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uniform distribution theory

DISTRIBUTION OF THE SEQUENCE $p_n/n \bmod 1$

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ABSTRACT. In this paper we show that the sequence $p_n/n \bmod 1$, $n = 1, 2, \dots$, where p_n is the nth prime and the sequence $\log n \bmod 1$, $n = 1, 2, \dots$ have the same set of distribution functions. The presented proof is divided into two steps. In Step 1 we derive that $p_n/n \bmod 1$ has the same set of distribution functions as the sequence $\log(n \log n) \bmod 1$ and in Step 2 we prove that $\log(n \log n) \bmod 1$, has the same set of distribution functions as $\log n \bmod 1$. Step 2 is based on a new method for computing distribution functions of $x_n + y_n \bmod 1$ from the distribution functions of the two–dimensional sequence $(x_n, y_n) \bmod 1$ by using the Riemann-Stieltjes integration of related two-dimensional functions. We apply this method to $x_n = \log n \bmod 1$ and $y_n = \log\log n \bmod 1$, $n = 2, 3, \dots$. In an alternative Step 2 we use some generalization of a result of Koksma on distribution functions.

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1. Introduction

In this paper we study the following sequences:

(i) $\frac{p_n}{n} \bmod 1, n = 1, 2, \dots$, where p_n denotes the nth prime;

(ii) $\log n \bmod 1, n = 1, 2, \dots$;

(iii) $\log(n \log n) \bmod 1, n = 2, 3, \dots$;

(iv) $(\log n, \log\log n) \bmod 1, n = 2, 3, \dots$.

We prove that the sequences (i), (ii) and (iii) have the same distribution. Our proof is divided into two steps. In Step 1 we derive that (i) and (iii) have the same distribution and in Step 2 we prove the coincidence of the distributions of (iii) and (ii) by using the distribution of (iv). Here we identify the notion of distribution of a sequence $x_n \bmod 1$, $n = 1, 2, \dots$, with the set $G(x_n \bmod 1)$ of all distribution functions (abbreviating d.f.s) of the sequence $x_n \bmod 1$,

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$n = 1, 2, \dots$, see definitions in Part 2. In Step 2 we present a new general method of finding the distribution of $(x_n + y_n) \bmod 1$ from the distribution of $(x_n, y_n) \bmod 1$. This is a main result of the paper, but its application to (iii) is complicated. In an alternative Step 2 we present a simpler method (not such general), using some extension of the Koksma theorem (1933)[K1] (see [KN, p. 58, Th. 7.7]). We also apply it to the sequence

(v) $\log(n \log^{(i)} n) \bmod 1$, $n = n_0, n_0 + 1, \dots$, where $\log^{(i)} x$ is the ith iterated logarithm, $i = 0, 1, 2, \dots$,

which again has the same distribution as (ii).

A plan of this paper is the following: After definitions and notations (Part 2), we describe our method and results without proofs (Part 3), and then we add proofs of results one by one (Part 4–10).

2. Definitions and notations

Consult monographs [KN], [DT] and [SP].

- $x \bmod 1$ or $\{x\}$ denotes the fractional part of the real number x.
- $[x]$ is the integer part of x.
- A function $g : [0, 1] \to [0, 1]$ will be called *distribution function* (shortly d.f.) if
  - (i) $g(0) = 0$, $g(1) = 1$, and
  - (ii) g is nondecreasing.

• Two d.f.s $g(x)$ and $\tilde{g}(x)$ are called equivalent if they have the same values for all common points of continuity, or equivalently, if $g(x) = \tilde{g}(x)$ a.e. In the following we shall not distinguish them.

• Let x_n be an infinite sequence of real numbers from the unit interval $[0, 1)$ (in the opposite case we will use $x_n \bmod 1$). For the initial segment $x_1, \dots, x_N$, let the *step* d.f. $F_N(x)$ be defined as

$$F_N(x) = \frac{1}{N} \sum_{n=1}^{N} c_{[0,x)}(x_n)$$

for all $x \in [0, 1)$ and $F_N(1) = 1$. Here $c_{[0,x)}$ is the *characteristic function* of the interval $[0, x)$.

• A d.f. $g(x)$ is a *distribution function of the sequence* x_n if there exists an increasing sequence of positive integers N_k, $k = 1, 2, \dots$, such that $\lim_{k\to\infty} F_{N_k}(x)$

$= g(x)$ a.e. on $[0,1]$, i.e., in all points x of continuity of $g(x)$ (this *weak* definition differs from the point-wise definition given in [KN, p. 53, Def. 7.2]).

• $G(x_n)$ denotes the set of all d.f.s of a given sequence x_n. If $G(x_n) = \{g\}$, then g is said to be *the limit law* or *asymptotic d.f.* (abbreviating a.d.f.) of x_n. If $g(x) = x$, then x_n is called *uniformly distributed* (shortly u.d.) sequence.

• The *lower d.f.* $\underline{g}(x)$ and the *upper d.f.* $\overline{g}(x)$ of the sequence x_n are defined by $\underline{g}(x) = \liminf_{N\to\infty} F_N(x)$ and $\overline{g}(x) = \limsup_{N\to\infty} F_N(x)$. Note that these d.f.s need not be d.f.s of x_n, see [SP, p. 1–11].

• A two-dimensional function $g : [0,1]^2 \to [0,1]$ is a d.f. (see[SP, p. 1–61]) if

(i) $g(0,0) = 0$, $g(1,1) = 1$,
(ii) $g(x,0) = 0$, $g(0,y) = 0$, if $0 \le x \le 1$, $0 \le y \le 1$, and
(iii) $g(x,y)$ is nondecreasing, i.e. for every point $(x,y) \in [0,1]^2$ its differential satisfies
$$\mathrm{d}g(x,y) = g(x,y) + g(x+\mathrm{d}x, y+\mathrm{d}y) - g(x+\mathrm{d}x, y) - g(x, y+\mathrm{d}y) \ge 0.$$

• In the following we shall not distinguish two d.f.s $g(x,y)$, $\tilde{g}(x,y)$ for which

(i) $g(x,y) = \tilde{g}(x,y)$ for all common points of continuity $(x,y) \in (0,1)^2$,
(ii) $g(x,1) = \tilde{g}(x,1)$ for all common points of continuity $x \in (0,1)$, and
(iii) $g(1,y) = \tilde{g}(1,y)$ for all common points of continuity $y \in (0,1)$.

