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Understanding the Vertical Structure of the Subtropical Thermocline

Jeffrey Anthony Polton

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Department of Meteorology

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Declaration

I confirm that this is my own work and the use of all material from other sources has been properly and fully acknowledged.

Jeff Polton
Abstract

Current understanding of the vertical structure of the subtropical thermocline is broadly based on one of two competing theories: the diffusive thermocline theory where diapycnal mixing balances downward advection, or the ventilated thermocline theory where the surface conditions are advected into the interior. Neither of these theories include an active western boundary current, assuming that it does not significantly influence the thermocline stratification in closing the circulation.

Closing the gyre with a western boundary current, however, imposes a strong constraint on the vertical structure, which can be conveniently interpreted as an integral constraint on potential vorticity fluxes. In particular, the vertical flux of potential vorticity through any steady-state closed Bernoulli contour must be zero.

In a planetary geostrophic ocean, the integral constraint is a balance between vertical fluxes of potential vorticity associated with vertical advection, buoyancy forcing, and dissipative friction. The integral constraint recovers the diffusive and ventilated thermocline theories and further highlights that friction, manifest in the western boundary current, can be essential in closing the balance.

The integral constraint is extended for application in an eddy-permitting ocean simulation. An additional potential vorticity flux attributed to eddies is shown to effectively replace the frictional flux and acts to either oppose the advective or the buoyancy potential vorticity fluxes in the western boundary current region, where the model effectively captures mesoscale variability.

These results show that the western boundary current is active in closing the gyre-scale circulation and quantifies, in terms of potential vorticity fluxes, the role of dissipative western boundary current processes in maintaining the subtropical thermocline. Furthermore, the diagnostics developed in this thesis, which show that eddies can affect the stratification of the subtropical gyre, could be used, in eddy-resolving data, to diagnose the extent of the eddy influence.
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"... It's so vast ..."
"... And that's just the surface."
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Chapter One

Introduction

Understanding the statistically-steady large-scale circulation and stratification is one of the principle goals of physical oceanography. In the subtropics, the combined action of wind and buoyancy forcing produces a basin-scale recirculating flow. This flow has a vertical component that advects warm, surface heated, water downward into the ocean, wherein at a depth of around 500 m there is a region in which potential temperature, salinity and potential density vary rapidly with depth. This is the thermocline.

Conventionally, for reasons of analytical tractability, theoretical descriptions of the dynamics that maintain the thermocline have focused on particular regions or aspects of the recirculating fluid. However, a significant complexity inherent in the general circulation problem is that, with neither start nor finish, fluid recirculates in an external feedback loop whereby water mass transformations in one region subsequently affect the dynamics of the next. Consequently, where these interactions have not been explored, there are significant gaps in understanding. In particular, it is not known whether dissipative processes in the western boundary current affect the thermocline structure in the interior.

Understanding the processes that control the thermocline structure is a fundamental problem in physical oceanography, as it also has significant implications for the global heat budget. Owing to the disparate heat capacities of air and water, even slight changes in the thermocline structure may lead to significant adjustment of the atmospheric heat content with ramifications for climate change.

In this thesis a holistic approach is taken to gain an improved understanding of the vertical structure of the subtropical thermocline. In this approach, western boundary
effects and interior processes are melded in the form of an integral constraint. In particular, the following issues are addressed:

- Does closing the gyre with a western boundary current impose a constraint on the vertical structure?
- What is the gyre-scale role of western boundary current dissipative processes? and can they be quantified?
- Do eddies play a role in setting the ocean thermocline?

In this chapter the thermocline problem is posed in Section 1.1. Broadly speaking, solutions to the thermocline problem fall into three categories: diffusive theories, given in Section 1.2; ventilated theories, given in Section 1.3; and a thermocline configuration that arises from eddy mixing of potential vorticity, given in Section 1.4. In the light of observational and numerical simulation data, the relative merits of each theory are discussed in Section 1.5. Finally, the structure of the thesis is presented in Section 1.6.

