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DISTRIBUTION FUNCTIONS OF RATIO SEQUENCES, II

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ABSTRACT. For an increasing sequence x_n, $n = 1, 2, \dots$, of positive integers define the block sequence $X_n = (x_1/x_n, \dots, x_n/x_n)$. We study the set $G(X_n)$ of all distribution functions of X_n, $n = 1, 2, \dots$. We find a special x_n such that $G(X_n)$ is not connected and we give some criterions for connectivity of $G(X_n)$. We also give an x_n such that $G(X_n)$ contains one-step distribution function with step 1 in 1 but does not contain one-step distribution function with step 1 in 0. We prove that if $G(X_n)$ is constituted by one-step distribution functions, at least two different, then it contains distribution functions with steps in 0 and 1.

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1. Introduction

Block sequences $X_n = (x_{n,1}, \dots, x_{n,N_n})$, $n = 1, 2, \dots$, are main tool for constructing examples of various types of distribution of sequences. In [ST] a new type of blocks X_n is introduced and the set $G(X_n)$ of distribution functions of X_n is studied (for definitions see below). In this paper we solve some open problems on $G(X_n)$ formulated in [ST]. Some examples on $G(X_n)$ can be found in [ST1]. In [TMF] the authors introduced and studied the so called dispersion of X_n (see also [FT, FT1, FMT]).

In this paper we use the following definitions and notations, see [SP, p. 1–28, 1.8.23]: Let x_n, $n = 1, 2, \dots$, be an increasing sequence of positive integers (by "increasing" we mean strictly increasing). The double sequence x_m/x_n, $m, n = 1, 2, \dots$ is called *the ratio sequence* of x_n; it was introduced by T. Šalát

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[Sa]. He studied its everywhere density. For further study of ratio sequences, O. Strauch and J.T. Tóth [ST] introduced a sequence X_n of blocks

$$X_n = \left(\frac{x_1}{x_n}, \frac{x_2}{x_n}, \dots, \frac{x_n}{x_n}\right), \quad n = 1, 2, \dots$$

and they defined distribution functions of X_n as follows:

• Denote by $F(X_n, x)$ the step distribution function

$$F(X_n, x) = \frac{\#\{i \le n; \frac{x_i}{x_n} < x\}}{n},$$

for $x \in [0, 1)$ and $F(X_n, 1) = 1$.

• A non-decreasing function $g : [0, 1] \to [0, 1]$, $g(0) = 0$, $g(1) = 1$ is called distribution function (abbreviating d.f.). We shall identify any two d.f. coinciding at common points of continuity.

• A d.f. $g(x)$ is the d.f. of the sequence of blocks X_n, $n = 1, 2, \dots$, if there exists an increasing sequence $n_1, n_2, \dots$ of positive integers such that

$$\lim_{k\to\infty} F(X_{n_k}, x) = g(x)$$

a.e. on $[0, 1]$. This is equivalent to the weak convergence, i.e., the preceding limit holding for every point $x \in [0, 1]$ of continuity of $g(x)$.

• Denote by $G(X_n)$ the set of all d.f.s of X_n, $n = 1, 2, \dots$. For a singleton $G(X_n) = \{g(x)\}$, the d.f. $g(x)$ is also called asymptotic d.f. (abbreviating a.d.f.) of X_n.

• We use the following special d.f.s: A one-step d.f. $c_\alpha(x)$ with the step 1 in $\alpha \in [0, 1]$

$$c_\alpha(x) = \begin{cases} 0 & \text{if } x \le \alpha, \\ 1 & \text{if } x \in (\alpha, 1), \\ 1 & \text{if } x = 1, \end{cases}$$

and a constant d.f.

$$h_\alpha(x) = \begin{cases} 0 & \text{if } x = 0, \\ \alpha & \text{if } x \in (0, 1), \\ 1 & \text{if } x = 1. \end{cases}$$

Thus, in any case $c_\alpha(0) = 0$, $c_\alpha(1) = 1$, $h_\alpha(0) = 0$ and $h_\alpha(1) = 1$. Clearly, $c_0(x) = h_1(x)$ and $c_1(x) = h_0(x)$.

• For the classical point sequence y_n, $n = 1, 2, \dots$, in $[0, 1]$, the step d.f. is defined by

$$F_N(x) = \frac{\#\{n \le N; y_n < x\}}{N},$$

$F_N(1) = 1$, and $g(x)$ is a d.f. of y_n if there exists an increasing sequence of positive integers N_k such that $F_{N_k}(x) \to g(x)$ a.e., as $k \to \infty$ (cf. [DT, p. 138]; [SP, p. 1–8]). Let $G(y_n)$ denote the set of all d.f.s of y_n. It is known (cf. R. Winkler [W]) that $G(y_n)$ is nonempty, closed and connected with respect to the L^2 metric

$$\rho(g_1, g_2) = \left(\int_0^1 (g_1(x) - g_2(x))^2 \mathrm{d}x \right)^{1/2}.$$

Moreover, for every nonempty closed and connected[1] set H of d.f.s there exists a sequence $y_n \in [0,1)$, $n = 1, 2, \dots$, such that $G(y_n) = H$.

Nonemptiness and closedness of $G(X_n)$ are clear, but the connectivity of $G(X_n)$ is proved in [ST] only under the sufficient condition that

$$\lim_{n\to\infty} \int_0^1 (F(X_{n+1}, x) - F(X_n, x))^2 \mathrm{d}x = 0. \tag{1}$$

This follows directly from the following theorem of H.G. Barone [B]:

Theorem 1. *If t_n is a sequence in a metric space (X, ρ) satisfying*

(i) *any subsequence of t_n contains a convergent subsequence, and*

(ii) $\lim_{n\to\infty} \rho(t_{n+1}, t_n) = 0$,

then the set of all limit points of t_n is connected in (X, ρ).

Putting $X=$ the set of all d.f.s on $[0,1]$, $t_n = F(X_n, x)$, and $\rho=$ the L^2 metric on X, then the Helly selection principle (see [SP, p. 4–5, Th. 4.1.0.10] implies (i), and thus (ii) $\rho(t_{n+1}, t_n) \to 0$ implies the connectivity of $G(X_n)$.

Here is the plan of this paper: In Part 2 we prove that (ii) $\rho(t_{n+1}, t_n) \to 0$ is not necessary. We give an example of x_n such that $\rho(t_{n_k+1}, t_{n_k}) \to 1$ and $G(X_n)$ is connected. Furthermore we find a permutation $\pi : \mathbb{N} \to \mathbb{N}$ such that (ii') $\rho(t_{\pi(n)}, t_{\pi(n+1)}) \to 0$. Applying the Barone theorem to permuted sequence $t_{\pi(n)}$, we obtain again the connectivity of such $G(X_n)$. In the second example we find a sequence x_n having non-connected $G(X_n)$. In Part 3 we study $\rho(t_{n+1}, t_n)$ independently, where we use the following basic expressions (see [ST]): For $m \le n$ and $x \in [0,1]$ we have

$$F(X_m, x) = \frac{n}{m} F\left(X_n, x\frac{x_m}{x_n} \right), \tag{2}$$

and for m,n=1,2,...

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[1] H is connected if for every two $g_1, g_2 \in H$ and every $\varepsilon > 0$ there exists $g_{n_1}, \dots, g_{n_k} \in H$ such that $\rho(g_1, g_{n_1}) < \varepsilon$, $\rho(g_{n_i}, g_{n_{i+1}}) < \varepsilon$, $i = 1, 2, \dots, k-1$, and $\rho(g_{n_k}, g_2) < \varepsilon$.

$$\rho^2(t_m,t_n) = \frac{1}{mn}\sum_{i=1}^{m}\sum_{j=1}^{n}\left|\frac{x_i}{x_m}-\frac{x_j}{x_n}\right| - \frac{1}{2m^2}\sum_{i,j=1}^{m}\left|\frac{x_i}{x_m}-\frac{x_j}{x_m}\right| - \frac{1}{2n^2}\sum_{i,j=1}^{n}\left|\frac{x_i}{x_n}-\frac{x_j}{x_n}\right|. \tag{3}$$

In Part 4 we study the X_n's which have one-step d.f.s $c_\alpha(x)$. We prove that if $c_1(x) \in G(X_n)$, then $c_0(x) \in G(X_n)$ for special class of "almost arithmetic" sequences x_n. We thank L. Mišík for letting us include the Example 4, showing that if $c_1(x) \in G(X_n)$, then $c_0(x)$ need not to belong to $G(X_n)$. Furthermore we prove that if $G(X_n)$ contains only $c_\alpha(x)$ and $\#G(X_n) \ge 2$, then it must be $\{c_0(x), c_1(x)\} \subset G(X_n)$. In Part 5 we study a transformation of X_n with respect to the mapping $1/x$ mod 1. In Part 6 we formulate some open problems. For the convenience of the reader we repeat the basis properties of $G(X_n)$ from [ST, ST1]:

(i) If $g(x) \in G(X_n)$ increases and is continuous at $x = \beta$ and $g(\beta) > 0$, then there exists $1 \le \alpha < \infty$ such that $\alpha g(x\beta) \in G(X_n)$. If every d.f. of $G(X_n)$ is continuous at 1, then $\alpha = 1/g(\beta)$.

