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ON DISTRIBUTION FUNCTIONS OF CERTAIN BLOCK SEQUENCES

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ABSTRACT. In this paper we study the sequence $x_1 < x_2 < \cdots <$ of positive integers for which $\frac{x_n}{x_{n+1}} \to 1$. In this case we prove that the set $G(X_n)$ of distribution functions of the sequence of blocks $X_n = \left(\frac{x_1}{x_n}, \frac{x_2}{x_n}, \dots, \frac{x_n}{x_n}\right)$ can be infinite. On the other hand if the ratio sequence $\frac{x_m}{x_n}$, $m = 1, 2, \dots, n$, $n = 1, 2, \dots$ is uniformly distributed, then $\frac{x_n}{x_{n+1}} \to 1$.

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1. Introduction

Let x_n, $n = 1, 2, \dots$ be an increasing sequence of positive integers. T. Šalát [10] in 1969 introduced and studied everywhere density in $[0, \infty)$ of the double sequence

$$\frac{x_m}{x_n}, \quad m, n = 1, 2, \dots \tag{1}$$

which is called ratio sequence. The density of (1) is equivalent to the everywhere density in $[0, 1]$ of the sequence

$$\frac{x_1}{x_1}, \frac{x_1}{x_2}, \frac{x_2}{x_2}, \frac{x_1}{x_3}, \frac{x_2}{x_3}, \frac{x_3}{x_3}, \dots, \frac{x_1}{x_n}, \frac{x_2}{x_n}, \dots, \frac{x_n}{x_n}, \dots \tag{2}$$

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It is also called ratio block sequence of x_n, $n = 1, 2, \ldots$ and we see that it is composed by blocks

$$X_n = \left(\frac{x_1}{x_n}, \frac{x_2}{x_n}, \ldots, \frac{x_n}{x_n}\right), \quad n = 1, 2, \ldots \tag{3}$$

which can be studied individually. The density of the sequence of individual blocks (3) implies the density of (1). Also, if the distribution functions of (2) or (3) are increasing, then again the sequence (1) is everywhere dense. This is a motivation for the study of sets $G(x_m/x_n)$ and $G(X_n)$ of distribution functions of (2) and (3), respectively (defined in Par. 2), cf. [14], [15] and [5].

The second motivation for study of block sequence (3) is that also other kinds of block sequences were studied by several authors, see [3], [6], [7], [8], [9], [11], [17], etc.

This note is focused on increasing integer sequences x_n, $n = 1, 2, \ldots$ satisfying

$$\lim_{n\to\infty} \frac{x_n}{x_{n+1}} = 1. \tag{4}$$

In Part 3 we prove that (4) does not imply that $G(X_n)$ is singleton. This gives the negative answer to the question recently published in [5, p. 76, Q.2] and in [12, Problem 1.9.2] [Unsolved Problem section (eds. O. Strauch and R. Nair)] placed on the home page http://udt.mat.savba.sk of the journal Uniform Distribution Theory.

In Part 4, assuming for singleton $G(x_m/x_n) = \{g(x)\}$ that $g(x) < 1$ for $x \in [0, 1)$, we prove (4). This proves that the uniform distribution of (2) implies (4).

2. Definitions and basic results

In the following we use standard notations and definitions from [7], [2], [13].

1. By distribution function we mean any function $g\colon [0, 1] \to [0, 1]$ such that $g(0) = 0$, $g(1) = 1$ and g is nondecreasing in $[0, 1]$.

2. For the block sequence X_n in (3) denote

$$A(X_n, x) = \#\{i \le n : \frac{x_i}{x_n} < x\} \quad \text{and} \quad F(X_n, x) = \frac{A(X_n, x)}{n}$$

for $x \in [0, 1)$ and $F(X_n, 1) = 1$.

3. Denote $G(X_n)$ the set of all distribution functions $g(x)$ for which there exists an increasing sequence of indices n_k, $k = 1, 2, \ldots$ such that

$$\lim_{k\to\infty} F(X_{n_k}, x) = g(x)$$

for all points $x \in [0,1]$ of continuity of $g(x)$, i.e. almost everywhere (abbrev. a.e.) in $[0,1]$. In other words $F(X_{n_k}, x)$ converges to $g(x)$ weakly.