1.1 The thermocline problem

Why do depth profiles show sharp changes in potential temperature, salinity and potential density in the upper 1km of the major global oceans? Why is the thermocline a prevalent feature of all the major world oceans? (Fig. 1.1). As is always the case in improving understanding of complex systems, the skill is in identifying the important controlling processes. In this problem, the global distribution of the thermoclines suggest that their existence cannot be attributed to local phenomena, such as geographical basin configurations, but must be due to globally distributed processes, such as wind-stress, surface heating and internal mixing. In the subtropical gyre, wind-stress in the surface Ekman layer induces a convergent Ekman transport which, by continuity, induces a vertical velocity, an Ekman pumping velocity, at the base of the Ekman layer. This pumps warm surface waters downward. On the other hand, in the interior, internal mixing, such as by
Figure 1.1: Potential temperature (°C) versus depth at mid-latitudes in western regions of various oceans. N = North, S = South; A = Atlantic, I = Indian, P = Pacific (from Robinson and Stommel, 1959).

Kelvin-Helmholtz instabilities, acts to warm the abyssal ocean. In a statistically-steady state this is balanced by abyssal upwelling of cold deep waters. How does the upwelling and downwelling combine to produce the thermocline structure?

The earliest attempts to analytically solve the vertical structure were the simultaneously published adiabatic, advective, similarity theory of Welander (1959) and the diffusive similarity theory of Robinson and Stommel (1959). Taking diffusion to represent diapycnal mixing, the Rossby number to be small and the effects of compressibility to be neglected, then subject to a prescribed surface density and a vertical density profile that asymptotes to a uniform value at depth, they both solve the steady-state planetary-scale fluid equations, or “thermocline equations”:

\[
\begin{align*}
\text{momentum equation,} & \quad \rho_0 f k \times \mathbf{v} = -\nabla_h p; \\
\text{continuity,} & \quad \nabla \cdot \mathbf{u} = 0; \\
\text{hydrostatic balance,} & \quad \frac{\partial p}{\partial z} + \rho g = 0; \\
\text{thermodynamic equation,} & \quad \mathbf{u} \cdot \nabla \rho = \kappa_v \frac{\partial^2 \rho}{\partial z^2} + \kappa_h \nabla_h^2 \rho,
\end{align*}
\]

where \( \mathbf{u} \) and \( \mathbf{v} \) are three-dimensional and horizontal velocities relative to a local Cartesian frame of reference, \( xi + yj + zk \), \( p \) is pressure, \( \rho \) and \( \rho_0 \) are density and reference density, \( f \) is the Coriolis parameter, \( g \) is the gravitational acceleration and \( \kappa_v \) and \( \kappa_h \) are the vertical and horizontal mixing coefficients.
1.2 Diffusive thermocline theory

An internal thermocline that is controlled by diffusive dynamics ($\kappa_v \neq 0$) can be considered as an internal boundary layer formed at the interface between the wind-driven upper ocean and the thermally-driven abyssal ocean (see Fig. 1.2). Here, diffusion is necessary to smooth the vertical density gradient created by the warm downwelling and cold abyssal upwelling and to thus prevent a density discontinuity (Robinson and Stommel, 1959; Stommel and Webster, 1962; Welander, 1971b; Salmon, 1990).

Figure 1.2: A diffusive, internal, thermocline is found at the interface between downwelling warm surface water and upwelling cold abyssal water. Diffusion (representing diabatic mixing) smoothes the density gradient across the front and prevents a discontinuity.

Robinson and Stommel solved (1.1)–(1.4) by combining them into a single equation and assuming the solution fits a particular form. Defining the function $G$ such that

$$p = \rho_0 \frac{\partial G}{\partial z},$$

(1.5)
(1.1)–(1.4) can be written in terms of $G$. From (1.1),

\[ u = -\frac{1}{f} \frac{\partial^2 G}{\partial z \partial y}, \]
\[ v = \frac{1}{f} \frac{\partial^2 G}{\partial z \partial x}. \]

Substituting into (1.2) gives

\[ w = \frac{\beta}{f^2} \frac{\partial G}{\partial x}, \]
where $u$, $v$ and $w$ are the components of $\mathbf{u}$ and $\beta$ is the meridional gradient of the Coriolis parameter, $f = f_0 + \beta y$ (for $f_0$ a constant). Hydrostatic balance (1.3) is rewritten as

\[ \rho = -\frac{\rho_0 \partial^2 G}{g \partial z^2}. \]

