2^{\omega^*(m)} - 1} = g(m) + \sum_{k=1}^{K} \beta_k(m),$$

hence

$$0 \le \delta(m) - \left(g(m) + \sum_{k=1}^{K} \beta_k(m) \right) \le \delta(m) \le 2^{2-\omega(m)} = 4\,\theta_K(m),$$

using a simple inequality stated as a Lemma in Waterhouse [7]. □

The next step is to evaluate the asymptotic behavior of the sums over each of the functions $\beta_k(m)$.

Lemma 2. *For every $k \ge 1$, there exists a positive constant A_k, such that*

$$\sum_{1 \le m \le x} \beta_k(m) = A_k\, x(\log x)^{2^{-k}/3-1} + O\left(x(\log x)^{2^{-k}/3-3/2+\varepsilon} \right),$$

for each fixed $\varepsilon > 0$ and $x \to \infty$, the O-constant depending on k.

Proof. In view of Odoni's Lemma, it will suffice to find an asymptotics for the sum over primes. Following Schwarz and Waterhouse [5], we define an integer $B = B(x) := [\log\log x/\log 2]$, so that $2^B \asymp \log x$. We find that

$$\sum_{2<p\le x} \beta_k(p) = \sum_{\substack{p\le x:\\ 2^{k+1}\mid(p-1)}} 2^{k-\alpha(p)} = \sum_{j\ge k+1} \sum_{\substack{p\le x:\\ 2^j \,\|\, (p-1)}} 2^{k-j} =$$

$$= \sum_{j=k+1}^{B} \sum_{\substack{p\le x:\\ 2^j \,\|\, (p-1)}} 2^{k-j} + \sum_{\substack{p\le x:\\ 2^{B+1}\mid(p-1)}} 2^{k-\alpha(p)}. \tag{3.1}$$

By a crude estimate, the last sum is

$$\ll_k 2^{-B} \sum_{p\le x} 1 \ll x(\log x)^{-2}.$$

Using the Prime Number Theorem for arithmetic progressions (see K. Prachar [4], p. 144, Satz 8.3), we get for each of the inner sums

$$\sum_{\substack{p\le x:\\ 2^j \,\|\,(p-1)}} 2^{k-j} = 2^{k-j}\,\pi(x,2^{j+1},2^j+1) = \frac{2^{k-j}\,\mathrm{li}(x)}{\phi(2^{j+1})} + O_k\left(2^{-j}\,x\,e^{-C\sqrt{\log x}}\right) = $$
$$= \frac{2^{k-2j}x}{\log x} + O\left(\frac{2^{-j}x}{(\log x)^2}\right),$$

with $C>0$, uniformly in $j\le B$. Using this in (3.1), we arrive at

$$\sum_{p\le x}\beta_k(p) = \frac{2^k x}{\log x}\sum_{j=k+1}^{B} 2^{-2j} + O\left(\frac{x}{(\log x)^2}\right) = \frac{2^{-k}}{3}\,\frac{x}{\log x} + O\left(\frac{x}{(\log x)^2}\right).$$

In view of Odoni's Lemma, this result readily implies the assertion of Lemma 2. □

It remains to estimate the contribution of the error committed by the approximation of Lemma 1. I.e., we shall show that, for every fixed $K\ge 1$,

$$\sum_{m\le x}\theta_K(m) \ll_K x\,(\log x)^{2^{-K}/4-1}. \tag{3.2}$$

Similarly as before, we see that

$$\sum_{2<p\le x}\theta_K(p) = \sum_{\substack{p\le x:\\ 2^{K+2}\,|\,(p-1)}}\frac{1}{2} = \frac{1}{2}\,\pi(x,2^{K+2},1) = \frac{2^{-K-2}\,x}{\log x} + O\left(\frac{x}{(\log x)^2}\right),$$

by the Prime Number Theorem for a fixed arithmetic progression. Odoni's Lemma implies again more than claimed in (3.2).

Together with Lemmas 1 and 2, and with (2.2), this completes the proof of Theorem 1.

It is possible to write down an explicit representation for the constants A_k. To this end one may appeal to E. Wirsing [8], Satz 1 and the comments thereafter. This result is less precise than Odoni's, insofar that it does not provide an error estimate, but it gives the leading coefficient explicitly. Using this we readily find that

$$A_k = \frac{\exp(-E\,2^{-k}/3)}{\Gamma(2^{-k}/3)}\,P_k\,e^{-R_k},$$

where $E = 0.577215\ldots$ is the Euler-Mascheroni constant, and

$$P_k := \prod_p \left(e^{-\beta_k(p)/p} \left(1 + \sum_{j=1}^{\infty} \beta_k(p^j) p^{-j} \right) \right),$$

$$R_k := \lim_{x\to\infty} \left(\sum_{p\le x} \frac{\beta_k(p)}{p} - \frac{2^{-k}}{3} \log\log x \right),$$

the existence of the limit and the infinite product being guaranteed by Wirsing's analysis. For the coefficient A_0 such a representation has been given already by Schwarz and Waterhouse [5].