4. Put $N = \frac{k(k+1)}{2} + l$ with $0 \le l < k+1$. Abbreviating $\overline{A}(X_n, x) = \sum_{j=1}^{n} A(X_j, x)$, for the ratio block sequence x_m/x_n in (2) denote

$$F_N(x_m/x_n, x) = \frac{\#\{(i,j) : 1 \le i \le j \le k, \frac{x_i}{x_j} < x\} + \#\{i : i \le l, \frac{x_i}{x_{k+1}} < x\}}{N}$$
$$= \frac{\overline{A}(X_k, x) + O(k)}{N} = \frac{\overline{A}(X_k, x)}{N} + O\left(\frac{1}{\sqrt{N}}\right), \tag{5}$$

where $x \in [0,1)$ and $F_N(x_m/x_n, 1) = 1$.

5. Similarly $G(x_m/x_n)$ denotes the set of all distribution functions $g(x)$ of block sequence (2) for which there exists an increasing sequence of indices N_k, $k = 1, 2, \dots$ such that

$$\lim_{k\to\infty} F_{N_k}(x_m/x_n, x) = g(x)$$

a.e. in $[0,1]$.

6. On the set of all distribution functions the L^2 metric is defined by

$$\rho(g_1, g_2) = \left(\int_0^1 (g_1(x) - g_2(x))^2 \mathrm{d}x\right)^{1/2}.$$

It is known: (i) $G(x_m/x_n)$ is nonempty, closed and connected (see [18]), similarly for $G(X_n)$, but $G(X_n)$ is not connected in general, cf. [5].

(ii) If $G(X_n) = \{g(x)\}$ is singleton then again $G(x_m/x_n) = \{g(x)\}$ (see [14]).

3. Counterexample

In this part we prove that $\frac{x_n}{x_{n+1}} \to 1$ does not imply that $G(X_n)$ is a singleton. This is a negative answer to the Problem 1.9.2 in [12].

Example 1. Let a_k, n_k, $k = 1, 2, \dots$, and x_n, $n = 1, 2, \dots$ be three increasing integer sequences and $h_1 < h_2$ be two positive integers. Assume that

(i) $\frac{n_k}{n_{k+1}} \to 0$ for $k \to \infty$;

(ii) $\frac{a_k}{n_{k+1}} \to 0$ for $k \to \infty$;

(iii) for odd k we have
$a_k^{h_2} \le x_{n_k} = (a_{k-1} + n_k - n_{k-1})^{h_1} \le (a_k + 1)^{h_2}$ and
$x_i = (a_k + i - n_k)^{h_2}$ for $n_k < i \le n_{k+1}$;

(iv) for even k we have
$a_k^{h_1} \le x_{n_k} = (a_{k-1} + n_k - n_{k-1})^{h_2} \le (a_k + 1)^{h_1}$ and
$x_i = (a_k + i - n_k)^{h_1}$ for $n_k < i \le n_{k+1}$.

Then $\frac{x_n}{x_{n+1}} \to 1$ and the set $G(X_n)$ of all distribution functions of the sequence of blocks X_n is $G(X_n) = G_1 \cup G_2 \cup G_3 \cup G_4$, where[1]

$G_1 = \{x^{\frac{1}{h_2}}.t; t \in [0,1]\}$,
$G_2 = \{x^{\frac{1}{h_2}}(1-t) + t; t \in [0,1]\}$,
$G_3 = \{\max(0, x^{\frac{1}{h_1}} - (1 - x^{\frac{1}{h_1}})u); u \in [0,\infty)\}$ and
$G_4 = \{\min(1, x^{\frac{1}{h_1}}.v); v \in [1,\infty)\}$.

Proof. **1.** Firstly we prove that for any $h_1 < h_2$ the sequences a_k, n_k, x_n satisfying (i)–(iv) exist:

For $i = 1, \dots, n_1$ we put $x_i = i^{h_1}$ and then we find a_1 such that $a_1^{h_2} \le x_{n_1} \le (a_1+1)^{h_2}$. If we have selected, for an odd step k, all a_i, $i = 1, 2, \dots, k-1$, x_i, $i = 1, 2, \dots, n_k$, then we find a_k such that $a_k^{h_2} \le x_{n_k} < (a_k + 1)^{h_2}$, and then we put $x_i = (a_k + i - n_k)^{h_2}$ for $n_k < i \le n_{k+1}$, where we choose n_{k+1} sufficiently large to satisfy the limits (i) and (ii). For an even step k we proceed similarly replacing h_2 by h_1.

