

UNIFORM DISTRIBUTION OF SOME RATIOS INVOLVING THE NUMBER OF PRIME AND INTEGER DIVISORS

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ABSTRACT. We show that the fractional parts of the ratios $n/\omega(n)$, $n/a^{\omega(n)}$, $n/\tau(n)$ and $n/a^{\tau(n)}$, where $a \geq 2$ is a fixed integer and, as usual, $\omega(n)$ and $\tau(n)$ denote the number of prime divisors and the total number of divisors of $n > 1$, respectively, are uniformly distributed in the unit interval $[0, 1]$. This complements results of several authors about the scarcity of integral values taken by the above fractions.

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1. Introduction

In this paper, we study the distribution of fractional parts of the ratios

$$\rho(n) = \frac{n}{\omega(n)}, \quad \vartheta_a(n) = \frac{n}{a^{\omega(n)}}, \quad \xi(n) = \frac{n}{\tau(n)}, \quad \zeta_a(n) = \frac{n}{a^{\tau(n)}}, \quad (1)$$

where $a \geq 2$ is a fixed integer and, as usual, $\omega(n)$ and $\tau(n)$ denote the number of prime divisors and the total number of divisors of $n > 1$, respectively. We also put $\omega(1) = \rho(1) = 0$. It has been shown in [20] that $\rho(n) \in \mathbb{Z}$ for $(1 + o(1))x / \log \log x$ positive integers $n \leq x$ as $x \rightarrow \infty$.

Several more general results are obtained in [7, 11]. The integrality of the function $\vartheta_a(n)$ has been studied in [16], where the exact order of magnitude of the number of positive integers $n \leq x$ with $\vartheta_a(n) \in \mathbb{Z}$ is established.

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Several more problems with a similar flavor have been treated previously in [1, 2, 4, 7, 11, 15, 16, 20, 21, 22] (see also the references therein).

Here, we obtain nontrivial bounds for exponential sums with the functions (1) which imply that the sequences of fractional parts $\{\rho(n)\}$, $\{\vartheta_a(n)\}$, $\{\xi(n)\}$ and $\{\zeta_a(n)\}$, $n = 1, 2, \dots$, are uniformly distributed over the unit interval $[0, 1]$. Our method is a variant of that of [1].

The functions (1) are of rather different rates of growth and arithmetic structure. Accordingly, although the general scheme of derivation of each bound is the same, each case requires some specific adjustments and slightly different technical tools.

Throughout this paper, for any real number $x > 0$ and any integer $\nu \geq 1$, we write $\log_\nu x$ for the function defined inductively by $\log x = \max\{\ln x, 1\}$ (where $\ln x$ is the natural logarithm of x), and $\log_\nu x = \log(\log_{\nu-1} x)$ for $\nu > 1$ (with $\log_1 x = \log x$).

In what follows, we use the Landau symbol O , as well as the Vinogradov symbols \ll , \gg and \asymp with their usual meanings, with the understanding that any implied constants may occasionally, where obvious, depend on our parameters a and ε and are absolute otherwise. We recall that the notations $A \ll B$, $B \gg A$ and $A = O(B)$ are equivalent, and that $A \asymp B$ is equivalent to $A \ll B \ll A$. We always use the letters p and q to denote prime numbers, while m and n always denote positive integers.

2. Preliminary results

Here, we collect some known results that are used in this paper.

We use $\pi(x)$ to denote the number of primes $p \leq x$, and we use $\pi(x; f, d)$ to denote the number of primes $p \leq x$ in the fixed arithmetic progression $p \equiv f \pmod{d}$. By the classical *Page bound* (see Chapter 20 of [8]), and using partial summation (see the remark in Chapter 22 of [8]), it follows that for some absolute constant $A > 0$, the estimate

$$\pi(x; f, d) = \frac{x}{\varphi(d) \log x} + O\left(\frac{x}{\exp(A\sqrt{\log x})}\right) \quad (2)$$

holds provided that $1 \leq d \leq \sqrt{\log x}$ and $\gcd(f, d) = 1$.