• Similarly, for a two-dimensional sequence (x_n, y_n), $n = 1, 2, \dots, N$ in $[0,1)^2$, we denote the step d.f. by

$$F_N(x,y) = \frac{1}{N} \sum_{n=1}^{N} c_{[0,x)\times[0,y)}((x_n, y_n))$$

if $(x,y) \in [0,1)^2$.

• $G((x_n,y_n))$ denotes the set of all d.f.s $g(x,y)$ which are weak limits $F_{N_k}(x,y) \to g(x,y)$ for suitable sequences of indices N_k, $k \to \infty$, i.e. the following holds:

(i) $F_{N_k}(x,y) \to g(x,y)$ for all points $(x,y) \in (0,1)^2$ of continuity of $g(x,y)$,
(ii) $F_{N_k}(x,1) \to g(x,1)$ for all points $x \in (0,1)$ of continuity of $g(x,1)$,
(iii) $F_{N_k}(1,y) \to g(1,y)$, for all points $y \in (0,1)$ of continuity of $g(1,y)$.

See [SP, p. 1–61].

3. Methods and results

Step 1. Starting with an old result of M. Cipolla (1902)[C] (cf. P. Ribenboim (1995)[R, p. 249])

$$p_n = n \log n + n(\log \log n - 1) + o\left(\frac{n \log \log n}{\log n}\right) \tag{1}$$

we prove that

$$G(p_n/n \bmod 1) = G(\log(n \log n) \bmod 1).$$

To do this we use the following theorem (later proved in Part 8):

Theorem 1. *Let x_n and y_n be two real sequences. Assume that all d.f.s in $G(x_n \bmod 1)$ are continuous at 0 and 1. Then the zero limit of fractional parts*

$$\lim_{n\to\infty} (x_n - y_n) \bmod 1 = 0$$

implies $G(x_n \bmod 1) = G(y_n \bmod 1)$. The same implication follows from the continuity of d.f.s in $G(y_n \bmod 1)$ at 0 and 1.

We shall obtain the required continuity of $g(x) \in G(\log(n \log n) \bmod 1)$ at $x = 0$ and $x = 1$ by proving (in Part 6) the coincidence $G(\log(n \log n) \bmod 1) = G(\log n \bmod 1)$ and by using [KN, pp. 58–59]

$$G(\log n \bmod 1) = \left\{ g_u(x) = \frac{e^{\min(x,u)} - 1}{e^u} + \frac{1}{e^u}\frac{e^x - 1}{e - 1}; u \in [0,1] \right\}. \tag{2}$$

Step 2. To prove $G(\log(n \log n) \bmod 1) = G(\log n \bmod 1)$ using the first method we express (in Part 5 and directly by definition) d.f.s of the two-dimensional sequence $(\log n, \log \log n) \bmod 1$ as

$$\begin{aligned} G((\log n, \log \log n) \bmod 1) = \{g_{u,v}(x,y); u \in [0,1], v \in [0,1]\} \\ \cup \{g_{u,0,j,\alpha}(x,y); u \in [0,1], \alpha \in A, j = 1,2,\dots\} \\ \cup \{g_{u,0,0,\alpha}(x,y); u \in [\alpha,1], \alpha \in A\}, \end{aligned} \tag{3}$$

where A is the set of all limit points of the sequence $e^n \bmod 1$, $n = 1, 2, \dots$ [1] and, for $(x, y) \in [0, 1)^2$,

$$
\begin{aligned}
g_{u,v}(x, y) &= g_u(x) \cdot c_v(y), \\
g_{u,0,j,\alpha}(x, y) &= g_{u,0,j,\alpha}(x) \cdot c_0(y), \\
g_{u,0,0,\alpha}(x, y) &= g_{u,0,0,\alpha}(x) \cdot c_0(y), \text{ where} \\
c_v(y) &= \begin{cases} 0 & \text{if } 0 \le y \le v, \\ 1 & \text{if } v < y \le 1, \end{cases} \\
\text{and } c_v(0) &= 0, c_v(1) = 1, \\
g_{u,0,j,\alpha}(x) &= \frac{e^{\max(\alpha,x)} - e^{\alpha}}{e^{j+u}} + \frac{e^{\min(x,u)} - 1}{e^u} + \frac{1}{e^u}\frac{e^x - 1}{e - 1}\left(1 - \frac{1}{e^{j-1}}\right), \\
g_{u,0,0,\alpha}(x) &= \frac{e^{\max(\min(x,u),\alpha)} - e^{\alpha}}{e^u},
\end{aligned}
\tag{4}
$$

and, for $x \in [0, 1]$ and $y \in [0, 1]$, marginal d.f.s are

$$
\begin{aligned}
g_{u,v}(x, 1) &= g_u(x) = g_{u,v}(x, 1 - 0), \\
g_{u,v}(1, y) &= c_v(y) = g_{u,v}(1 - 0, y), \\
g_{u,0,j,\alpha}(x, 1) &= g_u(x) > g_{u,0,j,\alpha}(x, 1 - 0), j = 0, 1, 2, \dots, \\
g_{u,0,j,\alpha}(1, y) &= h_\beta(y) = g_{u,0,j,\alpha}(1 - 0, y), \text{ where } \beta = 1 - \frac{1}{e^{j+u-\alpha}}, \text{ and} \\
h_\beta(y) &= \beta \text{ if } y \in (0, 1), h_\beta(0) = 0, h_\beta(1) = 1,
\end{aligned}
\tag{5}
$$

where, the parameters u, v, j and α play the following role: Let $F_N(x, y)$ denote the step d.f. of the sequence $(\log n, \log\log n) \bmod 1$ (for definition see Part 2) and let $F_{N_k}(x, y) \to g(x, y)$ be a weak convergence as $k \to \infty$. Then

- $F_{N_k}(x, y) \to g_{u,v}(x, y)$ for $\{\log N_k\} \to u$ and $\{\log\log N_k\} \to v > 0$,
- $F_{N_k}(x, y) \to g_{u,0,j,\alpha}(x, y)$ for $\{\log N_k\} \to u$, $\{\log\log N_k\} \to 0$, $\{e^J\} \to \alpha$, $K - [e^J] = j > 0$, where $K = [\log N_k]$, $J = [\log\log N_k]$, and
- $F_{N_k}(x, y) \to g_{u,0,0,\alpha}(x, y)$ for $\{\log N_k\} \to u$, $\{\log\log N_k\} \to 0$, $\{e^J\} \to \alpha$, $K - [e^J] = 0$.