(ii) Assume that all d.f.s in $G(X_n)$ are continuous at 0 and $c_1(x) \notin G(X_n)$. Then for every $\tilde{g}(x) \in G(X_n)$ and every $1 \le \alpha < \infty$ there exists $g(x) \in G(X_n)$ and $0 < \beta \le 1$ such that $\tilde{g}(x) = \alpha g(x\beta)$ a.e.[2]

(iii) Assume that all d.f.s in $G(X_n)$ are continuous at 1. Then all d.f.s in $G(X_n)$ are continuous on $(0,1]$, i.e., the only possible discontinuity is in 0.

(iv) If $\underline{d}(x_n) > 0$, then for every $g(x) \in G(X_n)$ we have $(\underline{d}(x_n)/\overline{d}(x_n))x \le g(x) \le (\overline{d}(x_n)/\underline{d}(x_n))x$ for every $x \in [0,1]$. Thus $\underline{d}(x_n) = \overline{d}(x_n) > 0$ implies u.d. of the block sequence X_n, $n = 1,2,\dots$.

(v) If $\underline{d}(x_n) > 0$, then every $g(x) \in G(X_n)$ is continuous on $[0,1]$.

(vi) If $\underline{d}(x_n) > 0$, then there exists $g(x) \in G(X_n)$ such that $g(x) \ge x$ for every $x \in [0,1]$.

(vii) If $\overline{d}(x_n) > 0$, then there exists $g(x) \in G(X_n)$ such that $g(x) \le x$ for every $x \in [0,1]$.

(viii) Assume that $G(X_n)$ is a singleton, i.e., $G(X_n) = \{g(x)\}$. Then either $g(x) = c_0(x)$ for $x \in [0,1]$; or $g(x) = x^\lambda$ for some $0 < \lambda \le 1$ and $x \in [0,1]$. Moreover, if $\overline{d}(x_n) > 0$, then $g(x) = x$

[2] This is a corrected version of [ST, Th. 3.3].

(viii') A block sequence X_n has a.d.f. $g(x) = x^\lambda$, $0 < \lambda \le 1$, if and only if $\frac{x_{nk}}{x_n} \to k^{1/\lambda}$ as $n \to \infty$ and for every $k = 1, 2, \dots$.[3]

(ix) $\max_{g\in G(X_n)} \int_0^1 g(x)\mathrm{d}x \ge \frac{1}{2}$.

(x) Assume that every d.f. $g(x) \in G(X_n)$ has a constant value on some fixed interval $(u, v) \subset [0, 1]$ (these values for different $g(x)$ can be different). If $\underline{d}(x_n) > 0$ then all d.f.s in $G(X_n)$ have constant values on infinitely many intervals.

2. The connectivity of $G(X_n)$

In the first example we give an integer sequence x_n, for which $G(X_n)$ is connected, $\limsup_{n\to\infty} \rho(t_{n+1}, t_n) = 1$ and we find a permutation π of $\mathbb{N}$ such that (ii') $\rho(t_{\pi(n+1)}, t_{\pi(n)}) \to 0$ holds.

Example 1. Let x_n, $n = 1, 2, \dots$, be an increasing sequence of positive integers for which there exists a sequence n_k, $k = 1, 2, \dots$, of positive integers such that (as $k \to \infty$)

(i) $\frac{n_{k-1}}{n_k} \to 0$,

(ii) $\frac{n_k}{x_{n_k}} \to 0$,

(iii) $\frac{x_{n_{k-1}}}{x_{n_k}} \to 0$, and

(iv) $x_{n_k - i} = x_{n_k} - i$ for $i = 0, 1, \dots, n_k - n_{k-1} - 1$.

Then the sequence of blocks

$$X_n = \left(\frac{x_1}{x_n}, \frac{x_2}{x_n}, \dots, \frac{x_n}{x_n}\right)$$

has

$$G(X_n) = \{h_\alpha(x); \alpha \in [0, 1]\}.$$

Proof. For given $\theta \in [0, 1]$ and $n = n_k - [\theta(n_k - n_{k-1})]$ and by (iv) we have

$$x_n = x_{n_k} - [\theta(n_k - n_{k-1})].$$

For $i \le n$ we distinguish two cases: $x_i \in (x_{n_{k-1}}, x_n]$ and $x_i \le x_{n_{k-1}}$.

(I) For $x_i \in (x_{n_{k-1}}, x_n]$ we have

$$\frac{x_i}{x_n} \in \left[\frac{x_{n_k} - (n_k - n_{k-1}) + 1}{x_{n_k} - [\theta(n_k - n_{k-1})]}, 1\right] \to [1, 1]$$

[3] See F. Filip and J.T. Tóth [FT1].

as $n \to \infty$ and for any $\theta \in [0,1]$. The number of such x_i's is

$$(n_k - n_{k-1}) - [\theta(n_k - n_{k-1})] = (1-\theta)(n_k - n_{k-1}) + O(1).$$

(II) For $x_i \le x_{n_{k-1}}$ we have

$$\frac{x_i}{x_n} \in \left[0, \frac{x_{n_{k-1}}}{x_{n_k} - [\theta(n_k - n_{k-1})]}\right] \to [0,0].$$

We thus get, for any $x \in (0,1)$ and any sufficiently large n,

$$F(X_n, x) = \frac{n_{k-1}}{n} = \frac{n_{k-1}}{n_{k-1} + (1-\theta)(n_k - n_{k-1}) + O(1)}.$$

This gives:

(a) If $\theta \le \varepsilon_0 < 1$, for some fixed ε_0, then

$$F(X_n, x) \to c_1(x).$$

(b) If $\theta = 1$, then

$$F(X_n, x) \to c_0(x).$$

(c) For any $\alpha \in (0,1)$ there exists a sequence $\theta_k \to 1$, as $k \to \infty$, such that

$$\frac{n_{k-1}}{n_{k-1} + (1-\theta_k)(n_k - n_{k-1})} \to \alpha,$$

and in this case

$$F(X_n, x) \to h_\alpha(x).$$

□

Note that the sequences $n_k = 2^{k^2}$ and $x_{n_k} = 2^{(k+1)^2}$ satisfy the assumptions (i), (ii), (iii) and (iv). We also see that $G(X_n)$ is connected but

$F(X_{n_k+1}, x) \to c_0(x)$, and

$F(X_{n_k}, x) \to c_1(x)$,

a.e. on $[0,1]$ and thus $\rho(t_{n_k+1}, t_{n_k}) \to 1$. Using the permutation $\pi : \mathbb{N} \to \mathbb{N}$

$$1, 2, \dots, n_1, n_2, n_2 - 1, n_2 - 2, \dots, n_1 + 1, n_2 + 1, n_2 + 2, \dots n_3, n_4, n_4 - 1,$$
$$n_4 - 2, \dots, n_3 + 1, n_4 + 1, n_4 + 2, \dots, n_5, n_6, n_6 - 1, n_6 - 2, \dots, n_5 + 1, \dots$$

we have $\rho(t_{\pi(n+1)}, t_{\pi(n)}) \to 0$ as $n \to \infty$, because the "neighbouring" d.f. of $t_{\pi(n)}$ satisfies the scheme

$$c_1(x), c_1(x), \dots, c_0(x), c_0(x), \dots, c_1(x), c_1(x), \dots, c_0(x), c_0(x), \dots,$$
$$c_1(x), c_1(x), \dots, c_0(x), c_0(x), \dots$$

In the following example, using two given sequences x_n and y_n we construct a third sequence z_n for which $G(Z_n)$ is not connected.