4. The r-th power moment of $\delta(m)$

From the analysis developed so far, it is in fact an easy consequence to establish an analogous result for the sum over $(\delta(m))^r$, where $r > 1$ is an arbitrary fixed integer. For any integers $K \ge 1$, $r > 1$, $m > 2$, it is immediate from Lemma 1 that

$$(\delta(m))^r = \sum_{\substack{r_0,\ldots,r_{K+1}\ge 0 \\ r_0+\cdots+r_{K+1}=r}} \frac{r!}{r_0!\ldots r_{K+1}!} (g(m))^{r_0} \prod_{k=1}^{K} (\beta_k(m))^{r_k} \ (\Delta_K(m))^{r_{K+1}} \tag{4.1}$$

(with the usual convention that $0^0 := 1$). To sum each term of the right hand side over $m \le x$, one may again restrict the summation to primes, and then appeal to Odoni's Lemma.

Part 1. *The terms of* (4.1) *with* $r_{K+1} = 0$. Throughout, in what follows, $\sum^{(4.2)}$ is supposed to mean that summation is restricted to those integers j which satisfy

$$j > \max\{k :\ 1 \le k \le K,\ r_k \ne 0\ \}. \tag{4.2}$$

(As usual, $\max\emptyset := -\infty$.) For any $(r_0, \ldots, r_K) \in \mathbb{Z}_{\ge 0}^{K+1}$ with $r_0+\cdots+r_K = r$, and a prime p, suppose that $2^j \parallel (p-1)$ with j fulfilling (4.2). Then,

$$(g(p))^{r_0} \prod_{k=1}^{K} (\beta_k(p))^{r_k} = \exp_2\left(-jr_0 + \sum_{k=1}^{K} (k-j) r_k \right) = \exp_2\left(-jr + \sum_{k=1}^{K} k r_k \right),$$

writing $\exp_2(z) := 2^z$ to express powers of 2 with complicated exponents. Therefore,

$$\sum_{2<p\le x} (g(p))^{r_0} \prod_{k=1}^{K} (\beta_k(p))^{r_k} = \exp_2\left(\sum_{k=1}^{K} k\, r_k\right) \sum_{j\ge 1}^{(4.2)} 2^{-jr} \sum_{\substack{p\le x:\\ 2^j \,\|\, (p-1)}} 1. \tag{4.3}$$

Let $B = B(x) = [\log\log x/\log 2]$ as before, so that $2^B \asymp \log x$. The terms in (4.3) corresponding to $j > B$ are altogether

$$\ll_{K,r} \; 2^{-B} \sum_{p\le x} 1 \ll \frac{x}{(\log x)^2}.$$

From the right-hand side of (4.3), there thus remains

$$\exp_2\left(\sum_{k=1}^{K} k\, r_k\right) \sum_{1\le j\le B}^{(4.2)} 2^{-jr}\, \pi(x, 2^{j+1}, 2^j + 1).$$

Again by the Prime Number Theorem for arithmetic progressions (K. Prachar [4], p. 144, Satz 8.3), this is equal to

$$c(\mathbf{r}_K)\, \frac{x}{\log x} + O_{K,r}\left(\frac{x}{(\log x)^2}\right),$$

where

$$c(\mathbf{r}_K) := \exp_2\left(\sum_{k=1}^{K} k\, r_k\right) \sum_{j\ge 1}^{(4.2)} 2^{-j(r+1)}, \quad \mathbf{r}_K := (r_0, \ldots, r_K). \tag{4.4}$$

For given $\mathbf{r}_K$, let K' denote the largest index for which $r_{K'} \ne 0$. Then we notice for later use that

$$\begin{aligned} c(\mathbf{r}_K) &= \exp_2\left(\sum_{k=1}^{K'} k\, r_k\right) \sum_{j=K'+1}^{\infty} 2^{-j(r+1)} = \\ &= \exp_2\left(\sum_{k=0}^{K'} (k-K')\, r_k - K'\right) \left(2^{r+1}-1\right)^{-1} \le \left(2^{r+1}-1\right)^{-1}, \end{aligned}$$

with equality if and only if $K' = 0$, i.e., $\mathbf{r}_K = (r, 0, \dots, 0)$. By Odoni's Lemma, there exists a positive constant $A(\mathbf{r}_K)$ so that

$$\begin{aligned}\sum_{2<m\le x} (g(m))^{r_0} \prod_{k=1}^{K} (\beta_k(m))^{r_k} &= A(\mathbf{r}_K)\, x(\log x)^{c(\mathbf{r}_K)-1} + \\ &\quad + O_{K,r}\left(x(\log x)^{c(\mathbf{r}_K)-3/2+\varepsilon}\right),\end{aligned} \tag{4.5}$$

for each fixed $\varepsilon > 0$.