Finally, $u$, $v$, $w$ and $\rho$ can be eliminated from the steady-state thermodynamic equation (1.4) to leave:

\[ -\frac{1}{f} \left( \frac{\partial^2 G}{\partial y \partial z} \frac{\partial^3 G}{\partial x^2 \partial z^2} - \frac{\partial^2 G}{\partial x \partial z} \frac{\partial^3 G}{\partial y \partial z^2} - \frac{\beta}{f} \frac{\partial G}{\partial x} \frac{\partial^3 G}{\partial z^3} \right) = -\kappa_v \frac{\partial^4 G}{\partial z^4} - \kappa_h \left( \frac{\partial^4 G}{\partial x^2 \partial z^2} + \frac{\partial^4 G}{\partial y^2 \partial z^2} \right). \]

Special solutions of the form $G = x F(z)$ might then be sought under appropriate boundary conditions (for example, Welander, 1959; Robinson and Stommel, 1959; Stommel and Webster, 1962).

Scalings for the depth, $D$, and thickness, $\delta_i$, can be obtained for the internal thermocline (illustrated in Fig. 1.3). Assuming that in the upper ocean the vertical velocity is dominated by the action of the surface Ekman pumping velocity $w_{ek}$, such that diffusive phenomena, though present at all depths, are only considered
to be significant beneath the vertical extent of the wind-driven influence. Then the depth, $D$, of the internal thermocline is given by the vertical extent of the wind-driven influence and can be obtained by combining scalings for adiabatic advection with thermal wind balance to get

$$D \sim w_{ek}^{1/2}$$

(Welander, 1971b; Stommel and Webster, 1962). This favourably compares with the thermocline depth (1.39) in the Luyten et al. (1983) adiabatic layered model.

The scaling for the thickness of the internal thermocline is therefore governed by the vertical diffusivity coefficient, $\kappa_v$, and is obtained by combining a scaling for thermal wind balance, across the thermocline, with a scaling for the thermodynamic equation, to obtain

$$\delta_i \sim \kappa_v^{1/2}$$

(Salmon, 1990, after Stommel and Webster (1962)). Note that these simple scaling arguments do not include the possible interactions between the western boundary current and the gyre recirculation (Spall, 1992), nor do they accommodate deep convective mixed layer events that are found in the cooler northerly waters.

### 1.3 Adiabatic thermocline theory

Ventilated thermocline theories are the adiabatic solutions, $\kappa_v = \kappa_h = 0$, to (1.1)–(1.4). The first solution to be found was a similarity solution, derived by Welander (1959), and is given in Section 1.3.1. It was, however, too complex and contrived to evoke the physical insight for which the subsequent Luyten et al. (1983) layered model, described in Section 1.3.2, is known.

#### 1.3.1 Similarity solution

A typical solution to (1.1)–(1.4) can be most easily demonstrated by bypassing (1.10). Instead, a solution is obtained by following Welander (1971a) whereby the large-scale potential vorticity,

$$Q = -f \frac{\partial \rho}{\partial z},$$

(1.13)
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Montgomery potential,

\[ M = p + \rho g z, \quad (1.14) \]

and density, \( \rho \), are each conserved following the flow. Thus the potential vorticity can be expressed as

\[ Q = Q(\rho, M). \quad (1.15) \]

In order to solve (1.15) analytically and obtain an expression for the vertical density profile, it is further assumed that potential vorticity can be globally written as a linear function of density and Montgomery potential,

\[ Q = a \rho + b M, \quad (1.16) \]

for constants \( a \) and \( b \), which are imposed to fit the observed thermocline depth scale, \( D_{\text{sim}} \) (1.23), and the thermocline thickness scale, \( \delta_{\text{sim}} \) (1.24). The vertical structure of the potential vorticity is then obtained as follows. Firstly, differentiating (1.16) with respect to \( z \),

\[ \frac{\partial Q}{\partial z} = a \frac{\partial \rho}{\partial z} + b \left( \frac{\partial p}{\partial z} + g z \frac{\partial \rho}{\partial z} + \rho g \right). \quad (1.17) \]

Then eliminating \( p \) using hydrostatic balance (1.3),

\[ \frac{\partial Q}{\partial z} = a \frac{\partial \rho}{\partial z} + b g z \frac{\partial \rho}{\partial z}, \quad (1.18) \]

and substituting for \( Q \) (1.13), gives

\[ \frac{\partial Q}{\partial z} = -\frac{Q}{f} (a + b g z). \quad (1.19) \]