The bound (2) is completely effective. However, we also need the more precise but not effective *Siegel–Walfisz bound* which asserts that for any positive

constants A and C

$$\left| \pi(x; f, d) - \frac{x}{\varphi(d) \log x} \right| \leq \frac{x}{(\log x)^A} \quad (3)$$

holds provided that $1 \leq d \leq (\log x)^C$, $\gcd(f, d) = 1$ and $x \geq x_0(A, C)$, where $x_0(A, C)$ depends only on A and C (see Chapter 22 of [8]).

We denote by $\pi_\nu(x)$ the number of positive integers $n \leq x$ with $\omega(n) = \nu$.

A particular case of the version of the classical *Hardy and Ramanujan* inequality given in [18] and [19] implies that the estimate

$$\pi_\nu(x) \asymp \frac{x(\log_2 x)^{\nu-1}}{(\nu-1)! \log x} \quad (4)$$

holds uniformly for $0.5 \log_2 x \leq \nu \leq 2 \log_2 x$.

Let $\mathbf{e}(x) = \exp(2\pi i x)$ for all $x \in \mathbb{R}$.

Finally, we also need the following bound for exponential sums with prime numbers,

$$\max_{\gcd(b,d)=1} \left| \sum_{\substack{p \leq x \\ p \text{ prime}}} \mathbf{e}(bp/d) \right| \ll x(d^{-1/2} + x^{-1/4} d^{1/8} + d^{1/2} x^{-1/2}) \log^3 x \quad (5)$$

which holds for any real $x \geq 1$ and integer $d \geq 1$ and follows immediately from Theorem 2 of [24] by partial summation (see [3]).

3. Exponential sums

For any integers c and N with $N \geq 1$, we consider the exponential sums with the functions (1) given by

$$\begin{aligned} S(c; N) &= \sum_{n=1}^N \mathbf{e}(c\rho(n)), & T_a(c; N) &= \sum_{n=1}^N \mathbf{e}(c\vartheta_a(n)), \\ U(c; N) &= \sum_{n=1}^N \mathbf{e}(c\xi(n)), & V_a(c; N) &= \sum_{n=1}^N \mathbf{e}(c\zeta_a(n)), \end{aligned}$$

where as before $\mathbf{e}(x) = \exp(2\pi i x)$ for all $x \in \mathbb{R}$.

THEOREM 1. *For every integer $c \neq 0$, the following inequality holds:*

$$S(c; N) \ll |c|^{1/2} \frac{N \log_3 N}{(\log_2 N)^{1/4}}.$$

Proof. Let $P(n)$ denote the largest prime divisor of $n \geq 2$, and put $P(1) = 1$. As usual, we say that an integer $n \geq 1$ is *y-smooth* if and only if $P(n) \leq y$, and we put

$$\psi(x, y) = \#\{1 \leq n \leq x : n \text{ is } y\text{-smooth}\}.$$

Following [1], we now define the following sets \mathcal{E}_i , $i = 1, \dots, 6$.

We choose $Q = N^{1/u}$, where

$$u = \frac{2 \log_3 N}{\log_4 N},$$

and we denote by \mathcal{E}_1 the set of Q -smooth positive integers $n \leq N$.

Next, we denote by \mathcal{E}_2 the set of the positive integers $n \leq N$ not in \mathcal{E}_1 such that $P(n)^2 \mid n$.

Now let

$$k = \lfloor (1 - \gamma) \log_2 N \rfloor \quad \text{and} \quad K = \lfloor (1 + \gamma) \log_2 N \rfloor,$$

where $\gamma > 0$ is a sufficiently small absolute constant and let \mathcal{E}_3 denote the set of positive integers $n \leq N$ such that either $\omega(n) < k$ or $\omega(n) > K$.

Let \mathcal{E}_4 denote the set of positive integers $n \leq N / \log_2 N$.