By using the following Theorem 2 (which will be proved later in Part 9) we obtain the set $G(\log(n \log n) \bmod 1)$ from $G((\log n, \log\log n) \bmod 1)$.

[1] The exact form of A is a well-known open problem.

Theorem 2. *Let x_n and y_n be two sequences in $[0,1)$ and $G((x_n,y_n))$ denote the set of all d.f.s of the two-dimensional sequence (x_n,y_n). If $z_n = x_n + y_n \bmod 1$, then the set $G(z_n)$ of all d.f.s of z_n has the form*

$$G(z_n)=\left\{g(t)=\int\limits_{0\le x+y<t} 1.\mathrm{d}g(x,y)+\int\limits_{1\le x+y<1+t} 1.\mathrm{d}g(x,y); g(x,y)\in G((x_n,y_n))\right\}$$

assuming that all the used Riemann-Stieltjes integrals exist.

Applying this theorem to $x_n = \log n \bmod 1$ and $y_n = \log\log n \bmod 1$, we obtain in Part 6 that the resulting $G(\log(n\log n) \bmod 1)$ coincides with $G(\log n \bmod 1)$.

Some properties of the sequence $\log(n\log n) \bmod 1$ can also be obtained immediately, e.g., it is not uniformly distributed. This follows from the well known Niederreiter theorem (cf. [N]):

Theorem 3. *If x_n, $n = 1,2,\dots$, is a monotone sequence that is uniformly distributed modulo 1, then*

$$\lim_{n\to\infty}\frac{|x_n|}{\log n}=\infty.$$

Also the result that this sequence does not satisfy any limit law follows from the fact that its upper and lower distribution functions are different. They can be obtained by the following theorem of Koksma [K1], [K2, Kap. 8] (cf. [KN, p. 58, Th. 7.7]):

Theorem 4. *Let the real-valued continuous function $f(x)$ be strictly increasing and let $f^{-1}(x)$ be its inverse function. Assume that, for $k\to\infty$,*

(i) $f^{-1}(k+1)-f^{-1}(k)\to\infty$,

(ii) $\lim_{k\to\infty}\frac{f^{-1}(k+x)-f^{-1}(k)}{f^{-1}(k+1)-f^{-1}(k)}=\underline{g}(x)$ *exists for* $x\in[0,1]$,

(iii) $\liminf_{k\to\infty}\frac{f^{-1}(k)}{f^{-1}(k+x)}=\chi(x)$.

Then the sequence

$$f(n) \bmod 1,\quad n=1,2,\dots$$

has the lower d.f. $\underline{g}(x)$ and the upper d.f.

$$\overline{g}(x)=1-\chi(x)(1-\underline{g}(x)),\ \textit{for } x\in[0,1].$$

Using this we find the lower and upper d.f. of $\log(n\log n) \bmod 1$ as

$$\underline{g}(x)=\frac{e^x-1}{e-1},\quad \overline{g}(x)=\frac{e-e^{1-x}}{e-1}$$

and they are the same as for $\log n \bmod 1$, cf. [KN, pp. 58-59]. This gives an impulse to the following generalization (later proved in Part 10) which we use in

Alternate Step 2.

Theorem 5. *Let the real-valued function $f(x)$ be strictly increasing for $x \geq 1$ and let $f^{-1}(x)$ be its inverse function. Assume that*

(i) $\lim_{x\to\infty} f'(x) = 0$,

(ii) $\lim_{k\to\infty} f^{-1}(k+1) - f^{-1}(k) = \infty$,

(iii) $\lim_{k\to\infty} \frac{f^{-1}(k+w(k))}{f^{-1}(k)} = \psi(w)$ *for every sequence* $w(k) \in [0,1]$ *for which* $\lim_{k\to\infty} w(k) = w$, *where this limit defines the function* $\psi : [0,1] \to [1, \psi(1)]$,

(iv) $\psi(1) > 1$.

Then

$$G(f(n) \bmod 1) = \left\{ \tilde{g}_w(x) = \frac{\min(\psi(x), \psi(w)) - 1}{\psi(w)} + \frac{1}{\psi(w)} \frac{\psi(x) - 1}{\psi(1) - 1}; w \in [0,1] \right\}.$$

The lower d.f. $\underline{g}(x)$ *and the upper d.g.* $\overline{g}(x)$ *of* $f(n) \bmod 1$ *are*

$$\underline{g}(x) = \frac{\psi(x) - 1}{\psi(1) - 1}, \quad \overline{g}(x) = 1 - \frac{1}{\psi(x)}(1 - \underline{g}(x).)$$

Here $\underline{g}(x) = \tilde{g}_0(x) = \tilde{g}_1(x) \in G(f(n) \bmod 1)$ *and* $\overline{g}(x) = \tilde{g}_x(x) \notin G(f(n) \bmod 1)$.

The parameter w has the following meaning: Let $F_N(x)$ denote the step d.f. of the sequence $f(n) \bmod 1$, $n = 1, 2, \dots, N$ and $w(k) = \{f(N_k)\} \to w$, then $F_{N_k} \to \tilde{g}_w(x)$ for every $x \in [0,1]$.

Applying Theorem 5 to $f(x) = \log(x \log x)$ we again obtain that

$$G(\log(n \log n) \bmod 1) = G(\log n \bmod 1).$$

Furthermore, Theorem 5 can also be applied to $f(x) = \log x$ and to $f(x) = \log(x \log^{(i)} x)$, $i = 1, 2, \dots$ (see Part 7) and which have the same distribution.

4. Lower and upper d.f.s of $\log(n \log n) \bmod 1$

Theorem 3 implies directly that $\log(n \log n) \bmod 1$ is not u.d. For computing $\underline{g}, \overline{g}$ of $\log(n \log n) \bmod 1$ we apply Theorem 4 to the function $f(x) = \log(x \log x)$. Since

$$0 < \log f^{-1}(k+1) - \log f^{-1}(k) < f(f^{-1}(k+1)) - f(f^{-1}(k)) = 1$$

we have $\frac{\log f^{-1}(k+1)}{\log f^{-1}(k)} \to 1$ as $k \to \infty$ and because

$$k+1-k = f(f^{-1}(k+1)) - f(f^{-1}(k)) = \log\left(\frac{f^{-1}(k+1)\log f^{-1}(k+1)}{f^{-1}(k)\log f^{-1}(k)}\right)$$

we find $\frac{f^{-1}(k+1)}{f^{-1}(k)} \to e$. Similarly

$$k+x-k = f(f^{-1}(k+x)) - f(f^{-1}(k)) = \log\left(\frac{(f^{-1}(k+x))\log(f^{-1}(k+x))}{f^{-1}(k)\log f^{-1}(k)}\right)$$

gives $\frac{f^{-1}(k+x)}{f^{-1}(k)} \to e^x$. Summary:

$$\frac{f^{-1}(k+x) - f^{-1}(k)}{f^{-1}(k+1) - f^{-1}(k)} \to \frac{e^x - 1}{e-1}, \qquad \frac{f^{-1}(k)}{f^{-1}(k+x)} \to \frac{1}{e^x},$$

and for the sequence $\log(n\log n)$ mod 1 Theorem 4 gives

$$\underline{g}(x) = \frac{e^x-1}{e-1}, \quad \overline{g}(x) = \frac{e-e^{1-x}}{e-1}$$

and they are the same as lower and upper d.f. of $\log n$ mod 1, cf. [KN, pp. 58-59].