Example 2. Let x_n and y_n, $n = 1, 2, \dots$, be two strictly increasing sequences of positive integers such that for the related block sequences

$$X_n = \left(\frac{x_1}{x_n}, \dots, \frac{x_n}{x_n}\right) \text{ and } Y_n = \left(\frac{y_1}{y_n}, \dots, \frac{y_n}{y_n}\right),$$

we have

$$F(X_n, x) \to g_1(x) \text{ and } F(Y_n, x) \to g_2(x) \text{ a.e.,}$$

i.e., the sets of d.f. are singletons,[4] $G(X_n) = \{g_1(x)\}$ and $G(Y_n) = \{g_2(x)\}$. Furthermore, let n_k, $k = 1, 2, \dots$, be an increasing sequence of positive integers such that $N_k = \sum_{i=1}^{k} n_i$ satisfies

$$\frac{N_{k-1}}{N_k} \to 0 \text{ as } k \to \infty \text{ which is equivalent to } \frac{n_k}{N_k} \to 1.$$

Denote by z_n the following increasing sequence of positive integers composed by blocks (here we use the notation $a(b, c, d, \dots) = (ab, ac, ad, \dots)$)

$$(x_1, \dots, x_{n_1}), x_{n_1}(y_1, \dots, y_{n_2}), x_{n_1}y_{n_2}(x_1, \dots, x_{n_3}), x_{n_1}y_{n_2}x_{n_3}(y_1, \dots, y_{n_4}), \dots$$

Then the sequence of blocks

$$Z_n = \left(\frac{z_1}{z_n}, \dots, \frac{z_n}{z_n}\right)$$

has the set of d.f.s

$$\begin{aligned} G(Z_n) = {}& \{g_1(x), g_2(x), c_0(x)\} \\ & \cup \{g_1(xy_n); n = 1, 2, \dots\} \\ & \cup \{g_2(xx_n); n = 1, 2, \dots\} \\ & \cup \left\{\frac{1}{1+\alpha}c_0(x) + \frac{\alpha}{1+\alpha}g_1(x); \alpha \in [0, \infty)\right\} \\ & \cup \left\{\frac{1}{1+\alpha}c_0(x) + \frac{\alpha}{1+\alpha}g_2(x); \alpha \in [0, \infty)\right\}, \end{aligned}$$

where $g_1(xy_n) = 1$ if $xy_n \geq 1$, similarly for $g_2(xx_n)$.

[4] By (viii) in Part 1, for any singleton $G(X_n) = \{g(x)\}$ either $g(x) = c_0(x)$ or $g(x) = x^{\lambda}$ for some $0 < \lambda \leq 1$.

P r o o f. For every $n = 1, 2, \dots$ there exists an integer k such that

$$N_{k-1} < n \le N_k$$

(here $N_0 = 0$). Put $n' = n - N_{k-1}$. For every n we have

$$z_n = \begin{cases} x_{n_1} y_{n_2} \dots x_{n_{k-1}} y_{n'} & \text{if } k \text{ is even,} \\ x_{n_1} y_{n_2} \dots y_{n_{k-1}} x_{n'} & \text{if } k \text{ is odd.} \end{cases}$$

Firstly we assume that k is even. Then Z_n has the form

$$Z_n = \left(\dots, \frac{x_{n_1} y_{n_2} \dots y_{n_{k-2}} (x_1, \dots, x_{n_{k-1}})}{x_{n_1} y_{n_2} \dots x_{n_{k-1}} y_{n'}}, \frac{x_{n_1} y_{n_2} \dots x_{n_{k-1}} (y_1, \dots, y_{n'})}{x_{n_1} y_{n_2} \dots x_{n_{k-1}} y_{n'}} \right) =$$

$$\left(\dots, \frac{1}{x_{n_{k-1}} y_{n'}} \left(\frac{y_1}{y_{n_{k-2}}}, \dots, \frac{y_{n_{k-2}}}{y_{n_{k-2}}} \right), \frac{1}{y_{n'}} \left(\frac{x_1}{x_{n_{k-1}}}, \dots, \frac{x_{n_{k-1}}}{x_{n_{k-1}}} \right), \left(\frac{y_1}{y_{n'}}, \dots, \frac{y_{n'}}{y_{n'}} \right) \right)$$

and thus for $x > \frac{1}{x_{n_{k-1}}}$ we have

$$\begin{aligned} F(Z_n, x) &= \frac{N_{k-2} + n_{k-1} F(X_{n_{k-1}}, x y_{n'}) + n' F(Y_{n'}, x)}{N_{k-1} + n'} \\ &= \frac{N_{k-2}}{N_{k-1} + n'} + \frac{\frac{n_{k-1}}{N_{k-1}}}{1 + \frac{n'}{N_{k-1}}} F(X_{n_{k-1}}, x y_{n'}) + \frac{1}{1 + \frac{N_{k-1}}{n'}} F(Y_{n'}, x). \end{aligned}$$

If $n \to \infty$, then the first term tends to zero. If $F(Z_n, x) \to g(x)$ for some sequence of n, we can select a subsequence of n's such that $\frac{n'}{N_{k-1}} \to \alpha$ for some $\alpha \in [0, \infty)$, or $\frac{n'}{N_{k-1}} \to \infty$. For such n' we distinguish the following cases:

(a) If $n' =$ constant, then

$$\frac{\frac{n_{k-1}}{N_{k-1}}}{1 + \frac{n'}{N_{k-1}}} F(X_{n_{k-1}}, x y_{n'}) \to g_1(x y_{n'}) (\text{ here } g_1(x y_{n'}) = 1 \text{ for } x y_{n'} > 1)$$

$$\frac{1}{1 + \frac{N_{k-1}}{n'}} F(Y_{n'}, x) \to 0$$

and thus $F(Z_n, x) \to g_1(x y_{n'})$.

(b) If $n' \to \infty$, then $F(X_{n_{k-1}}, x y_{n'}) \to 1$; precisely $F(X_{n_{k-1}}, x y_{n'}) \to c_0(x)$.

(b1) If $\frac{n'}{N_{k-1}} \to 0$, then $F(Z_n, x) \to c_0(x)$.

(b2) If $\frac{n'}{N_{k-1}} \to \alpha \in (0, \infty)$, then $F(Z_n, x) \to \frac{1}{1+\alpha} c_0(x) + \frac{\alpha}{1+\alpha} g_2(x)$.

(b3) If $\frac{n'}{N_{k-1}} \to \infty$, then $F(Z_n, x) \to 0 + g_2(x)$.
For k-odd we use a similar computation. □

Now, identify $x_n = y_n$ and select x_n such that $g_1(x) = x$ (e.g., $x_n = n$ or $x_n = p_n$, the nth prime) and put $n_k = 2^{k^2}$ for $k = 1, 2, \dots$. Then the set of all d.f.s

$$G(Z_n) = \{g_1(x), c_0(x)\} \cup \{g_1(xx_n); n = 1, 2, \dots\}$$
$$\cup \left\{ \frac{1}{1+\alpha} c_0(x) + \frac{\alpha}{1+\alpha} g_1(x); \alpha \in [0, \infty) \right\}$$

is disconnected, as can be seen in the following Fig. 1.

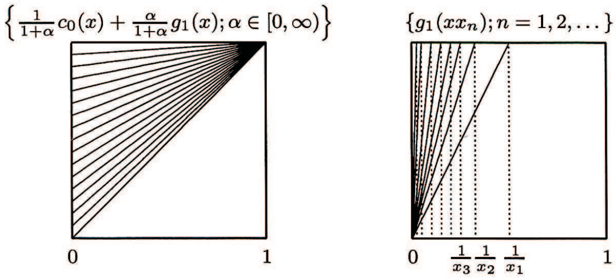

Figure 1

In [ST, Th. 3.2],is proved that if $g(x) \in G(X_n)$, $g(x)$ increases at $\beta \in [0, 1)$, $g(\beta) > 0$, then there exists $\alpha \in [1, \infty)$ such that $\alpha g(x\beta) \in G(X_n)$. Using this fact, we can define on $G(X_n)$ the relation $\tilde{g}(x) \prec g(x)$ if there exist α, β such that $\tilde{g}(x) = \alpha g(x\beta)$. For every element $g(x) \in G(X_n)$ we define $[g(x)]$ as the set of all $\tilde{g}(x) \in G(X_n)$ for which $\tilde{g}(x) \prec g(x)$. Assuming that all d.f.'s in $G(X_n)$ are continuous and strictly increasing, then we have

$$[g(x)] = \{g(x\beta)/g(\beta); \beta \in (0, 1]\}.$$

Denote as $G(g(x))$ the set of all possible limits $\lim_{k\to\infty} g(x\beta_k)/g(\beta_k)$, where $\beta_k \to 0$ and put

$$[g(x)]^* = [g(x)] \cup G(g(x)).$$

Theorem 2. *Assume that all d.f.s in $G(X_n)$ are continuous and strictly increasing. If $G(X_n) = \cup_{i=1}^{k} [g_i(x)]^*$, then $G(X_n)$ is connected if and only if $g_i(x)$, $i = 1, 2, \dots, k$ can be reordered into $g_{i_n}(x)$, $n = 1, 2, \dots, k$ such that*

(i) $[g_{i_n}(x)]^* \cap [g_{i_{n+1}}(x)]^* \neq \emptyset$, $n = 1, 2, \dots, k-1$.

P r o o f. 1^0. Firstly we prove that $[g(x)]^*$ is nonempty, closed and connected, for every $g(x) \in G(X_n)$. Note that, in the following we say that we can go connectively $g_1(x) \to g_2(x)$ through the set H if for every $\varepsilon > 0$ there exists a chain $g_{i_n}(x) \in H$, $n = 1, 2, \dots, m$ such that $\rho(g_1, g_{i_1}) < \varepsilon$, $\rho(g_{i_2}, g_{i_3}) < \varepsilon, \dots$, $\rho(g_{i_m}, g_2) < \varepsilon$.