Now let $n \leq N$ be a positive integer not in $\cup_{i=1}^4 \mathcal{E}_i$. This integer n has a unique representation of the form $n = mp$, where m is such that $m < N/Q$, and $p = P(n)$ is a prime number in the half-open interval $p \in \mathcal{L}_m$, where

$$L_m = \max \left\{ Q, P(m), \frac{N}{m \log_2 N} \right\} \quad \text{and} \quad \mathcal{L}_m = (L_m, N/m].$$

Let \mathcal{E}_5 be the set of those $n \leq N$ such that $L_m = Q$.

Finally, let \mathcal{E}_6 be the set of those positive integers $n \leq N$ which are not in $\cup_{i=1}^5 \mathcal{E}_i$ and such that $L_m = P(m)$.

For each $i = 1, \dots, 6$, the estimate

$$\#\mathcal{E}_i \ll \frac{N}{\log_2 N} \tag{6}$$

has been established in [1].

Finally, we put

$$\Delta = \left\lfloor |c|^{-1/2} (\log_2 N)^{3/4} \log_3 N \right\rfloor,$$

and remark that if $\Delta < 1$, then the bound asserted by Theorem 1 is trivial.

We define \mathcal{E}_7 as the set of those positive integers $n \leq N$ which are not in $\cup_{i=1}^6 \mathcal{E}_i$ and such that $\gcd(n, \omega(n)) \geq \Delta$. Let us fix a positive integer $\nu \in [k, K]$, and a divisor δ of ν . We see that if for $n \leq N$ we have $\gcd(n, \nu) = \delta$ and $\omega(n) = \nu$, then $n = m\delta$, where $\omega(m) = \omega(n) + O(\omega(\delta)) = \nu + O(\log \nu)$.

Since

$$\pi_\eta(x) \ll \frac{x}{\sqrt{\log_2 x}}$$

uniformly over $\eta \in \mathbb{Z}$ (see [10, Theorem 21.5]), we have

$$\#\mathcal{E}_7 \ll \sum_{\nu=k}^K \sum_{\substack{\delta|\nu \\ \delta > \Delta}} \frac{N \log \nu}{\delta \sqrt{\log_2 N}} \ll \frac{N \log_3 N}{\Delta \sqrt{\log_2 N}} \sum_{\nu=k}^K \tau(\nu).$$

Using Theorem 320 of [13], we estimate the sum over ν as $O(K \log K)$ and derive

$$\#\mathcal{E}_7 \ll \frac{N \sqrt{\log_2 N} (\log_3 N)^2}{\Delta}. \quad (7)$$

We now let \mathcal{N} be the set of positive integers $n \leq N$ which do not belong to any of the sets \mathcal{E}_i for $i = 1, \dots, 7$. We see from (6) and (7) that

$$S(c; x) = \sum_{n \in \mathcal{N}} \mathbf{e}(c\rho(n)) + O\left(\frac{N \sqrt{\log_2 N} (\log_3 N)^2}{\Delta}\right). \quad (8)$$

Again, as in [1], we note that each $n \in \mathcal{N}$ admits a unique representation of the form $n = pm$, where $p > P(m)$. Moreover, by our choice of the sets \mathcal{E}_5 and \mathcal{E}_6 , we see that in this case $p \in \mathcal{L}_m = (N/(m \log_2 N), N/m]$. Let \mathcal{M} be the set of permissible values for m .

Let \mathcal{N}_ν be the subset of $n \in \mathcal{N}$ with $\omega(n) = \nu$. Note that each such n is of the form $n = pm$, where $m \in \mathcal{M}$ has $\omega(m) = \nu - 1$ and $p \in \mathcal{L}_m$. We write $\mathcal{M}_{\nu-1}$ for the set of those $m \in \mathcal{M}$ with $\omega(m) = \nu - 1$.