5. D.f.s of $(\log n, \log\log n)$ mod 1

To find $G(\log(n\log n) \bmod 1)$ we shall apply Theorem 2 and thus we need to find the set $G((\{\log n\}, \{\log\log n\}))$ which we shall compute directly by definition:

For $0 \le x \le 1$, $0 \le y \le 1$ and a positive integer N denote by $F_N(x,y)$ the step d.f.

$$F_N(x,y) = \frac{\#\{3 \le n \le N; (\{\log n\}, \{\log\log n\}) \in [0,x) \times [0,y)\}}{N}$$

and desired d.f.s are all possible weak limits $\lim_{k\to\infty} F_{N_k}(x,y) = g(x,y)$, for suitable sequences $N_1 < N_2 < \dots$. Put

- $K(N) = [\log N]$,
- $J(N) = [\log\log N]$,
- $\mathbf{N} = \{1, 2, \dots, N\}$.

Note that, for given positive integers J and K, we have $J(N) = J$ for every N in $(e^{e^J}, e^{e^{J+1}})$ and $K(N) = K$ for every $N \in (e^K, e^{K+1})$. Further, let $k = 1, 2, \dots$ and $j = 0, 1, 2, \dots$. Since

$$0 \le \{\log n\} < x \iff \exists_k (0 \le \log n - k < x) \iff \exists_k (e^k \le n < e^{k+x})$$

$$0 \le \{\log\log n\} < y \iff \exists_j (0 \le \log\log n - j < y) \iff \exists_j (e^{e^j} \le n < e^{e^{j+y}}),$$

we have

$$F_N(x,y) = \frac{\sum_{j=0}^{J} \#\{[e^{e^j}, e^{e^{j+y}}) \cap (\cup_{k=1}^{K} [e^k, e^{k+x})) \cap \mathbf{N}\}}{N}.$$

Denote by $|X|$ the Lebesgue measure of $X \subset \mathbb{R}$. Using the facts that

- $\#\{[e^k, e^{k+x}) \cap \mathbf{N}\} = e^{k+x} - e^k + O(1)$;
- $1 \le k \le K = [\log N]$;
- $\frac{e^{e^{J-1+y}}}{e^{e^J}} \to 0$ as $J \to \infty$ and for fixed $0 \le y < 1$;
- $N > e^{e^J}$;

we have $F_N(x,y) = \tilde{F}_N(x,y) + o(1)$, where

$$\tilde{F}_N(x,y) = \frac{|[e^{e^J}, e^{e^{J+y}}) \cap (\cup_{k=[e^J]}^{K} [e^k, e^{k+x})) \cap [e^{e^J}, N]|}{N}$$

for every $0 \le x \le 1$ and every $0 \le y < 1$. Put

- $u(N) = \{\log N\}$,
- $v(N) = \{\log\log N\}$.

For any sequence N_k for which $F_{N_k}(x,y) \to g(x,y)$, there exists a subsequence N_k' such that $u(N_k') \to u$, $v(N_k') \to v$ and in this case we shall write

$$\lim_{k\to\infty} F_{N_k'}(x,y) = g_{u,v}(x,y).$$

On the other hand, the sequence $(\{\log n\}, \{\log\log n\})$, $n = 2, 3, \dots$, is everywhere dense in $[0,1]^2$. Proof: Let $(u,v) \in [0,1]^2$ and let J_k, $k = 1, 2, \dots$, be an increasing sequence of positive integers. Put $N_k = \left[e^{e^{J_k+v}}\right]$. Expressing $N_k = e^{e^{J_k+v_k}}$, then we have $v_k \to v$ as $k \to \infty$. Put $K_k = [\log N_k]$ and $N_k' = \left[e^{K_k+u}\right]$. Express $N_k' = e^{K_k+u_k'} = e^{e^{J_k+v_k'}}$, then $u_k' \to u$ and $v_k' \to v$. The final limit follows from the mean-value theorem

$$N_k' - N_k = e^{e^{J_k+v_k'}} - e^{e^{J_k+v_k}} = (v_k' - v_k) e^{e^{J_k+t}} e^{J_k+t}$$

for $t \in (v_k, v_k')$ and since $N_k, N_k' \in (e^{K_k}, e^{K_k+1})$ we have $(v_k' - v_k) \to 0$.

Thus for every $(u, v) \in [0,1]^2$ there exists a sequence of indices N_k such that $u(N_k) \to u$ and $v(N_k) \to v$ and moreover, by Helly theorem, there exists a subsequence N'_k of N_k such that $F_{N'_k}(x, y)$ weakly converges to $g_{u,v}(x, y)$. We shall see that for $v \neq 0$ the d.f. $g_{u,v}(x, y)$ is defined uniquely.

Thus, in what follows we assume $u(N) \to u$, $v(N) \to v$, $F_N(x, y) \to g_{u,v}(x, y)$ weakly (i.e. $\tilde{F}_N(x, y) \to g_{u,v}(x, y)$ weakly) for a suitable $N = N_k$, $k \to \infty$. We distinguish two main cases: $v > 0$ and $v = 0$.

1. $v > 0$.

1.a. $0 \le y < v$. In this case

$$\tilde{F}_N(x, y) < \frac{e^{e^{J+y}}}{e^{e^{J+v(N)}}} \to 0$$

for a suitable $N \to \infty$ and thus

$$g_{u,v}(x, y) = 0$$

for every $x \in (0, 1)$.