2. In contrary to the independence of a_k and n_{k+1} we have

$$\frac{a_k}{n_k^{\frac{h_1}{h_2}}} \to 1 \text{ for odd } k \to \infty, \quad \frac{a_k}{n_k^{\frac{h_2}{h_1}}} \to 1 \text{ for even } k \to \infty. \tag{6}$$

This follows from (iii) and (iv), directly, e.g. from (iii) we have

$$\frac{a_k^{h_2}}{n_k^{h_1}} < \left(\frac{a_{k-1}}{n_k} + 1 - \frac{n_{k-1}}{n_k}\right)^{h_1} < \frac{(a_k+1)^{h_2}}{n_k^{h_1}}.$$

As an application of (6) we have

$$\frac{a_k}{n_k} \to 0 \text{ for odd } k \to \infty, \quad \frac{a_k}{n_k} \to \infty \text{ for even } k \to \infty. \tag{7}$$

3. Now we prove $\frac{x_i}{x_{i+1}} \to 1$ as $i \to \infty$. Let $i \in (n_k, n_{k+1})$ and let e.g. k be odd. Then by (iii)

$$\frac{x_i}{x_{i+1}} = \left(1 - \frac{1}{a_k + i + 1 - n_k}\right)^{h_2} > \left(1 - \frac{1}{a_k}\right)^{h_2}$$

[1] All functions in $G_1, \dots, G_4$ are defined for $x \in (0,1)$ and for $x = 0$ or $x = 1$ are assumed to be equal to 0 or 1, respectively.

and for $i = n_k$ again

$$\frac{x_{n_k}}{x_{n_k+1}} > \frac{a_k^{h_2}}{(a_k+1)^{h_2}} > \left(1 - \frac{1}{a_k}\right)^{h_2}$$

which implies the limit 1 as odd $k \to \infty$. Similarly for even k.

4. Let $N \in [n_k, n_{k+1}]$ be an integer sequence (we shall omit the index in N_k) for $k \to \infty$. For $x \in (0,1)$ we have

$$F(X_N, x) = \frac{\#\{1 \le i \le n_{k-1}; \frac{x_i}{x_N} < x\}}{N} + \frac{\#\{n_{k-1} < i \le n_k; \frac{x_i}{x_N} < x\}}{N} + \frac{\#\{n_k < i \le N; \frac{x_i}{x_N} < x\}}{N} = o(1) + \frac{A}{N} + \frac{B}{N}. \tag{8}$$

To compute $\frac{A}{N}$ for odd k we use

$$\frac{x_i}{x_N} = \frac{(a_{k-1} + i - n_{k-1})^{h_1}}{(a_k + N - n_k)^{h_2}} < x \Longleftrightarrow i - n_{k-1} < x^{\frac{1}{h_1}}(a_k + N - n_k)^{\frac{h_2}{h_1}} - a_{k-1}$$

and we have

$$\frac{A}{N} = \frac{\min(n_k - n_{k-1}, \max(0, [x^{\frac{1}{h_1}}(a_k + N - n_k)^{\frac{h_2}{h_1}} - a_{k-1}]))}{N}. \tag{9}$$

Similarly, for even k

$$\frac{A}{N} = \frac{\min(n_k - n_{k-1}, \max(0, [x^{\frac{1}{h_2}}(a_k + N - n_k)^{\frac{h_1}{h_2}} - a_{k-1}]))}{N}. \tag{10}$$

For $\frac{B}{N}$ and odd k we use

$$\frac{x_i}{x_N} = \left(\frac{a_k + i - n_k}{a_k + N - n_k}\right)^{h_2} < x \Longleftrightarrow i - n_k < x^{\frac{1}{h_2}}(a_k + N - n_k) - a_k$$

which gives

$$\frac{B}{N} = \frac{\min(N - n_k, \max(0, [x^{\frac{1}{h_2}}(a_k + N - n_k) - a_k]))}{N}. \tag{11}$$

Similarly, for even k we have

$$\frac{B}{N} = \frac{\min(N - n_k, \max(0, [x^{\frac{1}{h_1}}(a_k + N - n_k) - a_k]))}{N}. \tag{12}$$