Note that

$$\#\mathcal{N} = \sum_{m \in \mathcal{M}} \pi(\mathcal{L}_m) \quad (9)$$

and for $k \leq \nu \leq K$ we have

$$\#\mathcal{N}_\nu = \sum_{m \in \mathcal{M}_{\nu-1}} \pi(\mathcal{L}_m), \quad (10)$$

where we use $\pi(\mathcal{L}_m)$ to denote the number of primes in the interval \mathcal{L}_m .

With the above notations, we can write

$$\sum_{n \in \mathcal{N}} \mathbf{e}(c\rho(n)) = \sum_{\nu=k}^K \sum_{m \in \mathcal{M}_{\nu-1}} \sum_{p \in \mathcal{L}_m} \mathbf{e}(cmp/\nu).$$

By the Page bound (2), for any integers b and d with $\gcd(b, d) = 1$ and $1 \leq d \leq \sqrt{\log x}$, we have

$$\sum_{p \leq x} \mathbf{e}(bp/d) = \frac{x}{\varphi(d) \log x} \sum_{\substack{1 \leq f \leq d \\ \gcd(f, d)=1}} \mathbf{e}(bf/d) + O\left(\frac{x}{\exp(0.5A\sqrt{\log x})}\right)$$

The sum over f is the classical *Ramanujan sum* which equals $\mu(d)$ because $\gcd(b, d) = 1$ (see, for example, Theorem 272 of [13]). Here, $\mu(d)$ denotes the Möbius function. Consequently, writing $\nu_c(m) = \nu / \gcd(cm, \nu)$ and noting that

$$\nu_c(m) \leq \nu \leq 2 \log_2 N < \log^{1/2} \left(\frac{Q}{\log_2 N} \right) < \log^{1/2} \left(\frac{N}{m \log_2 N} \right)$$

holds for all $m \in \mathcal{M}$ and all sufficiently large N , we derive that

$$\sum_{n \in \mathcal{N}} \mathbf{e}(c\rho(n)) \ll \sum_{\nu=k}^K \sum_{m \in \mathcal{M}_{\nu-1}} \left(\frac{\pi(\mathcal{L}_m)}{\varphi(\nu_c(m))} + \frac{N\nu_c(m)}{m \exp(0.5A\sqrt{\log Q})} \right).$$

We have

$$\begin{aligned} \sum_{\nu=k}^K \sum_{m \in \mathcal{M}_{\nu-1}} \frac{N\nu_c(m)}{m \exp(0.5A\sqrt{\log Q})} &\ll \frac{N \log_2 N}{\exp(0.5A\sqrt{\log Q})} \sum_{m < N} \frac{1}{m} \\ &\ll \frac{N \log N \log_2 N}{\exp(0.5A\sqrt{\log Q})} \ll \frac{N}{\log_2 N}. \end{aligned}$$

Therefore

$$\sum_{n \in \mathcal{N}} \mathbf{e}(c\rho(n)) \ll \sum_{\nu=k}^K \sum_{m \in \mathcal{M}_{\nu-1}} \frac{\pi(\mathcal{L}_m)}{\varphi(\nu_c(m))} + \frac{N}{\log_2 N}. \quad (11)$$

We now substitute the inequality

$$\varphi(\nu_c(m)) = \varphi\left(\frac{\nu}{\gcd(cm, \nu)}\right) \geq \frac{\varphi(\nu)}{\gcd(cm, \nu)} \geq \frac{\varphi(\nu)}{|c|\Delta}$$

in (11), which, in combination with (10) implies

$$\sum_{n \in \mathcal{N}} \mathbf{e}(c\rho(n)) \ll \sum_{\nu=k}^K \frac{|c|\Delta}{\varphi(\nu)} \#\mathcal{N}_\nu + \frac{N}{\log_2 N}. \quad (12)$$

Using the trivial inequality $\#\mathcal{N}_\nu \leq \pi_\nu(N)$ in combination with (4) (we assume that $\gamma < 1/2$, thus (4) applies for $k \leq \nu \leq K$), we deduce that

$$\sum_{n \in \mathcal{N}} \mathbf{e}(c\rho(n)) \ll \frac{|c|\Delta N}{\log N} \sum_{\nu=k}^K \frac{(\log_2 N)^{\nu-1}}{\varphi(\nu)(\nu-1)!} + \frac{N}{\log_2 N}. \quad (13)$$