1.b. $0 < v < y < 1$. In this case the contributions of the following intervals to the limit $\tilde{F}_N(x, y) \to g_{u,v}(x, y)$ are:

- The first interval $[e^{[e^J]}, e^{[e^J]+1})$ gives
$$\frac{e^{[e^J]+1} - e^{[e^J]}}{e^{e^{J+v(N)}}} \to 0;$$
- the interval $[e^K, e^{K+1})$ gives
$$\frac{e^{\min(K+x, K+u(N))} - e^K}{e^{K+u(N)}} \to \frac{e^{\min(x,u)} - 1}{e^u};$$
- Since $K - [e^J] = ([e^{J+v(N)}] - [e^J]) \to \infty$, the intervals $[e^k, e^{k+x})$, $k = [e^J] + 1, \dots, K - 1$ contribute with
$$\frac{1}{e^{K+u(N)}} \sum_{k=[e^J]+1}^{K-1} (e^{k+x} - e^k) \to \frac{1}{e^u} \frac{e^x - 1}{e - 1}.$$

1.c. $v = y$. Taking in consideration the cases 1.a. and 1.b. we see that $g_{u,v}(x, y)$ has for $y = v$ the step d.f. $g_u(x)$.

As summary, in the case $v > 0$ we have $g_{u,v}(x, y) = g_u(x) \cdot c_v(y)$, where d.f.

$$g_u(x) = \frac{e^{\min(x,u)} - 1}{e^u} + \frac{1}{e^u} \frac{e^x - 1}{e - 1}$$

for every $x \in [0, 1)$ and $g_u(1) = 1$. Here $c_v(y) = c_{(v,1]}(y)$, i.e., $c_v(y) = 0$ for $0 < y \le v$, $c_v(y) = 1$ for $v < y \le 1$, and always $c_v(0) = 0$, $c_v(1) = 1$.

2. $v = 0$.

We shall see that in this case we need two new parameters $j(N)$ and $\alpha(N)$ defined

- $j(N) = K - [e^J]$,
- $\alpha(N) = \{e^J\}$,

and we again assume that for the sequence of indices $N = N_k$, $(k \to \infty)$ we have $\tilde{F}_N(x,y) \to g(x,y)$, $j(N) \to j$, $\alpha(N) \to \alpha$ and in this case we shall write

$$\tilde{F}_N(x,y) \to g_{u,0,j,\alpha}(x,y).$$

We distinguish three cases: $j = \infty$, $0 < j < \infty$ and $j = 0$.

2.a. $j = \infty$. This case is the same as **1.b**, and thus

$$g_{u,0,\infty,\alpha}(x,y) = g_u(x) \cdot c_0(y).$$

2.b. $0 < j < \infty$. In this case the contributions of intervals to $\tilde{F}_N(x,y)$ are:

- The first interval $[e^{[e^J]}, e^{[e^J]+1})$ gives
$$\frac{\max(e^{e^J}, e^{[e^J]+x}) - e^{e^J}}{e^{K+u(N)}} \to \frac{e^{\max(\alpha,x)} - e^{\alpha}}{e^{j+u}};$$
- the interval $[e^K, e^{K+1})$ gives again
$$\frac{e^{\min(K+x,K+u(N))} - e^K}{e^{K+u(N)}} \to \frac{e^{\min(x,u)} - 1}{e^u};$$
- Since $K - [e^J] = j =$ constant , the intervals $[e^k, e^{k+x})$, $k = [e^J] + 1, \ldots, K-1$ contribute with
$$\frac{1}{e^{K+u(N)}} \sum_{k=[e^J]+1}^{K-1} (e^{k+x} - e^k) \to \frac{1}{e^u} \frac{e^x - 1}{e - 1}\left(1 - \frac{1}{e^{j-1}}\right).$$

Thus in the case $0 < j < \infty$ we have $g_{u,0,j,\alpha}(x,y) = g_{u,0,j,\alpha}(x) \cdot c_0(y)$, where
$$g_{u,0,j,\alpha}(x) = \frac{e^{\max(\alpha,x)} - e^{\alpha}}{e^{j+u}} + \frac{e^{\min(x,u)} - 1}{e^u} + \frac{1}{e^u}\frac{e^x - 1}{e - 1}\left(1 - \frac{1}{e^{j-1}}\right)$$
and $g_{u,0,j,\alpha}(1) = 1$.

2.c. $j = 0$. In this case we consider only the interval $[e^{[e^J]}, e^{[e^J]+1})$ and its contribution to $\tilde{F}_N(x,y)$ is

$$\frac{\max\left(\min(e^{K+x}, e^{K+u(N)}), e^{e^J}\right) - e^{e^J}}{e^{K+u(N)}} \to \frac{e^{\max(\min(x,u),\alpha)} - e^{\alpha}}{e^u}$$

and thus we have $g_{u,0,0,\alpha}(x,y) = g_{u,0,0,\alpha}(x)\cdot c_0(y)$, where

$$g_{u,0,0,\alpha}(x) = \frac{e^{\max(\min(x,u),\alpha)} - e^{\alpha}}{e^u}$$

and $g_{u,0,0,\alpha}(1) = 1$.

We now show that the following triples (u,j,α) can occur: Suppose that for integer sequence $J_1 < J_2 < \dots$ we have $\{e^{J_k}\} \to \alpha$. Then for every fixed j, $(j = 0,1,2,\dots)$ and $u \in [0,1]$ (if $j = 0$, then u need to satisfy $\alpha \le u$) there exist two integer sequences K_k and N_k such that

- $K_k - [e^{J_k}] = j$,
- $N_k \in (e^{K_k}, e^{K_k+1})$,
- $\{\log N_k\} \to u$,
- $N_k \in (e^{e^{J_k}}, e^{e^{J_k+1}})$,
- $\{\log\log N_k\} \to 0$.

It suffices to put

- $K_k = [e^{J_k}] + j$, and
- $N_k = [e^{K_k+u_k}]$, where
- $u_k \to u$, $0 < u_k < 1$,
- $u_k > \{e^{J_k}\}$ if $j = 0$,
- $u_k > \log\left(1 + \frac{1}{e^{K_k}}\right)$ if $u = 0$.

Denoting by A the set of all limit points of $\{e^J\}$, $J = 1,2,\dots$, we have 3.

Finally, by definition, we have $g(0,y) = g(x,0) = 0$ and for computing $g(x,y) \in G((\{\log n\},\{\log\log n\}))$ in the case $y = 1$ we can not use $\tilde{F}_N(x,y)$ instead of $F_N(x,y)$. In this case

$$F_N(x,1) = \frac{(\cup_{k=1}^{K}[e^k, e^{k+x})) \cap \mathbf{N}}{N}$$

for every $0 \le x \le 1$ and if $u(N) \to u$, then $F_N(x,1) \to g_u(x)$. Thus $G(\{\log n\}) = \{g_u(x); u \in [0,1]\}$, which also can be found in [KN, p. 59].

