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uniform distribution theory

ON THE RIEMANN ZETA-FUNCTION AND THE DIVISOR PROBLEM IV

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ABSTRACT. Let $\Delta(x)$ denote the error term in the Dirichlet divisor problem, and $E(T)$ the error term in the asymptotic formula for the mean square of $|\zeta(\frac{1}{2}+it)|$. If $E^*(t) = E(t)-2\pi\Delta^*(t/2\pi)$ with $\Delta^*(x) = -\Delta(x)+2\Delta(2x)-\frac{1}{2}\Delta(4x)$, then it is proved that

$$\int_0^T |E^*(t)|^3 \,\mathrm{d}t \ll_\varepsilon T^{3/2+\varepsilon}$$

and $\zeta(\frac{1}{2}+it) \ll_\varepsilon t^{\rho/2+\varepsilon}$ if $E^*(t) \ll_\varepsilon t^{\rho+\varepsilon}$.

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1. Introduction and statement of results

This paper is the continuation of the author's works [5], [6], where the analogy between the Riemann zeta-function $\zeta(s)$ and the divisor problem was investigated. As usual, let the error term in the classical Dirichlet divisor problem be

$$\Delta(x) \;=\; \sum_{n\le x} d(n) - x(\log x + 2\gamma - 1), \tag{1.1}$$

and the error term in the mean square formula for $|\zeta(\frac{1}{2}+it)|$ be defined by

$$E(T) \;=\; \int_0^T |\zeta(\tfrac{1}{2}+it)|^2 \,\mathrm{d}t - T\left(\log\left(\frac{T}{2\pi}\right) + 2\gamma - 1\right). \tag{1.2}$$

Here, as usual, $d(n)$ is the number of divisors of n, $\zeta(s)$ is the Riemann zeta-function, and $\gamma = -\Gamma'(1) = 0.577215\ldots$ is Euler's constant. The analogy between $\zeta(s)$ and the divisor problem is more exact if, instead with $\Delta(x)$, we

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work with the modified function $\Delta^*(x)$ (see M. Jutila [8], [9] and T. Meurman [11], [12]), where

$$\Delta^*(x) := -\Delta(x)+2\Delta(2x)-\tfrac{1}{2}\Delta(4x) = \tfrac{1}{2}\sum_{n\le 4x}(-1)^n d(n)-x(\log x+2\gamma-1). \tag{1.3}$$

M. Jutila (op. cit.) investigated both the local and global behaviour of the difference

$$E^*(t) \;:=\; E(t) - 2\pi\Delta^*\left(\frac{t}{2\pi}\right). \tag{1.4}$$

This function may be thought of as a discrepancy between $E^*(t)$ and $\Delta^*(x)$. In particular, Jutila in [9] proved that

$$\int_0^T (E^*(t))^2\,\mathrm{d}t \;\ll\; T^{4/3}\log^3 T, \tag{1.5}$$

which was sharpened in [6] by the author to the full asymptotic formula

$$\int_0^T (E^*(t))^2\,\mathrm{d}t \;=\; T^{4/3}P_3(\log T) + O_\varepsilon(T^{7/6+\varepsilon}), \tag{1.6}$$

where $P_3(y)$ is a polynomial of degree three in y with positive leading coefficient, and all the coefficients may be evaluated explicitly. Here and later ε denotes positive constants which are arbitrarily small, but are not necessarily the same ones at each occurrence, while $a \ll_\varepsilon b$ (same as $a = O_\varepsilon(b)$) means that the $\ll$–constant depends on ε. In Part II of [5] it was proved that

$$\int_0^T |E^*(t)|^5\,\mathrm{d}t \;\ll_\varepsilon\; T^{2+\varepsilon}, \tag{1.7}$$

while in Part III we investigated the function $R(T)$ defined by the relation

$$\int_0^T E^*(t)\,\mathrm{d}t = \frac{3\pi}{4}T + R(T), \tag{1.8}$$

and proved, among other things, the asymptotic formula

$$\int_0^T R^2(t)\,\mathrm{d}t = T^2 Q_3(\log T) + O_\varepsilon(T^{11/6+\varepsilon}), \tag{1.9}$$

where $Q_3(y)$ is a cubic polynomial in y with positive leading coefficient, whose all coefficients may be evaluated explicitly.