This is solved, for \( Q \), by separating the variables and then integrating,

\[ \int \frac{dQ}{Q} = - \int \frac{a + b g z}{f} dz, \quad (1.20) \]

such that

\[ \ln Q = - \frac{2 a z + b g z^2}{2 f} + k(\rho, M), \quad (1.21) \]

where \( k \) is a function of \( \rho \) and \( M \). This gives

\[ Q = Q_{\text{sim}} e^{-\left( \frac{a + b g z}{D_{\text{sim}}} \right)^2}, \quad (1.22) \]

where \( Q_{\text{sim}} \) is the potential vorticity at \( z = -D_{\text{sim}} \), with the thermocline depth scale, \( D_{\text{sim}} \), defined as

\[ D_{\text{sim}} = \frac{a}{bg}, \quad (1.23) \]
and the latitudinally varying thermocline thickness scale, $\delta_{\text{sim}}$, defined as

$$\delta_{\text{sim}} = \left(\frac{2f}{bg}\right)^{1/2}. \quad (1.24)$$

The restrictions imposed by the \textit{a priori} assumption, of a global similarity form for the vertical structure are such that this type of theory cannot satisfy the full physical boundary conditions. These boundary conditions include western boundary current processes, arbitrary surface densities and arbitrary surface Ekman pumping. It must also be borne in mind that similarity solutions are not solely modelling physical mechanisms and that success in modelling the exponential depth profiles, seen in Fig. 1.1, must in part be attributed to curve fitting.

### 1.3.2 Layered model

An inherent weakness of the similarity solution approach lies in the inability to represent multiple dynamical regimes within the ocean circulation system. Luyten \textit{et al.} (1983), however, developed a conceptually simple alternative by approximating the upper ocean as a series of stacked two-dimensional adiabatic fluid slabs (Fig. 1.4). This not only reduces the complex three-dimensional problem to a series of coupled two-dimensional problems but also predicts multiple dynamical regimes. The layered model is forced by prescribed Ekman pumping along zonally outcropping isopycnals, such that fluid parcels experience direct forcing where they are not shielded by lighter, shallower, layers\(^1\) (shaded regions). The dynamics of this layered large-scale model is described by the material conservation of potential vorticity,

$$Q_n = \frac{f}{h_n}, \quad (1.25)$$

where $h$ is the layer thickness and the subscript $n$ denotes the layer number. This is essentially the same as the approach taken by Welander and outlined in the previous section but the layered dynamics make it physically more intuitive. The potential vorticity conservation property can be analytically derived from the model.

\(^1\)Diabatic mixed layer dynamics is implicitly included via the prescribed Ekman pumping velocity.
Figure 1.4: The Luyten et al. ventilated thermocline is established by adiabatic parcel advection, as indicated by the motion of the parcel shown. The parcel is subducted from the base of the mixed layer and is constrained to flow along an isopycnic surface according to potential vorticity conservation. Direct forcing, by Ekman pumping, only occurs when the fluid is not shielded by a lighter layer (shown by shaded volumes).

Equations:

\[ \text{momentum equation, } \rho_n f k \times v_n = -\nabla h p_n; \]  
\[ \text{continuity, } \nabla \cdot u_n = 0. \]  

(1.26)  
(1.27)

Eliminating pressure from the momentum equation (1.26), by taking the curl in the k direction, and then invoking continuity (1.27) gives

\[ u_n \cdot \nabla f = f \frac{\partial w_n}{\partial z}. \]  

(1.28)

Continuing, (1.28) is integrated over the depth of the n\(^{th}\) layer to give

\[ h_n u_n \cdot \nabla f = f (w_n - w_{n+1}). \]  

(1.29)

Noting, however, that the term in parentheses is defined as:

\[ w_n - w_{n+1} = \frac{D}{Dt} z_n - \frac{D}{Dt} z_{n+1} = \frac{D}{Dt} h_n, \]  

(1.30)

then in steady state, (1.29) can be re-expressed as

\[ h_n u_n \cdot \nabla f = f u_n \cdot \nabla h_n, \]  

(1.31)
and therefore,
\[ \frac{\mathbf{u}_n \cdot \nabla f}{h_n} - \frac{f \mathbf{u}_n \cdot \nabla h_n}{h^2_n} = 0. \] (1.32)

Hence,
\[ \mathbf{u}_n \cdot \nabla \left( \frac{f}{h_n} \right) = 0. \] (1.33)

That is, in the absence of direct forcing, the barotropic potential vorticity in each layer is conserved following the flow.