All computations of $g(x,y) \in G((\{\log n\},\{\log\log n\}))$ are also valid for $x = 1$ and we see that $g_{u,v}(1,y) = c_v(y)$ and

$$g_{u,0,j,\alpha}(1,y) = h_\beta(y),\ \text{where } \beta = 1 - \frac{e^\alpha}{e^{j+u}},\quad j = 0,1,2,\dots$$

and this β covers $[0,1]$ if u runs $[\alpha,1]$ for $j = 0$ and $[0,1]$ for $j = 1,2,\dots$. Thus

$$g(1,y) \in G(\{\log\log n\}) = \{c_v(y); v \in [0,1]\} \cup \{h_\beta(y); \beta \in [0,1]\},$$

where $h_\beta(y) = \beta$ for $0 < y < 1$ and $h_\beta(0) = 0$, $h_\beta(1) = 1$. It can be compared with [S1].

6. D.f.s of $\log(n \log n) \bmod 1$

We shall find $G(\log(n\log n) \bmod 1)$ in the form (2) by applying Theorem 2 to $G((\log n, \log\log n)) \bmod 1)$. To do this we use the well known theorem for bounded $f(x,y)$, which says that the Riemann-Stietjes integral

$$\int_{[0,1]^2} f(x,y)\mathrm{d}g(x,y)$$

exists if and only if the set of all discontinuity points of $f(x,y)$ has a zero measure with respect to $g(x,y)$. Denoting by

$$X(t) = \{(x,y) \in [0,1]^2; 0 \le x+y < t\} \cup \{(x,y) \in [0,1]^2; 1 \le x+y < 1+t\},$$
$$X^{(1)}(t) = \{(x,y); y = t-x, x \in [0,1]\},$$
$$X^{(2)}(t) = \{(x,y); y = t+1-x, x \in [0,1]\},$$
$$X^{(3)} = \{(x,y); y = 1-x, x \in [0,1]\},$$

in the case $f(x,y) = c_{X(t)}(x,y)$ the discontinuity points form the line segments $X^{(1)}(t)$, $X^{(2)}(t)$ and $X^{(3)}$. Since

$$\mathrm{d}g_{u,v}(x,y) = \mathrm{d}g_u(x)\mathrm{d}c_v(y)$$
$$\mathrm{d}g_{u,0,j,\alpha}(x,y) = \mathrm{d}g_{u,0,j,\alpha}(x)\mathrm{d}c_0(y)$$

and bearing in mind (5) we have

$\mathrm{d}g_{u,v}(x,y) \ne 0$ only for $(x,y) = (x,v)$,

$\mathrm{d}g_{u,0,j,\alpha}(x,y) \ne 0$ only for $(x,0)$ and $(1,y)$.

Furthermore we see that $X^{(1)}(t) \cup X^{(2)}(t) \cup X^{(3)}$ has zero measure with respect to every $g(x,y) \in G((\log n, \log\log n) \bmod 1)$. Thus all integrals in Theorem 2 can be computed and every d.f. $g(t) \in G(\log(n\log n) \bmod 1)$ has the form

$$g(t) = \begin{cases} \int_0^{t-v} 1.\mathrm{d}g_u(x) + \int_{1-v}^{1} 1.\mathrm{d}g_u(x) & \text{if } t \ge v, \\ \int_{1-v}^{1+t-v} 1.\mathrm{d}g_u(x) & \text{if } t < v, \end{cases}$$

for $v > 0$ and for $v = 0$ it has the form

$$g(t) = \int_0^t 1.\mathrm{d}g_{u,0,j,\alpha}(x) + \int_0^t 1.\mathrm{d}(g_u(x) - g_{u,0,j,\alpha}(x)), \quad j = 0,1,2,\ldots.$$

Thus we have $G(\log(n\log n) \bmod 1) = \{g_{u,v}(x); u \in [0,1], v \in [0,1]\}$, where

$$g_{u,v}(x) = \begin{cases} g_u(1+x-v) - g_u(1-v) & \text{if } 0 \le x \le v, \\ g_u(x-v) + 1 - g_u(1-v) & \text{if } v < x \le 1. \end{cases}$$

Directly by means of computation we see that $g_{u,v}(x) = g_w(x)$, for $w = u + v \bmod 1$.

7. D.f.s of $\log(n\log^{(i)} n) \bmod 1$

We apply Theorem 5 to the function $f(x) = \log(x\log^{(i)} x)$, where $\log^{(i)} x$ is the ith iterated logarithm $\log\dots\log x$.

By the same way as in Part 3 we prove that, for $k \to \infty$,

$$\frac{\log^{(i)} f^{-1}(k + w(k))}{\log^{(i)} f^{-1}(k)} \to 1$$

for every sequence $w(k) \in [0,1]$. Proof: we start with

$$0 < \log f^{-1}(k+1) - \log f^{-1}(k) < \log f^{-1}(k+1) - \log f^{-1}(k) + \\ + \log^{(i+1)} f^{-1}(k+1) - \log^{(i+1)} f^{-1}(k) = 1$$

what implies $\frac{\log f^{-1}(k+1)}{\log f^{-1}(k)} \to 1$, and this implies $\log^{(2)} f^{-1}(k+1) - \log^{(2)} f^{-1}(k) \to 0$, what implies $\frac{\log^{(2)} f^{-1}(k+1)}{\log^{(2)} f^{-1}(k)} \to 1$, e.t.c.

Now, every $w(k)$ can be expressed as

$$w(k) = f(f^{-1}(k + w(k))) - f(f^{-1}(k)) = \\ = \log\left(\frac{f^{-1}(k + w(k))}{f^{-1}(k)} \cdot \frac{\log^{(i)} f^{-1}(k + w(k))}{\log^{(i)} f^{-1}(k)}\right).$$

Assuming that $w(k) \to w$ as $k \to \infty$, we have (iii) in the form $\psi(w) = e^w$. To prove (ii) we see that $f(f^{-1}(k+1)) - f(f^{-1}(k)) = 1$ and by mean-value theorem $(f^{-1}(k+1) - f^{-1}(k))f'(x) = 1$ for some $x \in (k, k+1)$, where $f'(x) \to 0$ as $k \to \infty$. As summary, $G(\log(n\log^{(i)} n) \bmod 1) = G(\log n \bmod 1)$.