Connectivity: If $g_1(x) = g(x\beta_1)/g(\beta_1)$ and $g_2(x) = g(x\beta_2)/g(\beta_2)$ then we can go connectively $g_1(x) \to g_2(x)$ through $g(x\beta)/g(\beta)$, β between β_1 and β_2, since

$$\frac{g(x\beta)}{g(\beta)} - \frac{g(x\beta')}{g(\beta')} = \left(\frac{g(x\beta) - g(x\beta')}{g(\beta)} + g(x\beta')\frac{g(\beta') - g(\beta)}{g(\beta)g(\beta')}\right) \to 0$$

as $(\beta' - \beta) \to 0$, where $\beta, \beta' \geq \varepsilon > 0$.

If $g_1(x) = \lim_{k\to\infty} g(x\beta_k)/g(\beta_k)$ and $g_2(x) = \lim_{k\to\infty} g(x\beta'_k)/g(\beta'_k)$, then we can go connectively

$$g_1(x) \to g(x\beta_k)/g(\beta_k) \to g(x\beta'_k)/g(\beta'_k) \to g_2(x)$$

through $[g(x)]$. Similarly for the rest

$$g_1(x) = g(x\beta_1)/g(\beta_1) \text{ and } g_2(x) = \lim_{k\to\infty} g(x\beta_k)/g(\beta_k).$$

Closedness: If $\lim_{k\to\infty} g(x\beta_k)/g(\beta_k) = g_1(x)$, we can select β_k such that $\beta_k \to \beta$. If $\beta > 0$, then from continuity $g(x)$ we have $g_1(x) = g(x\beta)/g(\beta)$. The closedness of $G(g(x))$ follows from definition of $G(g(x))$.

2^0. Assume that (i) holds and select $g^*_n(x) \in [g_{i_n}(x)]^* \cap [g_{i_{n+1}}(x)]^*$, $i = 1, 2, \dots, k-1$. Let $g_1(x) \in [g_{i_1}(x)]^*$ and $g_2(x) \in [g_{i_3}(x)]^*$. Then we can go connectively

$$g_1(x) \to \frac{g_{i_1}(x\beta_1)}{g_{i_1}(\beta_1)} \to g^*_1(x) \to \frac{g_{i_2}(x\beta_2)}{g_{i_2}(\beta_2)} \to g^*_2(x) \to \frac{g_{i_3}(x\beta_3)}{g_{i_3}(\beta_3)} \to g_2(x),$$

similarly in a general case.

3^0. Assume that (i) does not hold. Then $[g_i(x)]^*$, $i = 1, 2, \dots, k$, can be divided into two parts such that

$$(\cup_{i\in A}[g_i(x)]^*) \cap (\cup_{i\in B}[g_i(x)]^*) = \emptyset,$$

where $A \cup B = \{1, 2, \dots, k\}$. From closedness of such sets follows $\rho(g, \tilde{g}) \geq \delta > 0$ for some δ and every $g(x) \in \cup_{i\in A}[g_i(x)]^*$ and $\tilde{g}(x) \in \cup_{i\in B}[g_i(x)]^*$, which contradicts the connectivity of $G(X_n)$. □

3. The distances $\rho(t_{n+1}, t_n)$

In Part 2 we have proved that there exists x_n such that

$$\limsup_{n\to\infty} \rho(t_{n+1}, t_n) = 1.$$

This also can be proved by using (3) for $m = n + 1$

$$\rho^2(t_{n+1}, t_n) = \frac{1}{n(n+1)} \sum_{i=1}^{n+1} \sum_{j=1}^{n} \left| \frac{x_i}{x_{n+1}} - \frac{x_j}{x_n} \right| - \frac{1}{2(n+1)^2} \sum_{i,j=1}^{n+1} \left| \frac{x_i}{x_{n+1}} - \frac{x_j}{x_{n+1}} \right| - \frac{1}{2n^2} \sum_{i,j=1}^{n} \left| \frac{x_i}{x_n} - \frac{x_j}{x_n} \right|.$$

In addition, we prove that for every increasing sequence x_n of positive integers we have $\liminf_{n\to\infty} \rho(t_{n+1}, t_n) = 0$. To do this we use

Theorem 3. *For every increasing sequence of positive integers x_n and every $h = 1, 2, \dots$ we have*

(i) $\rho^2(t_{n+h}, t_n) \le \frac{1}{3} \left(\frac{x_{n+h} - x_n}{n+h} \right)^2 + \left(\frac{h}{n} \right)^2$;

(ii) $\rho^2(t_{n+h}, t_n) \le \frac{x_{n-1}}{x_n} + \left(\frac{h+1}{n+h} \right)^2$;

(iii) $\rho^2(t_{n+1}, t_n) \le o(1) + \left(1 - \frac{x_n}{x_{n+1}} \right) \frac{2}{(n+1)^2} \sum_{i=1}^{n} \frac{i x_i}{x_n}$;

(iv) $\rho^2(t_{n+1}, t_n) = o(1) + D_n$, *where* $D_n = \frac{1}{n^2} \sum_{1 \le i < j \le n} d_{n,i,j}$, *and*

$$d_{n,i,j} = \begin{cases} 0 & \text{if } \frac{x_j}{x_{n+1}} \ge \frac{x_i}{x_n} \\ 2\left(\frac{x_i}{x_n} - \frac{x_j}{x_{n+1}} \right) & \text{if } \frac{x_j}{x_{n+1}} < \frac{x_i}{x_n}. \end{cases}$$

Proof of (i). Since

$$F(X_n, x) = \frac{n+h}{n} F\left(X_{n+h}, x \frac{x_n}{x_{n+h}} \right)$$

and

$$0 \le F(X_{n+h}, x) - F\left(X_{n+h}, x\frac{x_n}{x_{n+h}}\right) = \frac{\#\{i \le n+h; x.x_n \le x_i < x.x_{n+h}\}}{n+h} \le x\frac{x_{n+h} - x_n}{n+h},$$

we have

$$\begin{aligned}
&\rho^2(t_{n+h}, t_n) = \\
&= \int_0^1 \left(F(X_{n+h}, x) - F\left(X_{n+h}, x\frac{x_n}{x_{n+h}}\right) - \frac{h}{n}F\left(X_{n+h}, x\frac{x_n}{x_{n+h}}\right)\right)^2 \mathrm{d}x \\
&\le \int_0^1 \left(F(X_{n+h}, x) - F\left(X_{n+h}, x\frac{x_n}{x_{n+h}}\right)\right)^2 \mathrm{d}x \\
&+ \left(\frac{h}{n}\right)^2 \int_0^1 \left(F\left(X_{n+h}, x\frac{x_n}{x_{n+h}}\right)\right)^2 \mathrm{d}x \\
&\le \int_0^1 \left(x\frac{x_{n+h} - x_n}{n+h}\right)^2 \mathrm{d}x + \left(\frac{h}{n}\right)^2
\end{aligned}$$

which gives (i). □

P r o o f o f (i i). As we can see, for every $x \in [0,1]$ we have

$$F(X_{n+h}, x) \ge F\left(X_{n+h}, x\frac{x_n}{x_{n+h}}\right)$$

and for $x = \frac{x_{n-1}}{x_n}$ we have

$$F\left(X_{n+h}, x\frac{x_n}{x_{n+h}}\right) = \frac{n-1}{n+h}$$

and thus

$$\begin{aligned}
&\int_0^1 \left(F(X_{n+h}, x) - F\left(X_{n+h}, x\frac{x_n}{x_{n+h}}\right)\right)^2 \mathrm{d}x \le \\
&\int_0^{\frac{x_{n-1}}{x_n}} 1 \cdot \mathrm{d}x + \int_{\frac{x_{n-1}}{x_n}}^1 \left(1 - \frac{n-1}{n+h}\right)^2 \mathrm{d}x
\end{aligned}$$

which gives (ii). □

Proof of (iii). Let us observe that

$$\rho^2(t_{n+1},t_n) =$$
$$= \frac{1}{n(n+1)}\sum_{i,j=1}^{n}\left|\frac{x_i}{x_{n+1}}-\frac{x_j}{x_n}\right| + \frac{1}{n(n+1)}\sum_{j=1}^{n}\left|1-\frac{x_j}{x_n}\right|$$
$$- \frac{1}{2(n+1)^2}\sum_{i,j=1}^{n}\left|\frac{x_i}{x_{n+1}}-\frac{x_j}{x_{n+1}}\right| - \frac{2}{2(n+1)^2}\sum_{j=1}^{n}\left|1-\frac{x_j}{x_{n+1}}\right|$$
$$\frac{1}{2n^2}\sum_{i,j=1}^{n}\left|\frac{x_i}{x_n}-\frac{x_j}{x_n}\right|$$

and

$$\frac{1}{n(n+1)}\sum_{i,j=1}^{n}\left|\frac{x_i}{x_{n+1}}-\frac{x_j}{x_n}\right| = \frac{1}{n(n+1)}\sum_{i,j=1}^{n}\left|\frac{x_i}{x_n}-\frac{x_j}{x_n}+\left(\frac{x_n}{x_{n+1}}-1\right)\frac{x_i}{x_n}\right| \le$$
$$\frac{1}{n(n+1)}\sum_{i,j=1}^{n}\left|\frac{x_i}{x_n}-\frac{x_j}{x_n}\right| + \left(1-\frac{x_n}{x_{n+1}}\right)\frac{1}{n(n+1)}\sum_{i,j=1}^{n}\frac{x_i}{x_n}.$$