Furthermore, the zonal dependence of the thermocline depth can be approximated with simple algebra. Assuming that the thermocline depth can be modelled by the depth of the deepest moving layer that flows over a motionless abyss, and assuming that this layer is ventilated, then in the zonal band where this layer experiences direct forcing, an expression for the layer depth, and hence the thermocline depth, can be deduced. This is done for layer 3 from Fig. 1.4. Rewriting the momentum equation (1.26) in the \( y \)-direction for the deepest moving layer gives
\[ v_3 = \frac{g^*_3}{f} \frac{\partial}{\partial x} h_3, \] (1.34)

where \( g^*_3 = g \frac{\rho_4 - \rho_3}{\rho_0} \) is the reduced gravity, \( \rho_0 \) is a reference density and \( \rho_3 \) and \( \rho_4 \) are the densities of layers 3 and 4. Rewriting the depth integrated linear vorticity balance (1.29) for the region where \( w_3 = w_{ek} \) and \( w_4 = 0 \) results in
\[ \beta h_3 v_3 = f w_{ek}. \] (1.35)

Substituting for \( v_3 \) from the above gives
\[ \frac{g^*_3 \beta h_3}{f} \frac{\partial}{\partial x} h_3 = f w_{ek}, \] (1.36)

or
\[ \frac{\partial}{\partial x} h_3^2 = \frac{2f^2 w_{ek}}{g^*_3 \beta}. \] (1.37)

Finally, solving for the depth, \( h_3 \), when \( h_3 = H_0 = \text{constant} \) on the eastern boundary at \( x = x_E \),
\[ h_3 = \sqrt{D_0^2 + H_0^2}, \] (1.38)

where
\[ D_0^2 = -\frac{2f^2}{g^*_3 \beta} \int_x^{x_E} w_{ek}(x, y) dx. \] (1.39)
This result suggests that the thermocline depth, $D \approx h_3$, varies with the Ekman pumping velocity, $w_{ek}$, such that $D \sim w_{ek}^{1/2}$.

A distinguishing feature that sets the layered model apart from the similarity solutions is the ability to represent different dynamical regimes (as depicted in Fig. 1.5). There is a ventilated regime, as already described, where the fluid conserves its potential vorticity following the flow when shielded from direct forcing. Beyond the southernmost extent of the ventilated region lies the “shadow zone”. Geostrophic flow cannot enter this region because the streamlines must emanate from the eastern boundary. Therefore, as there can be no flow through the boundary, the flow stagnates. The third regime is the homogeneous pool. Adjacent to the western boundary there is an area where the potential vorticity contours cannot be

![Figure 1.5: A plan view of layer 2 from Fig. 1.4 showing four different dynamical regimes of the Luyten et al. ventilated thermocline theory.](image-url)

(a) In the outcrop region the potential vorticity of a subducting parcel is set under the influence of direct forcing by Ekman pumping. (b) In the ventilated region, beneath layer 1, the parcel (shown) is shielded from the forcing and the dynamics is determined according to potential vorticity conservation. (c) Beyond the most southerly extent of the ventilated fluid lies the shadow zone, where the flow stagnates. (d) In the west there is a region where the potential vorticity is too low to be set by ventilation, instead other processes control the potential vorticity, such as eddy homogenisation of potential vorticity (Rhines and Young, 1982a), as described in Section 1.4.
traced back to the surface; instead they enter and exit from the western boundary current. Along these contours, where a potential vorticity conserving flow is possible, additional dynamics will be required to determine its solution (see Section 1.4). Despite being able to represent multiple dynamical regimes, the Luyten et al. (1983) model does not explicitly include a western boundary current which may be active in setting the thermocline structure. Instead, the recirculating fluid entering from the western boundary current has a potential vorticity distribution that is determined by an arbitrary choice of $H_0$, the thermocline depth at the eastern boundary.

Potential vorticity maps generated from climatological observations generally support the ventilated, adiabatic, thermocline theory (McDowell et al., 1982; Keffer, 1985; Talley, 1985). Taking Talley (1985) as an example, Fig. 1.6 shows potential vorticity calculated between potential density bands, overlayed with a dashed line that denotes the isopycnic extent of the homogeneous pool as calculated by an adiabatic layered model. Though the potential vorticity is not completely conserved following the time-mean flow (implying some eddy mixing), the observational data does favourably compare with the model simulation. This suggests that though mixing affects the material conservation of potential vorticity, it does not influence the first order approximation of the dynamics and structure.