It has been shown in [1] that if γ is sufficiently small, then

$$\sum_{\nu=k}^K \frac{(\log_2 N)^{\nu-1}}{\varphi(\nu)(\nu-1)!} \ll \frac{\log N}{\log_2 N},$$

which together with (8) and (13) leads to the bound

$$S(c; x) \ll \frac{N\sqrt{\log_2 N}(\log_3 N)^2}{\Delta} + \frac{|c|\Delta N}{\log_2 N}.$$

Recalling our choice of Δ , we finish the proof. \square

THEOREM 2. *Let $a \geq 2$ be integer. Then for any fixed $\varepsilon > 0$ and every integer $c \neq 0$ with $|c| \leq (\log N)^{\log a - \varepsilon}$, the following inequality holds:*

$$T_a(c; N) \ll \frac{N}{\log_2 N}.$$

Proof. We proceed as in the proof of Theorem 1. In particular, we put

$$\gamma = \frac{\varepsilon}{2\log a}$$

(we assume that $\varepsilon > 0$ is small enough, so $\gamma < 1/2$) and define the same parameters Q , k and K and the same sets $\mathcal{E}_1, \dots, \mathcal{E}_6$ for which we have the estimate (6).

Finally, we put

$$L = \left\lfloor \frac{k}{2} - \frac{\log |c|}{2\log a} \right\rfloor,$$

and define \mathcal{E}_7 as the set of those positive integers $n \leq N$ which are not in $\cup_{i=1}^6 \mathcal{E}_i$ and such that $a^L | n$. Clearly,

$$\#\mathcal{E}_7 \leq Na^{-L} \ll |c|^{1/2} Na^{-k/2} \ll N(\log N)^{-\varepsilon/4}. \quad (14)$$

We now let \mathcal{N} be the set of positive integers $n \leq N$ which do not belong to any of the sets \mathcal{E}_i for $i = 1, \dots, 7$. We see, from (6) and (14), that

$$T_a(c; x) = \sum_{n \in \mathcal{N}} \mathbf{e}(c\vartheta_a(n)) + O\left(\frac{N}{\log_2 N}\right). \quad (15)$$

As before, we define the sets \mathcal{N}_ν and $\mathcal{M}_{\nu-1}$. In particular, we have complete analogues of (9) and (10).

With the above notations we can write

$$\sum_{n \in \mathcal{N}} \mathbf{e}(c\vartheta_a(n)) = \sum_{\nu=k}^K \sum_{m \in \mathcal{M}_{\nu-1}} \sum_{p \in \mathcal{L}_m} \mathbf{e}(cmp/a^\nu).$$

By the Siegel–Walfisz bound (3), for any integers b and d with $\gcd(b, d) = 1$ and $1 \leq d \leq (\log x)^2$, we have

$$\sum_{p \leq x} \mathbf{e}(bp/d) = \frac{x}{\varphi(d) \log x} \sum_{\substack{1 \leq f \leq d \\ \gcd(f, d)=1}} \mathbf{e}(bf/d) + O\left(\frac{x}{\log^2 x}\right).$$

Consequently, writing $\lambda_c(m) = a^\nu / \gcd(cm, a^\nu)$ and noting that

$$\lambda_c(m) \leq a^\nu \leq a^K \leq (\log N)^2 < \log^3\left(\frac{Q}{\log_2 N}\right) < \log^3\left(\frac{N}{m \log_2 N}\right)$$

holds for all $m \in \mathcal{M}$ and all sufficiently large N , we have

$$\sum_{\nu=k}^K \sum_{m \in \mathcal{M}_{\nu-1}} \frac{N}{m \log^2 Q} \ll \frac{N \log N}{\log^2 Q} \ll \frac{N}{\log_2 N}.$$