In the following we will distinguish three cases

$$\frac{n_k}{N} \to t > 0, \quad \frac{n_k}{N} \to 0 \text{ and } \frac{N}{n_{k+1}} \to 0, \quad \text{and } \frac{N}{n_{k+1}} \to t > 0.$$

5. Now, let $\frac{n_k}{N} \to t > 0$ as $k \to \infty$.

a) Assume that k is odd and compute the limit of $\frac{A}{N}$ by (9). We have $\frac{n_k - n_{k-1}}{N} \to t$ and if $t < 1$ we see

$$x^{\frac{1}{h_1}} \left(\frac{a_k}{N^{\frac{h_1}{h_2}}} + \frac{N}{N^{\frac{h_1}{h_2}}} \left(1 - \frac{n_k}{N}\right) \right)^{\frac{h_2}{h_1}} - \frac{a_{k-1}}{N} \to \infty$$

since $\frac{N}{N^{\frac{h_1}{h_2}}}$ for $h_1 < h_2$ is unbounded and by (6)

$$\frac{a_k}{N^{\frac{h_1}{h_2}}} = \frac{a_k}{n_k^{\frac{h_1}{h_2}}} \left(\frac{n_k}{N}\right)^{\frac{h_1}{h_2}} \to t^{\frac{h_1}{h_2}}.$$

is bounded. Thus, for $0 < t < 1$, we have

$$\frac{A}{N} \to t \text{ for odd } k \to \infty. \tag{13}$$

a1) Let for the moment $t = 1$. We have $\frac{a_k}{n_k^{\frac{h_1}{h_2}}} \to 1$ and

$$x^{\frac{1}{h_1}} \left(\frac{a_k}{N^{\frac{h_1}{h_2}}} + \frac{N - n_k}{N^{\frac{h_1}{h_2}}} \right)^{\frac{h_2}{h_1}} - \frac{a_{k-1}}{N} \to x^{\frac{1}{h_1}} (1 + u)^{\frac{h_2}{h_1}}$$

assuming the limit $\frac{N - n_k}{N^{\frac{h_1}{h_2}}} \to u$, where $u \in [0, \infty)$ can be arbitrary. Put $v = (1 + u)^{\frac{h_2}{h_1}}$. Thus for $t = 1$ and corresponding $v \in [1, \infty)$ we have

$$\frac{A}{N} \to \min(1, x^{\frac{1}{h_1}} v) \text{ for odd } k \to \infty. \tag{14}$$

If $\frac{N - n_k}{N^{\frac{h_1}{h_2}}} \to \infty$, then

$$\frac{A}{N} \to 1 \text{ for odd } k \to \infty. \tag{15}$$

b) Now, again $0 < t \le 1$. For even k in (10) we have

$$x^{\frac{1}{h_2}} \left(\frac{a_k}{N^{\frac{h_2}{h_1}}} + \frac{N}{N^{\frac{h_2}{h_1}}} \left(1 - \frac{n_k}{N}\right) \right)^{\frac{h_1}{h_2}} - \frac{a_{k-1}}{N} \to x^{\frac{1}{h_2}}.t$$

since by (6)

$$\frac{a_k}{N^{\frac{h_2}{h_1}}} = \frac{a_k}{n_k^{\frac{h_2}{h_1}}} \left(\frac{n_k}{N}\right)^{\frac{h_2}{h_1}} \to t^{\frac{h_2}{h_1}}.$$

Thus

$$\frac{A}{N} \to x^{\frac{1}{h_2}}.t \text{ for even } k \to \infty. \tag{16}$$

c) For the limit $\frac{B}{N}$ as odd $k \to \infty$ we compute (11) by using $\frac{N-n_k}{N} \to 1-t$ and

$$x^{\frac{1}{h_2}}\left(\frac{a_k}{N}+1-\frac{n_k}{N}\right)-\frac{a_k}{N} \to x^{\frac{1}{h_2}}(1-t)$$

since by (7) we have $\frac{a_k}{N}=\frac{a_k}{n_k}\frac{n_k}{N} \to 0$. Thus

$$\frac{B}{N} \to x^{\frac{1}{h_2}}(1-t) \text{ for odd } k \to \infty. \tag{17}$$

d) Again by (7), for even k we have $\frac{a_k}{N}=\frac{a_k}{n_k}\frac{n_k}{N} \to \infty$, then (assuming $x<1$)

$$x^{\frac{1}{h_1}}\left(\frac{a_k}{N}+1-\frac{n_k}{N}\right)-\frac{a_k}{N} \to -\infty.$$