### 1.4 Eddy mixing of potential vorticity

The ocean circulation is active on all length scales ranging from the gyre scale to the mixing scale. The most energetic length scale is the Rossby deformation radius (see, for example, Stammer, 1997), which in the ocean is 20-100 km whereas in the atmosphere it is 1000 km. Fig. 1.7 (Stammer and Wunsch, 1994) demonstrates the richness of mesoscale structure in the surface ocean where altimeter measurements of surface elevation from TOPEX/POSEIDON data give the surface streamlines of geostrophic flow. Understanding the eddy mechanisms in the global ocean circulation is particularly important as, unlike in the atmosphere, the most energetic eddies in the ocean have a length scale that is too small to be feasibly resolved in numerical global ocean simulations.

Nevertheless, it is known that eddies generally act to mix potential vorticity
Figure 1.6: Potential vorticity, \((f/\rho)(\partial\rho/\partial z)\), calculated from Levitus (1982). Pacific climatology is interpolated onto \(\sigma_\theta\) surfaces for (a) \(\sigma_\theta = 25.525\), (b) \(\sigma_\theta = 26.025\) and (c) \(\sigma_\theta = 26.825\). The dashed contour shows the extent of the homogeneous pool region as calculated from a modified Luyten \textit{et al.} model with realistic forcing. Taken from Talley (1985).
Figure 1.7: Surface elevation over the world’s oceans, taken from the TOPEX/POSEIDON altimeter satellite. The green colours show depressions of up to 24 cm relative to the 2 year mean; the red colours show elevations of up to 24 cm relative to the 2 year mean. The global oceans are rich in mesoscale variability. Taken from Stammer and Wunsch (1994).
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on isopycnal surfaces (Green, 1970). Rhines and Young (1982a,b) argue that in many cases, beneath or removed from regions of external forcing (for example, heating or wind stress), lateral eddies can act to homogenise potential vorticity within closed time-mean streamlines, and that this mechanism could describe the ocean structure. In their quasi-geostrophic circulation theory of potential vorticity, which is homogenised within planetary gyres, Rhines and Young assume that, within a closed streamline, parcels can recirculate without significant change to their potential vorticity, even within boundary currents. This is in stark contrast to the classical homogeneous ocean picture (Stommel, 1948) where western boundary current dissipative changes to a fluid parcel’s potential vorticity (or more specifically, its vorticity) are necessary in order to balance the action of the wind-stress curl and close the gyre circulation. Rhines and Young assume that the eddy mechanism induces perturbations to an otherwise prescribed stratification, which only varies with depth, to give their homogenisation result that can be obtained as follows. In the absence of forcing, for a non dissipative and non divergent flow, where in statistically steady state there is no advection of potential vorticity,

\[ \nabla \cdot \overline{\mathbf{v}Q} = 0, \]  

(1.40)

the mean potential vorticity contours coincide with statistically steady streamlines such that

\[ \overline{Q} = \overline{Q(\psi)} \]  

(1.41)

(where “bar” terms denote time-mean quantities).

Integrating (1.40) over an area bounded by a time-mean streamline and applying the divergence theorem gives,

\[ \int \int_{\psi} \nabla \cdot \overline{\mathbf{v}Q} dA = \oint_{\psi} \mathbf{v}Q \cdot \mathbf{n} dl + \oint_{\psi} \mathbf{v'}Q' \cdot \mathbf{n} dl = 0 \]  

(1.42)

and therefore,

\[ \oint_{\psi} \mathbf{v'}Q' \cdot \mathbf{n} dl = 0, \]  

(1.43)

with \( \mathbf{n} \) given by

\[ n_i = \frac{1}{|\nabla \psi|} \frac{\partial \psi}{\partial x_i} \]  