The asymptotic formula (1.9) bears resemblance to (1.6), and it is proved by a similar technique. The exponents in the error terms are, in both cases, less than the exponent of T in the main term by $1/6$. This comes from the use of [6, Lemma 3], and in both cases the exponent of the error term is the limit of the method. Our first new result is an upper bound for the third moment of $|E^*(t)|$, which does not follow from any of the previous results. This is

Theorem 1. *We have*

$$\int_0^T |E^*(t)|^3 \,\mathrm{d}t \ \ll_\varepsilon \ T^{3/2+\varepsilon}. \tag{1.10}$$

In view of (1.6) it follows that (1.10) is, up to 'ε', best possible.

Corollary 1. *We have*

$$\int_0^T |\zeta(\tfrac12 + it)|^8 \,\mathrm{d}t \ \ll_\varepsilon \ T^{3/2+\varepsilon}.$$

The last result is, up to 'ε', the sharpest one known (see [3, Chapter 8]). It follows from Theorem 1.4 of [5, Part II], which says that the bound

$$\int_0^T |E^*(t)|^k \,\mathrm{d}t \ \ll_\varepsilon \ T^{c(k)+\varepsilon} \tag{1.11}$$

implies that

$$\int_0^T |\zeta(\tfrac12 + it)|^{2k+2} \,\mathrm{d}t \ \ll_\varepsilon \ T^{c(k)+\varepsilon}, \tag{1.12}$$

where $k \ge 1$ is a fixed real number.

Corollary 2. *We have*

$$\int_0^T (E^*(t))^4 \,\mathrm{d}t \ll_\varepsilon T^{7/4+\varepsilon}, \quad \int_0^T |\zeta(\tfrac12 + it)|^{10} \,\mathrm{d}t \ll_\varepsilon T^{7/4+\varepsilon}. \tag{1.13}$$

The first bound in (1.13) follows by the Cauchy-Schwarz inequality for integrals from (1.7) and (1.10). The second bound follows from (1.11)–(1.12) with $k = 4$ and represents, up to 'ε', the sharpest one known (see [3, Chapter 8]). The first exponent in (1.13) improves on $16/9 + \varepsilon$, proved in [5, Part I].

Corollary 3. *If, for $k > 0$ a fixed constant and $1 \ll G = G(T) \ll T$,*

$$J_k(T,G) := \frac{1}{\sqrt{\pi}G}\int_{-\infty}^{\infty} |\zeta(\tfrac12 + iT + iu)|^{2k} \mathrm{e}^{-(u/G)^2} \,\mathrm{d}u,$$

then

$$\int_T^{2T} J_1^4(t,G) \,\mathrm{d}t \ \ll_\varepsilon \ T^{1+\varepsilon} \tag{1.14}$$

holds for $T^{3/16} \le G = G(T) \ll T$.

Namely it was proved in [6] that, for $T^\varepsilon \ll G = G(T) \le T$ and fixed $m \ge 1$ we have

$$\int_T^{2T} J_1^m(t,G)\,\mathrm{d}t \ll G^{-1-m}\int_{-G\log T}^{G\log T}\left(\int_T^{2T}|E^*(t+x)|^m\,\mathrm{d}t\right)\mathrm{d}x + T\log^{2m}T. \tag{1.15}$$

Thus (1.14) follows from (1.13) and (1.15) with $m = 4$, and improves on the range $T^{7/36} \le G = G(T) \ll T$ stated in Theorem 1 of [6], since $3/16 < 7/36$.