Part 2. *The terms of* (4.1) *with* $r_{K+1} > 0$. These will give the error term in the final result. In fact, we estimate $\Delta_K(m)$ by $4\theta_K(m)$, in the sense of Lemma 1, and thus we have to evaluate (with Odoni's Lemma in the back of mind)

$$\sum_{2<p\le x} (g(p))^{r_0} \prod_{k=1}^{K} (\beta_k(p))^{r_k} (\theta_K(p))^{r_{K+1}}. \tag{4.6}$$

Now for $2^j \parallel (p-1)$, $j > K+1$, and $r_0 + \dots + r_{K+1} = r$, it follows that

$$(g(p))^{r_0} \prod_{k=1}^{K} (\beta_k(p))^{r_k} (\theta_K(p))^{r_{K+1}} = \exp_2\left(-jr_0 + \sum_{k=1}^{K} (k-j) r_k - r_{K+1}\right).$$

Unless $r_0 = \dots = r_K = 0$, we may repeat the above argument, including the breaking up of the sum at $j = B$. Thus we obtain

$$\sum_{2<p\le x} (g(p))^{r_0} \prod_{k=1}^{K} (\beta_k(p))^{r_k} (\theta_K(p))^{r_{K+1}} = c^*(\mathbf{r}_{K+1}) \frac{x}{\log x} + O_{K,r}\left(\frac{x}{(\log x)^2}\right),$$

with

$$\begin{aligned}c^*(\mathbf{r}_{K+1}) := \sum_{j>K+1} &\exp_2\left(-j(r_0+1) + \sum_{k=1}^{K} (k-j) r_k - r_{K+1}\right) = \\ &= \exp_2\left(-K - r - 1 - \sum_{k=0}^{K} (K-k) r_k\right) \left(2^{1+r-r_{K+1}} - 1\right)^{-1}.\end{aligned} \tag{4.7}$$

Hence, by Odoni's Lemma,

$$\sum_{2<m\le x} (g(m))^{r_0} \prod_{k=1}^{K} (\beta_k(m))^{r_k} (\theta_K(m))^{r_{K+1}} \ll_{K,r} x\,(\log x)^{c^*(\mathbf{r}_{K+1})-1}. \tag{4.8}$$

In the exceptional case that $r_0 = \dots = r_K = 0$, i.e., $r_{K+1} = r$, we simply conclude that

$$\sum_{2<p\le x} (\theta_K(p))^r = 2^{-r} \sum_{\substack{p\le x \\ 2^{K+2} \mid (p-1)}} 1 = \frac{2^{-K-r-1} x}{\log x} + O_{K,r}\left(\frac{x}{(\log x)^2}\right).$$

Thus in this case,

$$\sum_{2<m\le x} (\theta_K(m))^r \ll_{K,r} \; x\,(\log x)^{2^{-K-r-1}-1}\,. \tag{4.9}$$

But it is easy to see that the constant $c^*(\mathbf{r}_{K+1})$ defined in (4.7) satisfies $c^*(\mathbf{r}_{K+1}) \le 2^{-K-r-1}$. Therefore, collecting the results (4.1), (4.5), (4.8), and (4.9), we obtain the following asymptotics.

Theorem 2. *For arbitrary fixed positive integers r and K there exist positive constants $A^*(\mathbf{r}_K)$ such that, as $x \to \infty$,*

$$\sum_{2<m\le x} (\delta(m))^r = \; x \sum_{\substack{r_0,\dots,r_K\ge 0\\ r_0+\dots+r_K=r}} A^*(\mathbf{r}_K)(\log x)^{c(\mathbf{r}_K)-1} + O\left(x(\log x)^{2^{-K-r-1}-1}\right),$$

where the O-constant may depend on the integers r and K. Here $\mathbf{r}_K$ and $c(\mathbf{r}_K)$ have been defined in (4.4)*. In particular,*

$$\sum_{2<m\le x} (\delta(m))^r \;\sim\; \frac{A^*(r,0,\dots,0)\,x}{(\log x)^{1-1/(2^{r+1}-1)}}.$$

Remark. In general, not all of the constants $c(\mathbf{r}_K)$ exceed 2^{-K-r-1}, thus some of the terms $A^*(\mathbf{r}_K)\,x\,(\log x)^{c(\mathbf{r}_K)-1}$ may be absorbed by the error term.

To illustrate the established asymptotic formula, let us consider the special case $r=2$, $K=1$. By (4.4),

$$c(2,0) = \sum_{j=1}^{\infty} 2^{-3j} = \frac{1}{7},\;\; c(1,1) = 2\sum_{j=2}^{\infty} 2^{-3j} = \frac{1}{28},\;\; c(0,2) = 4\sum_{j=2}^{\infty} 2^{-3j} = \frac{1}{14}.$$

Of these constants only $c(2,0)$ and $c(0,2)$ exceed $2^{-K-r-1} = \frac{1}{16}$, and we obtain

$$\sum_{2<m\le x} (\delta(m))^2 = \frac{A^*(2,0)\,x}{r(\log x)^{6/7}} + \frac{A^*(0,2)\,x}{(\log x)^{13/14}} + O\left(\frac{x}{(\log x)^{15/16}}\right).$$

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Werner Georg Nowak
Institute of Mathematics
Department of Integrative Biology
Universität für Bodenkultur Wien
Gregor Mendel-Straße 33
A-1180 Vienna
AUSTRIA
E-mail: nowak@boku.ac.at
http://www.boku.ac.at/math/nth.html