(1.44)
(where “prime” terms denote deviations from the time-mean quantities). Assuming the potential vorticity flux, \( \vec{v}'Q' \), can be parameterised as a down-gradient flux of potential vorticity (Green, 1970):

\[
\vec{v}'_i Q'_j = -\kappa_{ij} \frac{\partial Q}{\partial \psi} \frac{\partial \psi}{\partial x_j},
\]

where \( \kappa_{ij} = A\delta_{ij} \) for \( A \), a constant (this is equivalent to pure strain, but a tensor with an antisymmetric component would not affect the subsequent result), then

\[
\oint \psi \left[ -\kappa_{ij} \frac{\partial Q}{\partial \psi} \frac{\partial \psi}{\partial x_j} \frac{\partial \psi}{\partial x_i} \right] dl | \nabla \psi | = 0,
\]

or

\[
- \frac{\partial Q}{\partial \psi} \oint \psi \kappa_{ij} \frac{\partial \psi}{\partial x_j} \frac{\partial \psi}{\partial x_i} \frac{dl}{| \nabla \psi |} = 0.
\]

But since

\[
\kappa_{ij} \frac{\partial \psi}{\partial x_j} \frac{\partial \psi}{\partial x_i} > 0,
\]

this implies \( \partial Q/\partial \psi = 0 \) and that \( \overline{Q} \) is a constant function of \( \overline{\psi} \). Therefore, within regimes bounded by statistically steady smooth streamlines, \( \overline{Q} \) must be uniform.

This scenario is numerically demonstrated in a quasi-geostrophic experiment (Holland, 1985). Fig. 1.8 shows that in the quiescent region, outside of the gyre, the potential vorticity reflects the planetary \( \beta \)-effect, whereas inside the gyre, within closed streamlines, the potential vorticity is homogenised by the action of eddies. Observational evidence also gives support for this mechanism (Coats, 1981; McDowell et al., 1982; Keffer, 1985; Talley, 1985). Fig. 1.6 illustrates the effect of the homogenisation becoming more and more influential with increasing depth; on the shallowest isopycnal (a) the figure suggests that a ventilation mechanism is responsible for a large part of the diagnosed potential vorticity pattern. This influence is lessened in (b) and on the deepest isopycnal shown (c) the whole of the North Pacific subtropical gyre has a nearly uniform potential vorticity.

Williams (1991), however, showed that ventilation can also lead to nearly uniform potential vorticity, suggesting that the observational evidence does not necessarily give irrevocable support for the Rhines and Young mechanism. Additionally, theoretical (Pedlosky and Young, 1983) and numerical (Pedlosky, 1983) simulations, modelling the interplay between ventilated surface waters and deeper eddy
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Figure 1.8: In a quasi-geostrophic model, the mean potential vorticity field is shown for a layer absent of forcing. It is plotted as (a) a contour plot and (b) a perspective mesh with $Q$ along the vertical axis. Note the uniform potential vorticity within the gyre and the dominance of planetary vorticity outside the gyre. Taken from Holland (1985)

processes that homogenise potential vorticity, show that even the presence of weak ventilation will significantly reduce the strength and size of homogeneous pool circulations. However, an alternative mechanism – eddy shedding that removes baroclinic instability – is found to homogenise potential vorticity in ventilated regions (Cox, 1985). Furthermore, the non dissipative western boundary current assumption of Rhines and Young is argued as inappropriate for the ocean circulation (Ierley and Young, 1983), though scaling analysis suggests the western boundary current is not strongly dissipative either (Young, 1984). Nevertheless, the presence of nearly uniform potential vorticity, as shown in observational data, has strong implications on the structure of the gyre in that region: it implies that the separation of density surfaces must be varying in such a way as to balance the corresponding changes in the Coriolis parameter – a strong restriction on the shape of the gyre.

Further discussion on the role of eddies in setting the vertical structure of the ocean is deferred until Chapter 4. However, the evidence seems to suggest that eddies are not of first order importance. The remaining thermocline theories are, then, the diffusive and adiabatic thermoclines theories.
1.5 Is the subtropical thermocline diffusive or adiabatic?

The discussion as to whether thermoclines are essentially ventilated and adiabatic or diffusive has not been categorically resolved by observations. Observational estimates show the diapycnal mixing rate, $\kappa_v$, to be small, typically $10^{-5} \text{ m}^2\text{s}^{-1}$ (Ledwell et al., 1993; Toole et al., 1994), suggesting that the thermocline may be considered as adiabatic to leading order. Salmon (1990), however, claims that a purely adiabatic model is inconsistent with the asymptotic limit of $\kappa_v \rightarrow 0$. He argues that in this limit the ocean would reduce to two vertical layers, with an internal boundary layer thermocline separating the warm surface waters of the subtropical gyre from the cold abyss. In the limit of no vertical diffusion the boundary layer arises because the order of the partial differential equations decreases in $z$, reducing the required number of boundary conditions. Salmon argues that it is then inconsistent to prescribe an arbitrary surface density field, as in the ventilated theory of Luyten et al. So, is the subtropical thermocline diffusive or adiabatic?