Both (1.6) and (1.10) imply that, in the mean sense, $E^*(t) \ll_\varepsilon t^{1/6+\varepsilon}$. The true order of this function is, however, quite elusive. If we define

$$\rho \;:=\; \inf\Big\{\; r > 0 \;:\; E^*(T) = O(T^r)\;\Big\}, \tag{1.16}$$

then we have unconditionally

$$1/6 \;\le\; \rho \le 131/416 = 0.314903\ldots, \tag{1.17}$$

and there is a big discrepancy between the lower and upper bound in (1.17). The lower bound in (1.17) comes from the asymptotic formula (1.6), which in fact gives $E^*(T) = \Omega(T^{1/6}(\log T)^{3/2})$. The upper bound comes from the best known bound for $\Delta(x)$ of M.N. Huxley [2] and $E(T)$ of N. Watt (unpublished). It remains yet to see whether a method can be found that would provide sharper bounds for ρ than for the corresponding exponents of $E(T)$ and $\Delta(x)$. This is important, as one can obtain bounds for $\zeta(\frac12 + it)$ from bounds of $E^*(t)$. More precisely, if as usual one defines the Lindelöf function for $\zeta(s)$ (the famous Lindelöf conjecture is that $\mu(\frac12) = 0$) by the relation

$$\mu(\sigma) \;=\; \liminf_{t\to\infty}\,\frac{\log|\zeta(\sigma+it)|}{\log t} \tag{1.18}$$

for any $\sigma \in \mathbb{R}$, then we have

Theorem 2. *If ρ is defined by (1.16) and $\mu(\sigma)$ by (1.18), then we have*

$$\mu(\tfrac12) \;\le\; \tfrac12\rho. \tag{1.19}$$

It may be remarked that, if $\rho \le 1/4$ holds, then $\theta = \omega$, where

$$\theta = \inf\Big\{\; c > 0 \;:\; E(T) = O(T^c)\;\Big\},\quad \omega = \inf\Big\{\; d > 0 \;:\; \Delta(T) = O(T^d)\;\Big\}.$$

Namely as $\theta \ge 1/4$ and $\omega \ge 1/4$ are known to hold (this follows e.g., from mean square results, see [4]) $\theta = \omega$ follows from (1.4) and $\omega = \sigma$, proved recently by Lau–Tsang [10], where

$$\sigma = \inf\Big\{\; s > 0 \;:\; \Delta^*(T) = O(T^s)\;\Big\}.$$

The reader is also referred to M. Jutila [8] for a discussion on some related implications. The limit of (1.19) is $\mu(\frac{1}{2}) \le 1/12$ in view of (1.17).

The plan of the paper is as follows. In Section 2 the necessary lemmas are given, while the proofs of Theorem 1 and Theorem 2 will be given in Section 3.

2. The necessary lemmas

In this section we shall state the lemmas which are necessary for the proof of our theorems.

Lemma 1 (O. Robert–P. Sargos [13]). *Let $k \ge 2$ be a fixed integer and $\delta > 0$ be given. Then the number of integers n_1, n_2, n_3, n_4 such that $N < n_1, n_2, n_3, n_4 \le 2N$ and*

$$|n_1^{1/k} + n_2^{1/k} - n_3^{1/k} - n_4^{1/k}| < \delta N^{1/k}$$

is, for any given $\varepsilon > 0$,

$$\ll_\varepsilon N^\varepsilon (N^4\delta + N^2). \tag{2.1}$$

This Lemma (with $k = 2$) is crucial in treating the fourth power of the sums in (2.5) and (2.12).

Lemma 2. *Let $T^\varepsilon \ll G \ll T/\log T$. Then we have*

$$E^*(T) \le \frac{2}{\sqrt{\pi}G}\int_0^\infty E^*(T+u)\,\mathrm{e}^{-u^2/G^2}\,\mathrm{d}u + O_\varepsilon(GT^\varepsilon), \tag{2.2}$$

and

$$E^*(T) \ge \frac{2}{\sqrt{\pi}G}\int_0^\infty E^*(T-u)\,\mathrm{e}^{-u^2/G^2}\,\mathrm{d}u + O_\varepsilon(GT^\varepsilon). \tag{2.3}$$

Lemma 2 follows on combining Lemma 2.2 and Lemma 2.3 of [4, Part I].