Replacing

$$\left|\frac{x_i}{x_{n+1}}-\frac{x_j}{x_{n+1}}\right| = \frac{x_n}{x_{n+1}}\left|\frac{x_i}{x_n}-\frac{x_j}{x_n}\right|$$

and summing up coefficients for $\left|\frac{x_i}{x_n}-\frac{x_j}{x_n}\right|$, i.e.,

$$\frac{1}{n(n+1)}-\frac{1}{2n^2}-\frac{1}{2(n+1)^2}\frac{x_n}{x_{n+1}} = -\frac{1}{2n^2(n+1)^2}+\frac{1}{2(n+1)^2}\left(1-\frac{x_n}{x_{n+1}}\right)$$

we find

$$\rho^2(t_{n+1},t_n) \le \left(1-\frac{x_n}{x_{n+1}}\right)\frac{1}{2(n+1)^2}\sum_{i,j=1}^{n}\left|\frac{x_i}{x_n}-\frac{x_j}{x_n}\right| +$$
$$\left(1-\frac{x_n}{x_{n+1}}\right)\frac{1}{n+1}\sum_{i=1}^{n}\frac{x_i}{x_n}+o(1). \tag{a}$$

Now we prove that

$$\frac{1}{n^2}\sum_{i,j=1}^{n}\left|\frac{x_i}{x_n}-\frac{x_j}{x_n}\right| = \frac{4}{n^2}\sum_{i=1}^{n}\frac{ix_i}{x_n}-\frac{2}{n}\sum_{i=1}^{n}\frac{x_i}{x_n}+o(1) \tag{b}$$

for every increasing sequence x_n, $n=1,2,\dots$ of positive integers.

P r o o f o f (b). We start with a general formula [SP, p. 4–9]

$$\int_0^1\int_0^1 |x-y|\mathrm{d}g(x)\mathrm{d}g(y) = 2\left(\int_0^1 g(x)\mathrm{d}x - \int_0^1 g^2(x)\mathrm{d}x\right)$$

which holds for every d.f. $g(x)$. Putting $g(x) = F(X_n, x)$ and summing up

$$\int_0^1 F(X_n, x)\mathrm{d}x = \sum_{i=1}^{n-1}\left(\frac{i}{n}\right)\left(\frac{x_{i+1}}{x_n} - \frac{x_i}{x_n}\right),$$

$$\int_0^1 F^2(X_n, x)\mathrm{d}x = \sum_{i=1}^{n-1}\left(\frac{i}{n}\right)^2\left(\frac{x_{i+1}}{x_n} - \frac{x_i}{x_n}\right),$$

we have

$$\frac{1}{n^2}\sum_{i,j=1}^{n}\left|\frac{x_i}{x_n} - \frac{x_j}{x_n}\right| = 2\sum_{i=1}^{n-1}\left(\frac{x_{i+1}}{x_n} - \frac{x_i}{x_n}\right)\left(\left(\frac{i}{n}\right) - \left(\frac{i}{n}\right)^2\right).$$

The Abel partial summation on the right hand side gives (b). □

Now, substituting (b) into (a) we find (iii). □

P r o o f o f (i v). We shall use the following reductions of $\rho(t_{n+1}, t_n)$: In the two summations one can replace $n+1$ by n. That is, the expression

$$\frac{1}{n(n+1)}\sum_{i=1}^{n}\sum_{j=1}^{n}\left|\frac{x_i}{x_{n+1}} - \frac{x_j}{x_n}\right|$$
$$-\frac{1}{2(n+1)^2}\sum_{i,j=1}^{n}\left|\frac{x_i}{x_{n+1}} - \frac{x_j}{x_{n+1}}\right| - \frac{1}{2n^2}\sum_{i,j=1}^{n}\left|\frac{x_i}{x_n} - \frac{x_j}{x_n}\right|$$

has the same inferior and superior limits as $\rho(t_{n+1}, t_n)$. Then, we replace $n+1$ by n in front of summation symbols

$$\frac{1}{n^2}\sum_{i=1}^{n}\sum_{j=1}^{n}\left|\frac{x_i}{x_{n+1}} - \frac{x_j}{x_n}\right| - \frac{1}{2n^2}\sum_{i,j=1}^{n}\left|\frac{x_i}{x_{n+1}} - \frac{x_j}{x_{n+1}}\right| - \frac{1}{2n^2}\sum_{i,j=1}^{n}\left|\frac{x_i}{x_n} - \frac{x_j}{x_n}\right|.$$

The last expression is equal to

$$\frac{1}{n^2}\sum_{i=1}^{n}\sum_{j=1}^{n}\left\{\left|\frac{x_i}{x_{n+1}} - \frac{x_j}{x_n}\right| - \frac{1}{2}\left(\frac{1}{x_{n+1}} + \frac{1}{x_n}\right)|x_i - x_j|\right\}$$

The contribution of the terms corresponding to $i = j$ verifies

$$0 \le \frac{1}{n^2} \sum_{i=1}^{n} \left| \frac{x_i}{x_{n+1}} - \frac{x_j}{x_n} \right| \le \frac{1}{n}$$

Thus, we may add the condition $i \ne j$ and finally we find that

$$D_n = \frac{1}{n^2} \sum_{1 \le i < j \le n} d_{n,i,j},$$

where

$$d_{n,i,j} = \left| \frac{x_i}{x_{n+1}} - \frac{x_j}{x_n} \right| + \left| \frac{x_j}{x_{n+1}} - \frac{x_i}{x_n} \right| - \left(\frac{1}{x_{n+1}} + \frac{1}{x_n} \right) (x_j - x_i),$$

verifies $\lim_{n\to\infty}(\rho(t_{n+1}, t_n) - D_n) = 0$. We recall that $i < j$ so that $x_i < x_j \le x_n$ and this yields (iv). □

Note that using the sequence x_n defined in Example 1, we can see that $d_{n_{k-1}+1,i,j} = 2(1-0) + o(1)$ for $n_{k-2} + 1 \le i < j \le n_{k-1} + 1$, since

$$\frac{x_j}{x_{n_{k-1}+1}} \in \left[\frac{x_{n_{k-2}+1}}{x_{n_{k-1}+1}}, \frac{x_{n_{k-1}}}{x_{n_{k-1}+1}} \right] \to 0, \quad \frac{x_i}{x_{n_{k-1}}} \in \left[\frac{x_{n_{k-2}+1}}{x_{n_{k-1}}}, \frac{x_{n_{k-1}}}{x_{n_{k-1}}} \right] \to 1,$$

which gives $D_{n_{k-1}+1} = 1 + o(1)$ and thus $\limsup_{n\to\infty} \rho(t_{n+1}, t_n) = 1$ for such special sequence x_n, $n = 1, 2, \dots$. On the other hand we prove

Theorem 4. *For every increasing sequence x_n, $n = 1, 2, \dots$, of positive integers we have*

$$\liminf_{n\to\infty} \rho(t_{n+1}, t_n) = 0.$$

P r o o f. Either, there exists an increasing sequence n_k, $k = 1, 2, \dots$, of indices such that $\frac{x_{n_k}}{x_{n_k+1}} \to 1$ and then by (iii) we have $\rho(t_{n_k+1}, t_{n_k}) \to 0$; or, there exists $\delta > 0$ such that $\frac{x_n}{x_{n+1}} \le \delta < 1$ for every $n = 1, 2, \dots$. Since

$$\frac{x_i}{x_n} = \frac{x_i}{x_{i+1}} . \frac{x_{i+1}}{x_{i+2}} \dots \frac{x_{n-1}}{x_n}$$

the sum in the right hand side of (iii) has as upper bound

$$\sum_{i=1}^{n} i\delta^{n-i} = \delta^n \sum_{i=1}^{n} i\delta^{-i} = \delta^{-1} \frac{(\delta^n - (n+1))(1-\delta^{-1}) + \delta^{-1}(\delta^n - 1)}{(1-\delta^{-1})^2}.$$

Thus $\frac{1}{(n+1)^2} \sum_{i=1}^{n} i\delta^{n-i} \to 0$ which by (iii) implies $\rho(t_{n+1}, t_n) \to 0$ as $n \to \infty$. □

4. One-step d.f.'s $c_\alpha(x)$

In [ST] there is proved that singleton $G(X_n) = \{c_1(x)\}$ does not exist, since (by Theorem 7.1) for every increasing sequence x_n of positive integers we have

$$\max_{g(x)\in G(X_n)} \int_0^1 g(x)\mathrm{d}x \geq \frac{1}{2}.$$

One may ask whether the following implication is true

$$c_1(x) \in G(X_n) \Longrightarrow c_0(x) \in G(X_n). \tag{4}$$

We prove (4) only for some "small" class of sequences x_n called "almost arithmetic"and disprove it in general. We use the following criteria.