8. Proof of Theorem 1

Since $\{x_n - y_n\} = x_n - y_n - [x_n - y_n] = (x_n - [x_n - y_n]) - y_n$ and $x_n - [x_n - y_n] \equiv x_n \bmod 1$, we can assume that $|x_n - y_n| \to 0$. For common increasing sequence of positive integers N_k, $k = 1, 2, \dots$, let

$$F_{N_k}(x) = \frac{1}{N_k}\sum_{n=1}^{N_k} c_{[0,x)}(\{x_n\}) \to g(x), \quad \tilde{F}_{N_k}(x) = \frac{1}{N_k}\sum_{n=1}^{N_k} c_{[0,x)}(\{y_n\}) \to \tilde{g}(x)$$

for every continuity point $x \in [0,1]$. Because, for every $h = \pm 1, \pm 2, \dots$, the limit $|x_n - y_n| \to 0$ implies $\lim_{k\to\infty} \frac{1}{N_k}\sum_{n=1}^{N_k} e^{2\pi i h x_n} = \lim_{k\to\infty} \frac{1}{N_k}\sum_{n=1}^{N_k} e^{2\pi i h y_n}$ and since

$$\frac{1}{N_k}\sum_{n=1}^{N_k} e^{2\pi i h x_n} = \int_0^1 e^{2\pi i h x}\mathrm{d}F_{N_k}(x), \frac{1}{N_k}\sum_{n=1}^{N_k} e^{2\pi i h y_n} = \int_0^1 e^{2\pi i h x}\mathrm{d}\tilde{F}_{N_k}(x),$$

the well-known Helly theorem[2] implies $\int_0^1 e^{2\pi i h x}\mathrm{d}g(x) = \int_0^1 e^{2\pi i h x}\mathrm{d}\tilde{g}(x)$ for every $h = \pm 1, \pm 2, \dots$. This gives, for every continuous $f : [0,1] \to \mathbb{R}$, $f(0) = f(1)$, that

$$\int_0^1 f(x)\mathrm{d}g(x) = \int_0^1 f(x)\mathrm{d}\tilde{g}(x), \text{ i.e. } \int_0^1 g(x)\mathrm{d}f(x) = \int_0^1 \tilde{g}(x)\mathrm{d}f(x).$$

For two common points $0 < x_1 < x_2 < 1$ of continuity of $g(x)$ and $\tilde{g}(x)$ and for sufficiently small $\Delta > 0$, define

$f(x) = 0$ for $x \in [0, x_1 - \Delta]$,
$f'(x) = 1/\Delta$ for $x \in (x_1 - \Delta, x_1)$,
$f(x) = 1$ for $x \in [x_1, x_2 - \Delta]$,
$f'(x) = -1/\Delta$ for $x \in (x_2 - \Delta, x_2)$, and
$f(x) = 0$ for $x \in [x_2, 1]$. Then

$$\int_0^1 g(x)\mathrm{d}f(x) \approx \frac{1}{\Delta}g(x_1)\Delta - \frac{1}{\Delta}g(x_2)\Delta, \int_0^1 \tilde{g}(x)\mathrm{d}f(x) \approx \frac{1}{\Delta}\tilde{g}(x_1)\Delta - \frac{1}{\Delta}\tilde{g}(x_2)\Delta,$$

and $\Delta \to 0$ gives

$$g(x_1) - \tilde{g}(x_1) = g(x_2) - \tilde{g}(x_2).$$

Now, we assume that $g(x_1) \neq \tilde{g}(x_1)$ and $g(x_2) \neq \tilde{g}(x_2)$. Fixing x_1 and letting $x_2 \to 1$ we have that one of $g, \tilde{g}$ must be discontinuous at 1, and fixing x_2, letting $x_1 \to 0$, one of $g, \tilde{g}$ must be discontinuous at 0. This gives Theorem 1 and the following modifications:

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[2] Second Helly theorem, see [SP, p. 4–5].

• If all d.f.s in $G(\{x_n\})$ and $G(\{y_n\})$ are continuous at 0, then

$$\{x_n - y_n\} \to 0 \Longrightarrow G(\{x_n\}) = G(\{y_n\}).$$

The same implies continuity at 1. Moreover
• The limit $\{x_n - y_n\} \to 0$ also implies

$$\{g \in G(\{x_n\}); g \text{ is continuous at } 0,1\} = \{\tilde{g} \in G(\{y_n\}); \tilde{g} \text{ is continuous at } 0,1\}.$$

• Assume that all d.f.s in $G(\{x_n\})$ are continuous at 0. Then

$$\{x_n - y_n\} \to 0 \Longrightarrow \{\tilde{g} \in G(\{y_n\}); \tilde{g} \text{ is continuous at } 0\} \subset G(\{x_n\}).$$

9. Proof of Theorem 2

Let (x_n, y_n), $n = 1, 2, \dots$, be a sequence in $[0,1)^2$ and $z_n = x_n + y_n \bmod 1$. Denote

$$F_N(t) = \frac{1}{N}\sum_{n=1}^{N} c_{[0,t)}(z_n), \quad F_N(x,y) = \frac{1}{N}\sum_{n=1}^{N} c_{[0,x)\times[0,y)}((x_n, y_n)).$$

Since

$$0 \le \{x + y\} < t \Longleftrightarrow (0 \le x + y < t) \text{ or } (1 \le x + y < 1 + t),$$

we have $F_N(t) = \frac{1}{N}\sum_{n=1}^{N} c_{X(t)}((x_n, y_n))$ and by means of Riemann-Stieltjes integration

$$F_N(t) = \int_{[0,1]^2} c_{X(t)}(x,y)\mathrm{d}F_N(x,y),$$

assuming that none of the points (x_n, y_n), $1 \le n \le N$, is a point of discontinuity of $c_{X(t)}(x,y)$. It is true that $(x_n, y_n) \notin X^{(1)}(t) \cup X^{(2)}(t)$ for almost all $t \in [0,1]$. By assumption the adjoining diagonal $X^{(3)}$ of $[0,1]^2$ has zero measure with respect to every $g(x,y) \in G((x_n, y_n))$, and thus the set of indices n for which $(x_n, y_n) \in X^{(3)}$ has zero asymptotic density and so it can be omitted. Here $X(t)$, $X^{(1)}(t)$, $X^{(2)}(t)$, and $X^{(3)}$ are defined in Part 6.