Thus

$$\frac{B}{N} \to 0 \text{ for even } k \to \infty. \tag{18}$$

e) Summing up (13), (16), (17) and (18) we find, for every $x \in (0,1)$,

$$F(X_N,x) \to \begin{cases} x^{\frac{1}{h_2}}(1-t)+t & \text{for odd } k \to \infty,\\ x^{\frac{1}{h_2}}.t & \text{for even } k \to \infty \end{cases} \tag{19}$$

for $\frac{n_k}{N} \to t$, $0<t<1$. For $\frac{n_k}{N} \to t=1$, $\frac{N-n_k}{N^{\frac{h_1}{h_2}}} \to u$ and $v=(1+u)^{\frac{h_2}{h_1}}$ we have applying (14)

$$F(X_N,x) \to \min(1, x^{\frac{1}{h_1}}.v) \quad \text{for odd } k \to \infty, \tag{20}$$

and for $\frac{N-n_k}{N^{\frac{h_1}{h_2}}} \to \infty$ we have

$$F(X_N,x) \to c_0(x) \quad \text{for odd } k \to \infty, \tag{21}$$

where $c_0(x)=1$ for $x \in (0,1)$.

6. In the case $\frac{n_k}{N} \to 0$ and $\frac{N}{n_{k+1}} \to 0$ we have $\frac{A}{N}=o(1)$ and then it suffices to compute the limit $\frac{B}{N}$ by (11) or (12).

a) Assume that odd $k \to \infty$. Since $\frac{N-n_k}{N} \to 1$ and by (7) we have $\frac{a_k}{N}=\frac{a_k}{n_k}\frac{n_k}{N} \to 0$ and thus

$$x^{\frac{1}{h_2}}\left(\frac{a_k}{N}+1-\frac{n_k}{N}\right)-\frac{a_k}{N} \to x^{\frac{1}{h_2}}. \tag{22}$$

b) Assume that even $k \to \infty$. In this case (by (6) and (ii)) we have

$$\frac{a_k}{N}=\frac{a_k}{n_k^{\frac{h_2}{h_1}}}\frac{n_k^{\frac{h_2}{h_1}}}{N}, \quad \frac{a_k}{n_k^{\frac{h_2}{h_1}}} \to 1, \quad \frac{a_k}{n_{k+1}} \to 0, \quad \text{then } \frac{n_k^{\frac{h_2}{h_1}}}{n_{k+1}} \to 0.$$

Thus, for any $u \in [0, \infty)$ we can find a subsequence of N such that

$$\frac{n_k^{\frac{h_2}{h_1}}}{N} \to u. \tag{23}$$

Then

$$x^{\frac{1}{h_1}} \left(\frac{a_k}{N} + 1 - \frac{n_k}{N} \right) - \frac{a_k}{N} \to x^{\frac{1}{h_1}} - (1 - x^{\frac{1}{h_1}})u. \tag{24}$$

c) Summing up (22) and (24) we find for every $x \in (0, 1)$

$$F(X_N, x) \to \begin{cases} x^{\frac{1}{h_2}} & \text{for odd } k \to \infty, \\ \max(0, x^{\frac{1}{h_1}} - (1 - x^{\frac{1}{h_1}})u) & \text{for even } k \to \infty \end{cases} \tag{25}$$

for $\frac{n_k}{N} \to 0$, $\frac{N}{n_{k+1}} \to 0$ and for $u \in (0, \infty)$ satisfying (23) if k is even. If $\frac{n_k^{\frac{h_2}{h_1}}}{N} \to \infty$ then

$$F(X_N, x) \to c_1(x) \quad \text{for even } k \to \infty, \tag{26}$$

where $c_1(x) = 0$ for $x \in (0, 1)$.