Therefore

$$\sum_{n \in \mathcal{N}} \mathbf{e}(c\vartheta_a(n)) \ll \sum_{\nu=k}^K \sum_{m \in \mathcal{M}_{\nu-1}} \frac{\pi(\mathcal{L}_m)}{\varphi(\lambda_c(m))} + \frac{N}{\log_2 N}. \quad (16)$$

We now substitute the inequality

$$\varphi(\lambda_c(m)) = \varphi\left(\frac{a^\nu}{\gcd(cm, a^\nu)}\right) \geq a^{\nu-L-1} |c|^{-1}$$

in (16), which, in combination with analogues of (9) and (10) implies

$$\begin{aligned} \sum_{n \in \mathcal{N}} \mathbf{e}(c\vartheta_a(n)) &\ll \sum_{\nu=k}^K \frac{|c|}{a^{\nu-L-1} |c|^{-1}} \#\mathcal{N}_\nu + \frac{N}{\log_2 N} \\ &\ll |c| a^{L-k} \sum_{\nu=k}^K \#\mathcal{N}_\nu + \frac{N}{\log_2 N} \ll |c| a^{L-k} N + \frac{N}{\log_2 N}. \end{aligned}$$

Recalling (15) and our choice of L , we finish the proof. \square

We now estimate the sums $U(c; N)$. Clearly, the functions $\vartheta_2(n)$ and $\xi(n)$ are quite similar to each other. So, both the resulting bound and its proof are very similar to those of Theorem 2.

THEOREM 3. *Let $a \geq 2$ be integer. Then for any fixed $\varepsilon > 0$ and every integer $c \neq 0$ with $|c| \leq (\log N)^{\log 2 - \varepsilon}$, the following inequality holds:*

$$U(c; N) \ll \frac{N}{\log_2 N}.$$

Proof. As have already noticed, the proof is very similar to that of Theorem 2. Thus, we only concentrate only on the new elements. In particular, we put

$$\gamma = \frac{\varepsilon}{2 \log 2}$$

(we assume that $\varepsilon > 0$ is small enough, so $\gamma < 1/2$), and define the same parameters Q , k and K and the same sets $\mathcal{E}_1, \dots, \mathcal{E}_6$ for which we have the estimate (6) and the set \mathcal{E}_7 (with respect to $a = 2$), for which we have the estimate (14).

We recall that an integer s is called *square-full* if $p^2 \mid s$ for each prime divisor p of s . We also consider the set \mathcal{E}_8 of those positive integers $n \leq N$ which are not in $\cup_{i=1}^7 \mathcal{E}_i$ and such that $s \mid n$ for some square-full integer $s \geq (\log_2 N)^2$. It is well known that the number of square-full $s \leq x$ is $O(x^{1/2})$ (see [5, 12, 23]). Therefore, by partial summation, we derive

$$\#\mathcal{E}_8 \leq \sum_{\substack{s \geq (\log_2 N)^2 \\ s \text{ square-full}}} \frac{N}{s^2} \ll \frac{N}{\log_2 N}. \quad (17)$$

We now let \mathcal{N} be the set of positive integers $n \leq N$ which do not belong to any of the sets \mathcal{E}_i for $i = 1, \dots, 8$. We see from (6), (14) and (17), that

$$U(c; x) = \sum_{n \in \mathcal{N}} \mathbf{e}(c\xi(n)) + O\left(\frac{N}{\log_2 N}\right).$$

For every square-full integer s we define the sets $\mathcal{N}_{s,\nu}$ of those $n \in \mathcal{N}$ which are of the form $n = sr$, where s is largest square-full divisor of n (thus $\gcd(s, r) = 1$ and r is square-free) and $\omega(r) = \nu$. We write every $n \in \mathcal{N}_{s,\nu}$ as $n = mp$ where $p = P(n)$ and put $\mathcal{M}_{s,\nu-1}$ for the set of all possible values of m in such representations.