Now, we assume that $F_{N_k}(x) \to g(x) \in G(z_n)$. Then by the first Helly theorem there exists subsequence N'_k of N_k such that $F_{N'_k}(x,y) \to g(x,y) \in G((x_n, y_n))$ and by the second Helly theorem[3] we have

$$g(t) = \int_{[0,1)^2} c_{X(t)}(x,y)\mathrm{d}g(x,y). \tag{6}$$

[3] We use the following form: For every bounded $f(x,y)$, the weak limit $F_N(x,y) \to g(x,y)$ implies $\int_{[0,1]^2} f(x,y)\mathrm{d}F_N(x,y) \to \int_{[0,1]^2} f(x,y)\mathrm{d}g(x,y)$ assuming that all the used Riemann-Stieltjes integrals exist.

Vice-versa, we assume that $F_{N_k}(x,y) \to g(x,y) \in G((x_n,y_n))$. Then by the first Helly theorem we can select subsequence N'_k of N_k such that $F_{N'_k} \to g(x)$. For such $g(x)$, the (6) is true, again.

10. Proof of Theorem 5

For a positive integer N define

- $K = K(N) = [f(N)]$,
- $w = w(N) = \{f(N)\}$,
- $A_N([x,y)) = \#\{n \le N; f(n) \in [x,y)\}$.

Clearly $f^{-1}(K+w) = N$ and for every $x \in [0,1]$ we have

$$F_N(x) = \frac{\sum_{k=0}^{K-1} A_N([k,k+x))}{N} + \\ + \frac{A_N([K,K+x) \cap [K,K+w))}{N} + \frac{O(A_N([1,f^{-1}(0))))}{N}$$

From monotonicity of $f(x)$ it follows $A_N([x,y)) = f^{-1}(y) - f^{-1}(x) + \theta$, where $|\theta| \le 1$. Thus

$$F_N(x) = \frac{\sum_{k=0}^{K-1}(f^{-1}(k+x) - f^{-1}(k))}{N} + \\ + \frac{\min(f^{-1}(K+x), f^{-1}(K+w)) - f^{-1}(K)}{N} + \frac{O(K)}{N}.$$

The assumption (ii) implies $1/(f^{-1}(k+1) - f^{-1}(k)) \to 0$ which together with Cauchy-Stolz (other name is Stolz-Cesàro, see [SP, p. 4–7]) lemma implies that $K/f^{-1}(K) \to 0$ and thus $K/N \to 0$. Furthermore we can express the first term in $F_N(x)$ as

$$\frac{\sum_{k=0}^{K-1}(f^{-1}(k+x) - f^{-1}(k))}{\sum_{k=0}^{K-1}(f^{-1}(k+1) - f^{-1}(k))} \cdot \frac{f^{-1}(K) - f^{-1}(0)}{f^{-1}(K+w)} \tag{7}$$

and the second term as

$$\frac{\min\left(\frac{f^{-1}(K+x)}{f^{-1}(K)}, \frac{f^{-1}(K+w)}{f^{-1}(K)}\right) - 1}{\frac{f^{-1}(K+w)}{f^{-1}(K)}}. \tag{8}$$

Using (ii) and (iii) the Cauchy-Stolz lemma implies

$$\lim_{K\to\infty}\frac{\sum_{k=0}^{K-1}(f^{-1}(k+x)-f^{-1}(k))}{\sum_{k=0}^{K-1}(f^{-1}(k+1)-f^{-1}(k))}=\lim_{k\to\infty}\frac{f^{-1}(k+x)-f^{-1}(k)}{f^{-1}(k+1)-f^{-1}(k)}=\frac{\psi(x)-1}{\psi(1)-1}, \tag{9}$$

where $K=K(N)$, $N=1,2,\dots$. Now, for increasing subsequence N_i of indices N, denote $K(N_i)=K_i$ and $w(N_i)=w_i$. If $w_i\to w$, then by (iii) $f^{-1}(K_i)/f^{-1}(K_i+w_i)\to 1/\psi(w)$, and $f^{-1}(K_i+x)/f^{-1}(K_i)\to\psi(x)$. Thus (7), (8), (9) imply

$$F_{N_i}\to\tilde g(x)=\frac{\psi(x)-1}{\psi(1)-1}\cdot\frac{1}{\psi(w)}+\frac{\min(\psi(x),\psi(w))-1}{\psi(w)}.$$

Vice-versa, if $F_{N_i}\to\tilde g(x)$, we can find a subsequence of N_i such that $w_i\to w$ for some $w\in[0,1]$, and every $w\in[0,1]$ can have such a form, because (i) implies that $|f^{-1}((K+w-\varepsilon,K+w+\varepsilon))|>1$ for sufficiently large K and related $\varepsilon\to 0$.

Since, for fixed $x\in[0,1]$, we have

$$\tilde g_w(x)=\begin{cases}1-\frac{1}{\psi(w)}\left(1-\frac{\psi(x)-1}{\psi(1)-1}\right) & \text{if } w\le x,\\ \frac{\psi(1)}{\psi(w)}\frac{\psi(x)-1}{\psi(1)-1} & \text{if } w\ge x,\end{cases}$$

thus $\inf_{w\le x}\tilde g_w(x)=\tilde g_0(x)$, $\inf_{w\ge x}\tilde g_w(x)=\tilde g_1(x)$ and since $\tilde g_0(x)=\tilde g_1(x)$ we have $\underline{g}(x)=\tilde g_0(x)=\tilde g_1(x)=\frac{\psi(x)-1}{\psi(1)-1}$.

Similarly, $\sup_{w\le x}\tilde g_w(x)=\tilde g_x(x)$, $\sup_{w\ge x}\tilde g_w(x)=\tilde g_x(x)$, and thus $\overline{g}(x)=\tilde g_x(x)\notin G(f(n)\bmod 1)$.

11. Concluding remarks

The question about distribution properties of sequences $p_n/n \bmod 1$ and $n/\pi(n) \bmod 1$, $n=1,2,\dots$, was formulated as an open problem in the Seminar on Number Theory of Prof. T. Šalát in Bratislava. We note that $p_n/n \bmod 1$ is a subsequence of $n/\pi(n) \bmod 1$ and by F. Luca the sequence $n/\pi(n) \bmod 1$, $n=1,2,\dots$ has the same d.f.s as $\log(n) \bmod 1$, cf. [S2]. In [GS] it is proved that $p_n/n \bmod 1$ is logarithmic weighted uniformly distributed sequence. The sequence p_n/n without mod1 is also studied by P. Erdős and K. Prachar (1961/62) [EP].

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