The next lemma is F.V. Atkinson's classical, precise asymptotic formula for $E(T)$ (see [1], [3] or [4]).

Lemma 3. *Let $0 < A < A'$ be any two fixed constants such that $AT < N < A'T$, and let $N' = N'(T) = T/(2\pi) + N/2 - (N^2/4 + NT/(2\pi))^{1/2}$. Then*

$$E(T) = \Sigma_1(T) + \Sigma_2(T) + O(\log^2 T), \tag{2.4}$$

where

$$\Sigma_1(T) = 2^{1/2}(T/(2\pi))^{1/4}\sum_{n\le N}(-1)^n d(n) n^{-3/4} e(T,n)\cos(f(T,n)), \tag{2.5}$$

$$\Sigma_2(T) = -2\sum_{n\le N'} d(n) n^{-1/2}(\log T/(2\pi n))^{-1}\cos(T\log T/(2\pi n) - T + \pi/4), \tag{2.6}$$

with

$$f(T,n) = 2T\operatorname{arsinh}\left(\sqrt{\pi n/(2T)}\right) + \sqrt{2\pi nT + \pi^2 n^2} - \pi/4$$
$$= -\tfrac14\pi + 2\sqrt{2\pi nT} + \tfrac16\sqrt{2\pi^3}n^{3/2}T^{-1/2} + a_5 n^{5/2}T^{-3/2} + a_7 n^{7/2}T^{-5/2} + \dots\,, \tag{2.7}$$

$$e(T,n) = (1+\pi n/(2T))^{-1/4}\Big\{(2T/\pi n)^{1/2}\operatorname{arsinh}\,(\sqrt{\pi n/(2T)}\Big\}^{-1}$$
$$= 1 + O(n/T) \qquad (1 \le n < T), \tag{2.8}$$

and $\operatorname{arsinh} x = \log(x + \sqrt{1+x^2}\,)$.

Lemma 4 (M. Jutila [8, Part II])**.** *For $A \in \mathbb{R}$ a constant we have*

$$\cos\Big(\sqrt{8\pi nT} + \tfrac16\sqrt{2\pi^3}n^{3/2}T^{-1/2} + A\Big) = \int_{-\infty}^{\infty}\alpha(u)\cos(\sqrt{8\pi n}(\sqrt{T}+u)+A)\,\mathrm{d}u, \tag{2.9}$$

where $\alpha(u) \ll T^{1/6}$ for $u \ne 0$,

$$\alpha(u) \ll T^{1/6}\exp(-bT^{1/4}|u|^{3/2}) \tag{2.10}$$

for $u < 0$, and

$$\alpha(u) = T^{1/8}u^{-1/4}\Big(d\exp(ibT^{1/4}u^{3/2}) + \bar{d}\exp(-ibT^{1/4}u^{3/2})\Big) + O(T^{-1/8}u^{-7/4}) \tag{2.11}$$

for $u \ge T^{-1/6}$ and some constants b (>0) and d.

We need also an explicit formula for $\Delta^*(x)$ (see [3, Chapter 15]). This is

Lemma 5. *For $1 \le N \ll x$ we have*

$$\Delta^*(x) = \frac{1}{\pi\sqrt2}x^{\frac14}\sum_{n\le N}(-1)^n d(n)n^{-\frac34}\cos(4\pi\sqrt{nx} - \tfrac14\pi) + O_\varepsilon(x^{\frac12+\varepsilon}N^{-\frac12}). \tag{2.12}$$

3. Proofs of the theorems

The proof of (1.10) of Theorem 1 is based on the method of [5]. We seek an upper bound for $R = R(V,T)$, the number of points

$$\{t_r\} \in [T,2T]\,(r=1,\dots,R), \quad V \le |E^*(t_r)| < 2V \quad (|t_r - t_s| \ge V \text{ if } r \ne s). \tag{3.1}$$