Theorem 5. *$c_1(x) \in G(X_n)$ is equivalent to the existence of two sequences $n'_k < n_k$, $k = 1, 2, \dots$, such that*

$$\left(\frac{x_{n'_k}}{x_{n_k}} \to 1\right) \text{ and } \left(\frac{n'_k}{n_k} \to 0\right), \tag{5}$$

and $c_0(x) \in G(X_n)$ is equivalent to the existence of $m'_k < m_k$, $k = 1, 2, \dots$, such that

$$\left(\frac{x_{m'_k}}{x_{m_k}} \to 0\right) \text{ and } \left(\frac{m'_k}{m_k} \to 1\right). \tag{6}$$

P r o o f. For a proof see Fig. 2

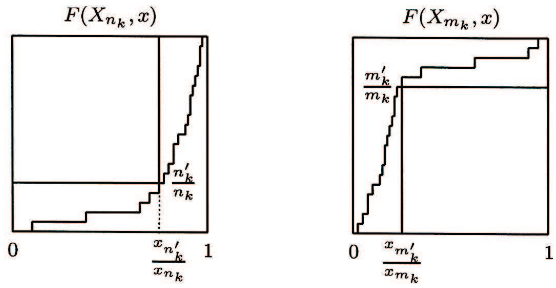

Figure 2

Example 3. Assume that the given integers $n'_k < n_k$ and $x_{n'_k} < x_{n_k}$, $k = 1, 2, \dots$, satisfy (5) and the other points $x_n \in [x_{n_{k-1}}, x_{n'_k}]$, for every fixed $k = 1, 2, \dots$ are equi-distributed, i.e.,

$$x_{n_{k-1}+i} = x_{n_{k-1}} + i\Delta_k, \quad i = 0, 1, 2, \dots, n'_k - n_{k-1} - 1. \tag{7}$$

Adding some consecutive intervals $[x_{n_{k-1}}, x_{n_k}]$, $[x_{n_k}, x_{n_{k+1}}]$, $[x_{n_{k+1}}, x_{n_{k+2}}]$ to an interval, we can obtain (without loss of generality) that

(i) $n_{k-1} < n'_k < n_k$ for $k = 1, 2, \dots$;

(ii) $\frac{n_{k-1}}{n_k} \to 0$;

(iii) $\frac{x_{n_{k-1}}}{x_{n_k}} \to 0$;

(iv) $\frac{n_{k-1}}{n'_k} \to 0$.

Because

$$\frac{x_{n'_k}}{x_{n_k}} \le \frac{x_{n_k} - (n_k - n'_k)}{x_{n_k}} = 1 - \frac{n_k}{x_{n_k}}\left(1 - \frac{n'_k}{n_k}\right)$$

the conditions (5) imply, for $k \to \infty$,

(v) $\frac{n_k}{x_{n_k}} \to 0$.

Using (5) and (i)-(iv)) we find

$$\Delta_k = \frac{x_{n'_k} - x_{n_{k-1}}}{n'_k - n_{k-1}} + O(1) = \frac{x_{n'_k}\left(1 - \frac{x_{n_{k-1}}}{x_{n'_k}}\right)}{n'_k\left(1 - \frac{n_{k-1}}{n'_k}\right)} + O(1) \sim \frac{x_{n_k}}{n'_k} \tag{8}$$

(here $f(k) \sim g(k)$ means that $\frac{f(k)}{g(k)} \to 1$). In the sequel we shall define m'_k and m_k as

$$m'_k = n_{k-1} \text{ and } m_k = n_{k-1} + i_k$$

for some $i_k \in [0, n'_k - n_{k-1} - 1]$. Then (6) is equivalent to

$$\frac{x_{n_{k-1}}}{x_{n_{k-1}+i_k}} = \frac{x_{n_{k-1}}}{x_{n_{k-1}} + i_k\Delta_k} = \frac{1}{1 + \frac{i_k\Delta_k}{x_{n_{k-1}}}} \to 0 \iff \frac{i_k\Delta_k}{x_{n_{k-1}}} \to \infty$$

$$\frac{n_{k-1}}{n_{k-1} + i_k} = \frac{1}{1 + \frac{i_k}{n_{k-1}}} \to 1 \iff \frac{i_k}{n_{k-1}} \to 0. \tag{9}$$

Denoting $\frac{i_k}{n_{k-1}} = \varepsilon_k$ and using (8) we can see that (9) is equivalent to

$$\varepsilon_k \frac{n_{k-1}}{n'_k}\frac{x_{n_k}}{x_{n_{k-1}}} \to \infty \wedge \varepsilon_k \to 0$$

and this is equivalent to

$$\frac{n_{k-1}}{n'_k}\frac{x_{n_k}}{x_{n_{k-1}}} \to \infty \tag{10}$$

since we can put

$$\varepsilon_k = \left(\frac{n_{k-1}}{n'_k}\frac{x_{n_k}}{x_{n_{k-1}}}\right)^{-1/2}.$$

Rewrite (10) in the form

$$\frac{1}{\frac{n'_k}{n_k}} \frac{\frac{x_{n_k}}{n_k}}{\frac{x_{n_{k-1}}}{n_{k-1}}} \to \infty$$

and since by (v) $\frac{x_{n_k}}{n_k} \to \infty$, we can assume, for some subsequences of n_k, that $\frac{x_{n_k}}{n_k}$ is increasing. Thus

$$\frac{\frac{x_{n_k}}{n_k}}{\frac{x_{n_{k-1}}}{n_{k-1}}} \geq 1$$

and since by (5) $\frac{1}{\frac{n'_k}{n_k}} \to \infty$, this implies (10) and finally (6).

□

L. Mišík (2004, personal communication) found the following sequence x_n for which $c_1(x) \in G(X_n)$ and $c_0(x) \notin G(X_n)$ and consequently (4) does not hold.

Example 4. Let x_n, $n = 1, 2, \dots$, be an increasing sequence of positive integers which satisfies the following conditions

(i) if $n_k = (k+1)(k-1)!2^{\frac{k(k-1)}{2}}$ for $k = 1, 2, \dots$, then $x_{n_k} = (k+1)n_k$,

(ii) if $n'_k = k(k-2)!2^{\frac{k(k-1)}{2}}$ then $x_{n'_k} = k^2 n'_k$,

(iii) if $n = 2^i n_{k-1} + j$, $0 \leq j < 2^i n_{k-1}$ and $0 \leq i < k-1$ for $k = 1, 2, \dots$, then $x_n = x_{n_{k-1}}(i+1)2^i + (i+3)kj$ (i.e. $n \in [n_{k-1}, n'_k]$),

(iv) if $n \in [n'_k, n_k]$ for $k = 1, 2, \dots$, then $x_n = x_{n'_k} + n - n'_k$.

Then for the sequence of blocks

$$X_n = \left(\frac{x_1}{x_n}, \frac{x_2}{x_n}, \dots, \frac{x_n}{x_n} \right)$$

we have $c_1(x) \in G(X_n)$ but $c_0(x) \notin G(X_n)$.[5]

P r o o f. We start with the following figure:

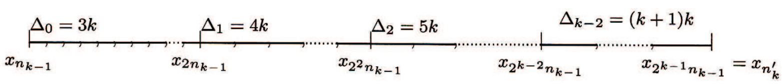

Figure 3

[5] This and the following Theorem 6 imply that $G(X_n) \not\subset \{c_\alpha(x); \alpha \in [0,1]\}$.

Here for n running through $[2^i n_{k-1}\ ,\ 2^{i+1}n_{k-1}]$, the x_n is equi-distributed in $[x_{2^i n_{k-1}}, x_{2^{i+1}n_{k-1}}]$ with difference Δ_i, where $i = 0, 1, \dots, k-2$.

1^0. Using the definition of x_n we can see that $\frac{x_{n'_k}}{x_{n_k}} \to 1$ and $\frac{n'_k}{n_k} \to 0$ and thus by Fig. 2 we have $c_1(x) \in G(X_n)$.