In particular, instead of (10) we have

$$\#\mathcal{N}_{s,\nu} = \sum_{m \in \mathcal{M}_{s,\nu-1}} \pi(\mathcal{L}_m).$$

We also note that for $n = sr \in \mathcal{N}_{s,\nu}$ we have

$$\omega(n) \geq \omega(r) \geq \omega(n) - \omega(s) = \omega(n) + O(\log_3 N / \log_4 N).$$

Thus, defining

$$k_0 = k - \lfloor \log_3 N \rfloor,$$

we see that for a sufficiently large N we have

$$\sum_{n \in \mathcal{N}} \mathbf{e}(c\xi(n)) = \sum_{\substack{s < (\log_2 N)^2 \\ s \text{ square-full}}} \sum_{\nu=k_0}^K \sum_{m \in \mathcal{M}_{s,\nu-1}} \sum_{p \in \mathcal{L}_m} \mathbf{e}(cmp/(\tau(s)2^\nu)).$$

The rest of the proof is fully analogous to that of Theorem 2 (one can easily verify that the extra summation over s does not change either the proof, or the final bound). \square

THEOREM 4. *Let $a \geq 2$ be integer. Then for any fixed $\varepsilon > 0$ and every integer $c \neq 0$ with $|c| \leq a^{(\log N)^{\log 2 - \varepsilon}}$, the following inequality holds:*

$$V_a(c; N) \ll \frac{N}{\log_2 N}.$$

Proof. Using the same notation and arguing exactly as in the proof of Theorem 3, we arrive to the identity

$$\begin{aligned} V_a(c; x) &= \sum_{n \in \mathcal{N}} \mathbf{e}(c\zeta_a(n)) + O\left(\frac{N}{\log_2 N}\right) \\ &= \sum_{\substack{s < (\log_2 N)^2 \\ s \text{ square-full}}} \sum_{\nu=k_0}^K \sum_{m \in \mathcal{M}_{s,\nu-1}} \sum_{p \in \mathcal{L}_m} \mathbf{e}(cmp/a^{\tau(s)2^\nu}) + O\left(\frac{N}{\log_2 N}\right). \end{aligned}$$

We now observe that

$$\gcd(cm, a^{\tau(s)2^\nu}) \leq |c| \gcd(m, a^{\tau(s)2^\nu}) \leq a^{2(\log N)^{\log 2 - \varepsilon}},$$

and using (5) after simple calculations we conclude the proof. \square

4. Concluding remarks

Clearly the bounds of Theorems 1, 2, 3 and 4 imply the desired uniformity of distribution property of the fractional parts $\{\rho(n)\}$, $\{\vartheta_a(n)\}$, $\{\xi(n)\}$ and $\{\zeta_a(n)\}$ for $n = 1, 2, \dots$. Moreover, coupled with the *Erdős–Turán inequality* (see [9, 14]), they give an explicit bound on the discrepancy of the above sequences.

There is no doubt that a more careful analysis of the numbers $m \in \mathcal{M}_{\nu-1}$ with a given value of either $\gcd(cm, \nu)$ or $\gcd(cm, a^\nu)$ may lead to improvements of Theorems 1 and 2, respectively (with respect to both the dependence on c and the

saving with respect to N). The same comments also apply to Theorems 3 and 4. However, here we mainly concentrated on giving simple proofs of the uniformity of distribution property rather than on extracting the best possible bounds on the discrepancy which would involve several more technical complications.

One can also use the same approach to study many other ratios of the form $f(n)/g(\omega(n))$ for some “natural” integer-valued functions f and g , for example polynomials with integer coefficients. However, it is not clear how to formulate a general result which would incorporate our Theorems 1, 2, 3 and 4. Certainly, the final bounds may depend on a variety of properties of f and g (including their rate of growth and multiplicative structure).

Finally, we note that our methods do not apply to study the distribution of the fractional parts $\{n^{1/\omega(n)}\}$ for $n = 1, 2, \dots$, which has been studied in [17] using a different approach.

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