We consider separately the points where $E^*(t_r)$ is positive or negative. Suppose the first case holds (the other one is treated analogously), using in either case the notation R for the number of points in question. Then from Lemma 2 we have

$$V \le E^*(t_r) \le \frac{2}{\sqrt{\pi}G}\int_0^\infty E^*(t_r+G+u)\,\mathrm{e}^{-u^2/G^2}\,\mathrm{d}u + O_\varepsilon(GT^\varepsilon), \tag{3.2}$$

and the integral may be truncated at $u = G\log T$ with a very small error. We may suppose that V satisfies

$$T^{1/6} \ \le\ V\ \le T^{1/4}. \tag{3.3}$$

Indeed, if

$$I_1(T) := \int_{T,|E^*|\le T^{1/6}}^{2T} |E^*(t)|^3\,\mathrm{d}t, \quad I_2(T) := \int_{T,|E^*|\ge T^{1/4}}^{2T} |E^*(t)|^3\,\mathrm{d}t,$$

then from (1.6) it follows that

$$I_1(T) \le T^{1/6}\int_T^{2T} |E^*(t)|^2\,\mathrm{d}t \ll T^{3/2}\log^3 T, \tag{3.4}$$

while from (1.7) we obtain that

$$I_2(T) \le T^{-1/2}\int_T^{2T} |E^*(t)|^5\,\mathrm{d}t \ll_\varepsilon T^{3/2+\varepsilon}. \tag{3.5}$$

Thus supposing that (3.3) holds we estimate

$$I(V,T) := \int_{T,V\le|E^*(t)|\le 2V}^{T} |E^*(t)|^3\,\mathrm{d}t$$

by splitting the interval $[T,\,2T]$ into $R\,(=R(V,T))$ disjoint subintervals J_r of length $\le V$, where in the r-th of these intervals we define t_r $(r = 1,\dots,R)$ by

$$|E^*(t_r)| = \sup_{t\in J_r} |E^*(t)|.$$

The proof of Theorem 1 will be a consequence of the bound

$$R\ \ll_\varepsilon\ T^{3/2+\varepsilon}V^{-4}, \tag{3.6}$$

provided that (3.1) holds (considering separately points with even and odd indices so that $|t_r - t_s| \ge V$ $(r \ne s)$ is satisfied). Namely we have

$$I(V,T) \ll V\sum_{j=1}^{R} |E^*(t_r)|^3 \ll_\varepsilon VT^{3/2+\varepsilon}V^{-4}V^3 = T^{3/2+\varepsilon}, \tag{3.7}$$

and from (3.4), (3.5) and (3.7) we obtain

$$\int_T^{2T} |E^*(t)|^3\,\mathrm{d}t\ \ll_\varepsilon\ T^{3/2+\varepsilon}. \tag{3.8}$$

The bound (1.10) follows from (3.8) if one replaces T by $T2^{-j}$ and sums the corresponding results for $j = 1, 2, \ldots$.

We continue the proof of Theorem 1 by noting that, like in [5, Part I], the integral on the right-hand side of (3.2) is simplified by Atkinson's formula (Lemma 3) and the truncated formula for $\Delta^*(x)$ (Lemma 5). We take $G = cVT^{-\varepsilon}$ (with sufficiently small $c > 0$) to make the O-term in (3.2) $\le \frac{1}{2}V$, and then we obtain

$$V \ll \sum_{j=4}^{6} V^{-1}T^{\varepsilon} \int_0^{G\log T} \sum\nolimits_j (t_r + G + u)\,\mathrm{e}^{-u^2/G^2}\,\mathrm{d}u \quad (r = 1, \ldots, R), \tag{3.9}$$

where we choose $X = T^{1/3-\varepsilon}$, $N = TG^{-2}\log T$ and, similarly to [5], for $t \asymp T$ we set (in the notation of Lemma 3)