2^0. On the contrary, assume that there exists increasing sequence $m'_l < m_l$, $l = 1, 2, \dots$, such that $m'_l \in [n_{k-1}, n_k]$, $k = k(l)$, (i)$\frac{x_{m'_l}}{x_{m_l}} \to 0$ and (ii)$\frac{m'_l}{m_l} \to 1$ as $l \to \infty$.

a) If $[2^j n_{k-1}, 2^{j+1}n_{k-1}] \subset [m'_l, m_l]$ for some $0 \le j \le k-2$, then

$$\frac{m'_l}{m_l} \le \frac{2^j n_{k-1}}{2^{j+1}n_{k-1}} = \frac{1}{2}$$

which contradicts (ii).

b) If $[m'_l, m_l] \subset [2^j n_{k-1}, 2^{j+2}n_{k-1}]$, then

$$\frac{x_{m'_l}}{x_{m_l}} \ge \frac{x_{2^j n_{k-1}}}{x_{2^{j+2}n_{k-1}}} = \frac{(j+1)2^j}{(j+3)2^{j+2}} = \left(1 - \frac{2}{j+3}\right)\frac{1}{4}$$

which contradicts (i).

c) If $[n'_k, n_k] \subset [m'_l, m_l]$, then

$$\frac{m'_l}{m_l} \le \frac{n'_k}{n_k} \to 0$$

which contradicts (ii).

d) If $m'_l \in [2^{k-2}n_{k-1}, n'_k]$ and $m_l \in [n'_k, n_k]$, i.e. $m_l = n'_k + i$, then (because $n'_k = 2^{k-1}m_{k-1}$ and $x_{m_l} = x_{n'_k} + i$)

$$\frac{x_{m'_l}}{x_{m_l}} \ge \frac{x_{2^{k-2}n_{k-1}}}{x_{m_l}} = \frac{x_{2^{k-2}n_{k-1}}}{x_{2^{k-1}n_{k-1}}} \cdot \frac{x_{n'_k}}{x_{m_l}} = \left(\frac{k-1}{k}\right) \cdot \frac{1}{2} \cdot \frac{1}{1 + \frac{i}{x_{n'_k}}},$$

$$\frac{m'_l}{m_l} \le \frac{n'_k}{m_l} = \frac{1}{1 + \frac{i}{n'_k}}.$$

Furthermore, (i) implies $\frac{i}{n'_k} \to 0$ and (ii) implies $\frac{i}{x_{n'_k}} = \frac{i}{k^2 n'_k} \to \infty$ which is impossible.

e) If $[2n_k, 2^2 n_k] \subset [m'_l, m_l]$ then

$$\frac{m'_l}{m_l} \le \frac{2n_k}{2^2 n_k} = \frac{1}{2}$$

which contradicts (ii).

f) Finally, assume that $m'_l \in [n'_k, n_k]$ and $m_l \in [n_k, 2n_k]$. Since $x_{2n_k} = 4x_{n_k}$, we have

$$\frac{x_{m'_l}}{x_{m_l}} \geq \frac{x_{n'_k}}{x_{2n_k}} = \frac{x_{n'_k}}{4x_{n_k}} \to \frac{1}{4}$$

which contradicts (i). □

Note that, for an increasing sequence of positive integers x_n, the implication (4) is equivalent to

$$\limsup_{n\to\infty} \frac{1}{nx_n} \sum_{i=1}^{n} x_i = 1 \Rightarrow \liminf_{n\to\infty} \frac{1}{nx_n} \sum_{i=1}^{n} x_i = 0. \tag{11}$$

In [ST] it is also proved (see Th. 8.4, 8.5) that

$$G(X_n) \subset \{c_\alpha(x); \alpha \in [0,1]\} \Longleftrightarrow \lim_{n\to\infty} \frac{1}{n^2 x_n} \sum_{i,j=1}^{n} |x_i - x_j| = 0,$$

$$G(X_n) = \{c_0(x)\} \Longleftrightarrow \lim_{n\to\infty} \frac{1}{mn} \sum_{i=1}^{m} \sum_{j=1}^{n} \left| \frac{x_i}{x_m} - \frac{x_j}{x_n} \right| = 0,$$

$$G(X_n) = \{c_0(x)\} \Longleftrightarrow \lim_{n\to\infty} \frac{1}{nx_n} \sum_{i=1}^{n} x_i = 0. \tag{12}$$

Furthermore, if $G(X_n) \subset \{c_\alpha(x); \alpha \in [0,1]\}$, then $\underline{d}(x_n) = 0$ (see (v) in Part 1) and $\overline{d}(x_n) > 0$ implies $c_1(x) \in G(X_n)$ (see (vii) in Part 1). Here we prove that

THEOREM 6. *Let x_n, $n = 1, 2, \dots$, be an increasing sequence of positive integers. Assume that $G(X_n) \subset \{c_\alpha(x); \alpha \in [0,1]\}$. Then $c_0(x) \in G(X_n)$ and if $G(X_n)$ contains two different d.f.s, then also $c_1(x) \in G(X_n)$.*

P r o o f. We start from the equation (2) (see [ST, p. 756, (1)])

$$F(X_m, x) = \frac{n}{m} F\left(X_n, x\frac{x_m}{x_n}\right),$$

which is valid for every $m \leq n$ and $x \in [0,1]$. Assuming, for two increasing sequences of indices $m_k \leq n_k$, that, as $k \to \infty$

(i) $F(X_{m_k}, x) \to c_{\alpha_1}(x)$ a.e.,

(ii) $F(X_{n_k}, x) \to c_{\alpha_2}(x)$ a.e.,

(iii) $\frac{n_k}{m_k} \to \gamma$,

(iv) $\frac{x_{m_k}}{x_{n_k}} \to \beta$, (such sequences $m_k \leq n_k$ exist by Helly theorem) then we have:

a) If $\beta > 0$ and $\gamma < \infty$ (see (3) in [ST]), then

$$c_{\alpha_1}(x) = \gamma c_{\alpha_2}(x\beta) \tag{13}$$

for almost all $x \in [0,1]$.

b) If $\beta = 0$ and $\gamma < \infty$, then by Helly theorem there exists subsequence (m'_k, n'_k) of (m_k, n_k) such that $F\left(X_{n'_k}, x\frac{x_{m'_k}}{x_{n'_k}}\right) \to h(x)$ a.e. and since

$$F\left(X_{n_k}, x\frac{x_{m'_k}}{x_{n'_k}}\right) \leq F(X_{n_k}, x\beta')$$

for every $\beta' > 0$ and sufficiently large k, we get $h(x) \leq c_{\alpha_2}(x\beta')$. Summarizing, we have

$$c_{\alpha_1}(x) \leq \gamma c_{\alpha_2}(x\beta') \tag{14}$$

for every $\beta' > 0$ a.e. on $[0,1]$.

We distinguish the following steps (notions (i)-(iv), a) and b) are preserve):

1^0. Let $c_{\alpha_1}(x) \in G(X_n)$, $0 \leq \alpha_1 < 1$, and let m_k, $k = 1, 2, \dots$, be an increasing sequence of positive integers for which

(i)$F(X_{m_k}, x) \to c_{\alpha_1}(x)$.

Relatively to the m_k, we choose an arbitrary sequence n_k, $m_k \leq n_k$, such that

(iii)$\frac{n_k}{m_k} \to \gamma$, $1 < \gamma < \infty$.

From (m_k, n_k) we select a subsequence (m'_k, n'_k) such that

(ii)$F(X_{n'_k}, x) \to c_{\alpha_2}(x)$ a.e. on $[0,1]$,

(iv)$\frac{x_{m'_k}}{x_{n'_k}} \to \beta$ for some $\beta \in [0,1]$.

a) If $\beta > 0$, then (13) $c_{\alpha_1}(x) = \gamma c_{\alpha_2}(x\beta)$ a.e. is impossible, because $\gamma > 1$ and for $x > \alpha_1$ we have $c_{\alpha_1}(x) = 1$. Thus $\beta = 0$.

b) The condition $\beta = 0$ implies (14) $c_{\alpha_1}(x) \leq \gamma c_{\alpha_2}(x\beta')$ for every $\beta' > 0$ and a.e. on $x \in [0,1]$. If $\alpha_2 > 0$, then $c_{\alpha_2}(x\beta') = 0$ for all $x < \frac{\alpha_2}{\beta'}$, which implies, using $\beta' \leq \alpha_2$, that $c_{\alpha_1}(x) = 0$ for $x \in (0,1)$, and this is contrary to the assumption $\alpha_1 < 1$.

Thus $\alpha_2 = 0$ and we have: If $0 \leq \alpha_1 < 1$ and $c_{\alpha_1}(x) \in G(X_n)$ then $c_0(x) \in G(X_n)$. Now, applying [ST, Th. 7.1] we have $\max_{c_\alpha(x)\in G(X_n)} \int_0^1 c_\alpha(x)\mathrm{d}x = 1 - \alpha \geq \frac{1}{2}$. Then the assumption $c_{\alpha_1}(x) \in G(X_n)$, $0 \leq \alpha_1 < 1$ is true, thus $c_0(x) \in G(X_n)$ holds.

2^0. In this case we start with the sequence n_k and we assume that $c_{\alpha_2}(x) \in G(X_n)$, $0 < \alpha_2 \leq 1$, and

(ii)$F(X_{n_k}, x) \to c_{\alpha_2}(x)$ a.e. on $[0,1]$.

Then we choose arbitrary m_k such that $m_k \le n_k$ and

(iii) $\frac{n_k}{m_k} \to \gamma$, $1 < \gamma < \infty$.