7. Finally, let $\frac{N}{n_{k+1}} \to t > 0$. Then $\frac{a_k}{N} \to 0$, because (ii) $\frac{a_k}{n_{k+1}} \to 0$. Computing the limit $\frac{B}{N}$ by (11) or (12) we find

$$F(N_N, x) \to \begin{cases} x^{\frac{1}{h_2}} & \text{for odd } k \to \infty, \\ x^{\frac{1}{h_1}} & \text{for even } k \to \infty. \end{cases} \tag{27}$$

8. Now, assume that $F(X_N, x) \to g(x)$ for some sequence of $N \in [n_k, n_{k+1}]$, i.e. $g(x) \in G(X_n)$. Then we can find subsequence of N (denoting again as N) such that $\frac{n_k}{N}$, $\frac{N - n_k}{N^{\frac{h_1}{h_2}}}$, $\frac{N}{n_{k+1}}$, and $\frac{n_k^{\frac{h_2}{h_1}}}{N}$ converge. Consequently $g(x)$ is contained in the collection of (19), (20), (21), (25), (26) and (27).

Thus the proof is finished. □

Remark 1. In [14, Th. 3.2] (see [5, p. 56, (i)]) it is proved that if $G(X_n)$ contains only continuous distribution functions then for every distribution function $g(x)$

$$g(x) \in G(X_n) \Longrightarrow \frac{g(xy)}{g(y)} \in G(X_n) \tag{28}$$

assuming $y \in (0, 1)$, $g(y) > 0$ and $g(x)$ increases at y.

Our $G(X_n) = G_1 \cup G_2 \cup G_3 \cup G_4$ is closed under implication (28), e.g. for $g(x) \in G_3$, i.e. $g(x) = \max(0, x^{\frac{1}{h_1}} - (1 - x^{\frac{1}{h_1}})u)$ and $y \in (0, 1)$ for which

$g(y) > 0$ we have

$$\frac{g(xy)}{g(y)} = \max\left(0, \frac{(xy)^{\frac{1}{h_1}} - (1-(xy)^{\frac{1}{h_1}})u}{y^{\frac{1}{h_1}} - (1-y^{\frac{1}{h_1}})u}\right) = \max\left(0, x^{\frac{1}{h_1}} + \frac{-u + x^{\frac{1}{h_1}}u}{y^{\frac{1}{h_1}} - (1-y^{\frac{1}{h_1}})u}\right)$$

$$= \max(0, x^{\frac{1}{h_1}} - (1-x^{\frac{1}{h_1}})u'), \qquad \text{where} \quad u' = \frac{-u + x^{\frac{1}{h_1}}u}{y^{\frac{1}{h_1}} - (1-y^{\frac{1}{h_1}})u}.$$

Remark 2. In [14, Th. 5.2, p. 762] it is proved that the condition $\frac{x_n}{x_{n+1}} \to 1$ implies the connectivity of $G(X_n)$ with respect to the L^2 metric (see also [5, Th.3(iii)]). Here a set H of distribution functions is connected if for every two $g_1, g_2 \in H$ and every $\varepsilon > 0$ there exists $g_{n_1}, \dots, g_{n_k} \in H$ such that $\rho(g_1, g_{n_1}) < \varepsilon$, $\rho(g_{n_i}, g_{n_{i+1}}) < \varepsilon$, $i = 1, 2, \dots, k-1$, and $\rho(g_{n_k}, g_2) < \varepsilon$. We shall say that from g_1 can be gone to g_2 through H.

In this remark we prove, directly, that the set $G(X_n) = G_1 \cup G_2 \cup G_3 \cup G_4$ is connected. It follows from:

(i) From $x^{\frac{1}{h_1}}$ can be gone to $c_0(x)$ through G_4;

(ii) from $c_0(x)$ can be gone to $x^{\frac{1}{h_2}}$ through G_2;

(iii) from $x^{\frac{1}{h_2}}$ can be gone to $c_1(x)$ through G_1, and

(iv) from $x^{\frac{1}{h_1}}$ can be gone to $c_1(x)$ through G_3, again.

4. Singleton $G(x_m/x_n)$

Theorem 1. *Let $G(x_m/x_n) = \{g(x)\}$ and $g(x) < 1$ for all $0 \le x < 1$. Then* $\lim_{n\to\infty} \frac{x_n}{x_{n+1}} = 1$.