$$\sum\nolimits_4(t) := t^{1/4} \sum_{X<n\le N} (-1)^n d(n) n^{-3/4} e(t+u,n) \cos(f(t+u,n)), \tag{3.10}$$

$$\sum\nolimits_5(t) := t^{1/4} \sum_{X<n\le N} (-1)^n d(n) n^{-3/4} \cos(\sqrt{8\pi n(t+u)} - \pi/4),$$

$$\sum\nolimits_6(t) := t^{-1/4} \sum_{n\le X} (-1)^n d(n) n^{3/4} \cos(\sqrt{8\pi n(t+u)} - \pi/4). \tag{3.11}$$

The sums in (3.10)–(3.11) over n are split into $O(\log T)$ subsums over the ranges $K < n \le K' \le 2K$. We denote these sums by $\Sigma_j(t,K)$ and let $\varphi(t)$ denote a smooth, nonnegative function supported in $[T/2,\, 5T/2]$, such that $\varphi(t) = 1$ when $T \le t \le 2T$. There must exist a set of $M = M(K)$ points $\{\tau_m\} \in \{t_r\}$ such that $M(K) \gg R/\log T$ for some j, K, so that it suffices to majorize $M(K)$, which we shall (with a slight abuse of notation) henceforth denote again by R. The contribution of $\sum_6(t,K)$ is estimated by raising the relevant portion of (3.9) to the fourth power and summing over r, noting that $|t_r - t_s| \ge V$ $(r \ne s)$, so that the sum of integrals over the intervals $[t_r+G, t_r+G+G\log T]$ is majorized by the integral over $[T/2,\, 5T/2]$. We proceed as in [5, Part I and Part II] integrating by parts, and using $\varphi^{(\ell)}(t) \ll_\ell T^{-\ell}$ $(\ell \ge 0)$. It transpires, when we develop $\sum_6^4(t,K)$ and set

$$\Delta := \sqrt{n_1} + \sqrt{n_2} - \sqrt{n_3} - \sqrt{n_4}\,,$$

that the contribution of $\Delta \ge T^{\varepsilon-1/2}$ is negligible (i.e., it is smaller than T^{-A} for any given $A > 0$). The contribution of $\Delta < T^{\varepsilon-1/2}$ is treated by Lemma 1 and trivial estimation of the ensuing integral. We obtain

$$\begin{aligned}
RV^4 &\ll V^{-1}T^{\varepsilon} \sup_{|u|\le G\log T} \int_{T/2}^{2T} \varphi(t) {\sum}_6^4(t,K)\, \mathrm{d}t \\
&\ll_\varepsilon T^{1+\varepsilon}V^{-1} \sup_{|u|\le G\log T, |\Delta|\le T^{\varepsilon-1/2}} T^{-1}K^3(K^4K^{-1/2}|\Delta| + K^2) \\
&\ll_\varepsilon T^{\varepsilon}V^{-1}(T^{-1/2}X^{13/2} + X^5) \ll_\varepsilon T^{5/3+\varepsilon}V^{-1},
\end{aligned}$$

since $K \ll X = T^{1/3-\varepsilon}$. This gives, since (3.3) holds,

$$R \ll_\varepsilon T^{5/3+\varepsilon}V^{-5} \ll_\varepsilon T^{3/2+\varepsilon}V^{-4},$$

which is the desired bound (3.6).

The contributions of $\sum_4(t,K)$ and of $\sum_5(t,K)$ are estimated analogously, with the remark that in the case of $\sum_4(t,K)$ one has to use Lemma 4 to deal with the complications arising from the presence of $\cos(f(t+u,n))$, coming from (2.5). This procedure was explained in detail in [5, Part I and Part II]. The non-negligible contribution of $\sum_5(t,K)$ will, again by raising the relevant expression to the fourth power, be for $\Delta \le T^{\varepsilon-1/2}$ again. The application of Lemma 1 gives in this case