From (m_k, n_k) we select a subsequence (m'_k, n'_k) such that

(ii) $F(X_{m'_k}, x) \to c_{\alpha_1}(x)$ a.e. on $[0,1]$,

(iv) $\frac{x_{m'_k}}{x_{n'_k}} \to \beta$ for some $\beta \in [0,1]$.

a) If $\beta > 0$, then by (13) $c_{\alpha_1}(x) = \gamma c_{\alpha_2}(x\beta)$ a.e. If $\alpha_1 < 1$, then $\gamma > 1$ implies $c_{\alpha_1}(x) > 1$ for some $x \in (0,1)$, a contradiction. Thus $\alpha_1 = 1$ (in this case $\beta \le \alpha_2$).

b) Now, $\beta = 0$ implies (14) $c_{\alpha_1}(x) \le \gamma c_{\alpha_2}(x\beta')$ for every $\beta' > 0$ and a.e. on $x \in [0,1]$ and the assumption $\alpha_2 > 0$ implies $c_{\alpha_2}(x\beta') = 0$ for all $x < \frac{\alpha_2}{\beta'}$, which gives $\alpha_1 = 1$. Summarizing, if $G(X_n)$ contains two different d.f.'s, then it contains $c_0(x)$ and $c_1(x)$ simultaneously. □

Remark 1. 1^0 a) implies $\frac{x_{m_k}}{x_{n_k}} \to 0$ and since

$$\frac{\#\{i \le n_k; \frac{x_i}{x_{n_k}} \le \frac{x_{m_k}}{x_{n_k}}\}}{n_k} = \frac{m_k}{n_k} \to \frac{1}{\gamma} > 0,$$

$G(X_n) \subset \{c_\alpha(x); \alpha \in [0,1]\}$ implies $F(X_{n_k}, x) \to c_0(x)$. E.g., for every $\gamma > 1$ the assumption (i) $F(X_{m_k}, x) \to c_{\alpha_1}(x)$, $0 \le \alpha_1 < 1$ implies $F(X_{[\gamma m_k]}, x) \to c_0(x)$.

Remark 2. If (ii) $F(X_{n_k}, x) \to c_{\alpha_2}(x)$, $0 < \alpha_2 \le 1$, then we can select sequences ε_k, $\underline{n}_k$, $\overline{n}_k$ such that

(j) $\varepsilon_k \to 0$,

(jj) $\frac{\overline{n}_k - \underline{n}_k}{n_k} \to 1$, and

(jjj) $\frac{x_{\underline{n}_k}}{x_{n_k}}, \frac{x_{\overline{n}_k}}{x_{n_k}} \in (\alpha_2 - \varepsilon_k, \alpha_2 + \varepsilon_k)$.

Then $F(X_{\overline{n}_k}, x) \to c_1(x)$, since for $i \in [\underline{n}_k, \overline{n}_k]$ we have

$$\frac{x_i}{x_{\overline{n}_k}} = \frac{x_i/x_{n_k}}{x_{\overline{n}_k}/x_{n_k}} \in \left[\frac{\alpha_2 - \varepsilon_k}{\alpha_2 + \varepsilon_k}, 1\right].$$

5. Transformation of X_n by $1/x \bmod 1$

The mapping $1/x \bmod 1$ transforms the block X_n to the block

$$Z_n = \left(\frac{x_n}{x_1}, \frac{x_n}{x_2}, \dots, \frac{x_n}{x_n}\right) \bmod 1.$$

For example, the block sequence $X_n = \left(\frac{1}{n}, \frac{2}{n}, \dots, \frac{n}{n}\right)$, $n = 1, 2, \dots$ which is u.d. is transformed to the block sequence

$$Z_n = \left(\frac{n}{1}, \frac{n}{2}, \dots, \frac{n}{n}\right) \bmod 1, \quad n = 1, 2, \dots$$

which has a.d.f.

$$g(x) = \int_0^1 \frac{1-t^x}{1-t}\mathrm{d}t = \sum_{n=1}^{\infty} \frac{x}{n(n+x)} = \gamma_0 + \frac{\Gamma'(1+x)}{\Gamma(1+x)},$$

where γ_0 is Euler's constant. This was proved by G. Pólya, (see I.J. Schoenberg [Sch]). The following theorem, which generalizes [KN, p.56, Th. 7.6] describes a relation between $G(X_n)$ and $G(Z_n)$.

Theorem 7. *If every $g(x) \in G(X_n)$ is continuous on $[0,1]$, then*

$$G(Z_n) = \left\{\tilde{g}(x) = \sum_{n=1}^{\infty} g(1/n) - g(1/(n+x)); g(x) \in G(X_n)\right\}.$$

P r o o f. For $f(x) = 1/x \bmod 1$ we have $f^{-1}([0,t)) = \cup_{i=1}^{\infty}(1/(t+i), 1/i]$. Thus $F(Z_n, t) = \sum_{i=1}^{\infty}(F(X_n, 1/i) - F(X_n, 1/(t+i)))$.

1^0. Assume that $F(X_{n_k}, x) \to g(x)$, where $g(x)$ is everywhere continuous on $[0,1]$. Thus

$$\sum_{i=1}^{K}(F(X_{n_k}, 1/i) - F(X_{n_k}, 1/(t+i))) \to \sum_{i=1}^{K}(g(1/i) - g(1/(t+i))),$$

$$\sum_{i=K+1}^{\infty} (F(X_{n_k}, 1/i) - F(X_{n_k}, 1/(t+i))) \le F(X_{n_k}, 1/(K+1)) \to g(1/(K+1)) \to 0.$$

Thus $F(Z_{n_k}, t) \to \tilde{g}(t) = \sum_{i=1}^{\infty}(g(1/i) - g(1/(t+i)))$ for $t \in [0,1]$.

2^0. Assume that $F(Z_{n_k}, t) \to \tilde{g}(t)$ weakly. From n_k there can be selected n'_k such that $F(X_{n'_k}, x) \to g(x)$. Assuming continuity of $g(x)$, we apply 1^0. □

Note that by [ST, Th. 4.1] all d.f.'s in $G(X_n)$ are continuous everywhere on $[0,1]$ if they are continuous at 0 and 1.

6. Open problems

The following questions remain open:

Q. 1. Characterize a nonempty closed set H of d.f.s for which there exists an increasing sequence of positive integers x_n such that $G(X_n) = H$.

Q. 2. Probably $\frac{x_n}{x_{n+1}} \to 1$ implies that $G(X_n)$ is a singleton. Note that $G(X_n)$ is a singleton if and only if the double limit $\lim_{m,n\to\infty} \rho(t_m, t_n) = 0$, cf. [ST, Th. 8.3].

Q. 3. Prove or disprove:

$$\lim_{n\to\infty} \frac{x_n}{x_1 + \cdots + x_n} = 0 \Longleftrightarrow c_0(x) \notin G(X_n).$$

If it is true, then $c_0(x) \notin G(X_n)$ gives necessary and sufficient conditions that the sequence $\omega = (Y_n)_{n=1}^{\infty}$ of blocks

$$Y_n = \left(\frac{1}{x_n}, \frac{2}{x_n}, \dots, \frac{x_n}{x_n}\right)$$

is u.d. For a theory of such block sequences Y_n, see Š. Porubský, T. Šalát and O. Strauch [PSS].

Q. 4. It remains open the theory of d.f. $G(X_n, Y_n)$ for two-dimensional blocks

$$(X_n, Y_n) = \left(\left(\frac{x_1}{x_n}, \frac{y_1}{y_n}\right), \left(\frac{x_2}{x_n}, \frac{y_2}{y_n}\right), \dots, \left(\frac{x_n}{x_n}, \frac{y_n}{y_n}\right)\right),$$

where x_n, $n = 1, 2, \dots$, and y_n, $n = 1, 2, \dots$ are increasing sequences of positive integers. E.g., it can be proved that the sequence $\left(\frac{p_i}{p_n}, \frac{i}{n}\right)$, $i = 1, 2, \dots, n$, is not u.d. in $[0,1]^2$, where p_n, $n = 1, 2, \dots$, is the increasing sequence of all primes (O. Strauch, a talk in 24th Journées Arithmétiques, Marseille, 2005).

Q. 5. Is the relation $\prec$ defined before Theorem 2 a partial ordering on $G(X_n)$? Note that if $G(X_n)$ contains only strictly increasing continuous d.f. and if $g_1(x) \prec g_2(x)$ and $g_2(x) \prec g_1(x)$, then for some $\beta \in (0,1]$ we have $\frac{g_1(x)}{g_1(y)} = \frac{g_1(x\beta)}{g_1(y\beta)}$ which implies $g_1(\beta^k) = (g_1(\beta)^k)$, $g_1((1-\beta)\beta) = g_1(1-\beta)g(\beta)$. The same for $g_2(x)$.

Q. 6. Characterize increasing sequences x_n, $n = 1, 2, \dots$, of positive integers for which $G(X_n)$ is connected.

Q. 7. Prove or disprove:

$$G(X_n) \subset \{c_\alpha(x); \alpha \in [0,1]\} \Longrightarrow G(X_n) = \{c_0(x)\},$$

for every increasing sequence x_n, $n = 1, 2, \dots$, of positive integers.

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