Proof. Suppose that $G(x_m/x_n) = \{g(x)\}$ and $g(x) < 1$ holds for every $0 \le x < 1$ and, in addition, $\liminf_{n\to\infty} \frac{x_n}{x_{n+1}} < 1$. We are going to derive a contradiction. Our assumptions obviously imply that (cf. (5))

$$\lim_{n\to\infty} \frac{\overline{A}(X_n, x)}{\frac{n(n+1)}{2}} = g(x) \tag{29}$$

holds for every $x \in [0,1)$ and also there exists an $\varepsilon \in (0,1)$ and an increasing sequence of positive integers m_k, $k = 1, 2, \dots$ such that the inequality $\frac{x_{m_k}}{x_{m_k+1}} < \varepsilon$ holds for all $k \in \mathbb{N}$. Put $x = \varepsilon$ and choose an α such that

$$\begin{cases} 0 < \alpha < \frac{2}{g(x)} - 2 & \text{if } g(x) \ne 0, \\ 0 < \alpha \text{ arbitrary} & \text{if } g(x) = 0. \end{cases} \tag{30}$$

The existence of such an α is guaranteed by $g(x) < 1$. Then the following equality obviously holds

$$\overline{A}(X_{[m_k+\alpha m_k]}, x) = \overline{A}(X_{m_k}, x) + \sum_{i=1}^{[\alpha m_k]} A(X_{m_k+i}, x)\,. \tag{31}$$

As $\frac{x_{m_k}}{x_{m_k+i}} \le \frac{x_{m_k}}{x_{m_k+1}} \le \varepsilon = x$ holds for every $i = 1, 2, \dots [\alpha m_k]$ we obtain $A(X_{m_k+i}, x) \ge m_k$ for all $i = 1, 2, \dots [\alpha m_k]$. This and (31) imply

$$\overline{A}(X_{[m_k+\alpha m_k]}, x) \ge \overline{A}(X_{m_k}, x) + [\alpha m_k] m_k\,.$$

Thus

$$g(x) = \lim_{k\to\infty} \frac{\overline{A}(X_{[m_k+\alpha m_k]}, x)}{\frac{[m_k+\alpha m_k]([m_k+\alpha m_k]+1)}{2}} \ge \lim_{k\to\infty} \frac{\overline{A}(X_{m_k}, x) + [\alpha m_k] m_k}{\frac{[m_k+\alpha m_k]([m_k+\alpha m_k]+1)}{2}}$$

$$= \lim_{k\to\infty} \frac{\frac{\overline{A}(X_{m_k}, x)}{\frac{m_k(m_k+1)}{2}} \frac{m_k(m_k+1)}{2} + [\alpha m_k] m_k}{\frac{[m_k+\alpha m_k]([m_k+\alpha m_k]+1)}{2}} = \frac{g(x)}{(1+\alpha)^2} + \frac{2\alpha}{(1+\alpha)^2}.$$

Rearranging the last equality we obtain

$$\alpha \ge \frac{2}{g(x)} - 2 \qquad \text{if} \qquad g(x) \ne 0$$

and

$$0 \ge \frac{2\alpha}{1+\alpha^2} \qquad \text{if} \qquad g(x) = 0\,,$$

contradicting (30). □

Corollary 1. *If the ratio sequence* $\frac{x_m}{x_n}$, $m = 1, 2, \dots, n$, $n = 1, 2, \dots,$ *is uniformly distributed, i.e.* $G(x_m/x_n) = \{g(x) = x\}$, *then* $\lim_{n\to\infty} \frac{x_n}{x_{n+1}} = 1$. *The same also holds for* $g(x) = x^\lambda$, $\lambda \in (0, 1]$.

Remark 3. For block sequence X_n, $n = 1, 2, \dots,$ with singleton $G(X_n) = \{g(x)\}$ in [14, Th. 8.2] is proved that either $g(x) = x^\lambda$, $\lambda \in (0, 1]$ or $g(x) = c_0(x)$, where $c_0(x) = 1$ for $x \in (0, 1]$ and $c_0(0) = 0$. Thus the condition $g(x) < 1$ implies $g(x) = x^\lambda$ and the limit $\frac{x_n}{x_{n+1}} \to 1$ follows directly from $G(X_n) = \{g(x)\} \Longrightarrow$

$G(x_m/x_n) = \{g(x)\}$ and from Corollary 1. Another method is given by [14, (1), p. 756] the relation $(m \le n)$ $F(X_m, x) = \frac{n}{m} F\left(X_n, x\frac{x_m}{x_n}\right)$ which for $m = m_k$, $n = m_k + 1$, $\frac{x_{m_k}}{x_{m_k+1}} \to \beta$ implies $\beta = 1$ since it gives $g(x) = g(x\beta)$ for $\beta > 0$.

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