$$\begin{aligned}
RV^4 &\ll_\varepsilon V^{-1}T^{1+\varepsilon}TK^{-3}(K^4T^{-1/2} + K^2) \\
&\ll_\varepsilon T^{2+\varepsilon}V^{-1}(K^{1/2}T^{1/2} + K^{-1}) \\
&\ll_\varepsilon T^{3/2+\varepsilon}V^{-1}K^{1/2} + T^{5/3+\varepsilon}V^{-1}, \qquad (3.12)
\end{aligned}$$

because $K \gg X = T^{1/3-\varepsilon}$ holds. For $K \le V^2$ the bound (3.12) reduces to (3.6), and we are done. If $V^2 < K \le T^{1+\varepsilon}V^{-2}$ (note that $V^2 < T^{1+\varepsilon}V^{-2}$ holds by (3.3)), then the relevant expression is squared, and not raised to the fourth power. We obtain

$$\begin{aligned}
RV^2 &\ll_\varepsilon V^{-1} \max_{|u|\le G\log T} \int_{T/2}^{5T/2} \varphi(t) {\sum}_5^2(t,K)\, \mathrm{d}t \\
&= T^{1/2}V^{-1} \max_{|u|\le G\log T} \int_{T/2}^{5T/2} \varphi(t) \times \\
&\times \sum_{K<m,n\le 2K} (-1)^{m+n} d(m)d(n)(mn)^{-3/4} \mathrm{e}^{i\sqrt{8\pi(t+u)}(\sqrt{m}-\sqrt{n})}\, \mathrm{d}t \\
&\ll T^{3/2}V^{-1} \sum_{m>K} d^2(m)m^{-3/2} + T^{1+\varepsilon}K^{-3/2}V^{-1} \sum_{K<m\ne n\le 2K} |\sqrt{m}-\sqrt{n}|^{-1}.
\end{aligned}$$

Here we used trivial estimation for the diagonal terms $m = n$, and the first derivative test ([3, Lemma 2.1]) for the remaining terms. Since $V^2 < K$ and

$$\sum_{K<m\neq n\le 2K} |\sqrt{m}-\sqrt{n}|^{-1} \ll \sum_{K<m\le 2K} \sqrt{K} \sum_{K<n\le 2K, n\neq m} |m-n|^{-1} \ll K^{3/2}\log K,$$

we obtain that

$$RV^2 \ll_\varepsilon T^{3/2}V^{-1}K^{-1/2}\log^3 T + T^{1+\varepsilon}V^{-1} \ll_\varepsilon T^{3/2+\varepsilon}V^{-2},$$

and (3.6) follows again. The proof of Theorem 1 is complete.

For the proof of Theorem 2 note that, by [4, Theorem 1.2], (1.4) and (1.19), we have

$$\begin{aligned}
|\zeta(\tfrac12+iT)|^2 &\ll \log T \int_{T-1}^{T+1} |\zeta(\tfrac12+it)|^2\,\mathrm{d}t + 1\\
&\ll \log T\Big(\log T + E(T+1) - E(T-1)\Big)\\
&\ll_\varepsilon \log T\left(\log T + 2\pi\Delta^*\Big(\frac{T+1}{2\pi}\Big) - 2\pi\Delta^*\Big(\frac{T-1}{2\pi}\Big)\right) + T^{\rho+\varepsilon}\\
&\ll_\varepsilon T^{\rho+\varepsilon},
\end{aligned}\tag{3.13}$$

since, from (1.3) and $d(n) \ll_\varepsilon n^\varepsilon$, it is seen that

$$\Delta^*(T+H) - \Delta^*(T) = O(H\log T) + \tfrac12 \sum_{4T<n\le 4(T+H)} (-1)^n d(n) \ll_\varepsilon HT^\varepsilon$$

holds for $1 \ll H \ll T$. Therefore (3.13) implies that

$$|\zeta(\tfrac12+iT)|^2 \ \ll_\varepsilon \ T^{\rho+\varepsilon},$$

and this gives $\mu(\frac12) \le \frac12\rho$, as asserted.

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