In a planetary geostrophic ocean model, Samelson and Vallis (1997a) simulate the double thermocline structure (Fig. 1.9) apparent in observations (Fig. 1.10) and also in more complex primitive equation models (Cox and Bryan, 1984). Therefore, the planetary geostrophic, non eddy-resolving, simplified dynamics is considered sufficient to capture the basic vertical structure of the ocean. Model parameter sensitivity tests show that the upper thermocline is controlled by ventilated dynamics and the lower thermocline is essentially diffusively controlled. Further details are deferred until Chapter 2.

The underlying physical reasons, however, for the existence of the two thermoclines are not identified by Samelson and Vallis. Marshall (2000) proposes that the existence of two thermoclines can be understood through an integral constraint that must be satisfied when a gyre is closed by a western boundary current. The terms in this balance resemble those in a buoyancy budget but have a natural physical interpretation as fluxes of potential vorticity associated with advection, diapycnal mixing, convection and friction. This integral constraint lies at the heart of this
Figure 1.9: Meridional cross section of $\sigma$ (2.1) from a solution of the Samelson and Vallis planetary geostrophic ocean model. This section is halfway across the gyre and exhibits a double thermocline structure, with a weakly stratified separating mode water, characteristic of the stratification found in the North Atlantic.

thesis and is used to investigate the effect of closing the gyre-scale circulation with a western boundary current.

1.6 Structure of the thesis

In Chapter 2, the Samelson and Vallis (1997a) planetary geostrophic ocean model is presented. Then in Chapter 3, the potential vorticity flux integral constraint is derived and applied to different levels within the planetary geostrophic model, highlighting the need for careful consideration of the western boundary current. In Chapter 4, the analysis is extended and applied to data from an eddy-permitting model. Finally, the conclusions are presented in Chapter 5.
Figure 1.10: A meridional cross section of $\sigma$ (2.1) in the North Atlantic showing a double thermocline structure at 50W. Adapted from McCartney (1982).
Chapter Two

The Thermocline in a Closed Planetary Geostrophic Ocean Model

The aim of this chapter is to introduce the Samelson and Vallis (1997a) planetary geostrophic ocean model. The model domain is explored in terms of its velocity and density fields to develop an understanding of the vertical structure that will serve as a conceptual frame of reference in the subsequent chapters. The Samelson and Vallis planetary geostrophic model is chosen for a number of reasons. Principally, unlike the thermocline theories in Section 1.2 and Section 1.3, the planetary geostrophic model is closed at the western boundary, meanwhile retaining the simplest and most computationally efficient dynamics that can capture a realistic density structure (c.f. Fig. 1.9 and Fig. 1.10). It is also useful to study a model of intermediate complexity before moving on to a full ocean general circulation model in Chapter 4.

In Section 2.1 the model formulation is described and in Section 2.2 the findings of Samelson and Vallis (1997a) are briefly summarised. The vertical structure of the subtropical gyre is explored further using a new trajectory buoyancy budget analysis in Section 2.3, and by considering the simple dynamics of thermocline genesis and development in Section 2.4. Also in this chapter, it is shown, in terms of simple dynamics, that closing the geostrophic interior with a dissipative western boundary current could, in principle, significantly modify the vertical structure of the interior.
2.1 Model formulation

The “thermocline equations” (1.1)–(1.4) model the steady-state response to surface forcing and though they implicitly model the large scale, by excluding eddy processes, the solutions still exhibit a certain richness. However, by their inability to accommodate western boundary currents, they are fundamentally incapable of completely modelling the large-scale circulation.

The requirement of no normal flow and no normal heat flux at rigid western boundaries raises the order of the thermocline equations, which are satisfied by the implementation of a combined friction and diffusion scheme. Friction in the form of linear drag (Killworth, 1985) closes the circulation and appears in the horizontal momentum equation (2.2), whilst horizontal biharmonic and Laplacian diffusion of density both appear in the thermodynamic equation (2.5). This apparently obscure choice of friction and diffusion terms is required to obtain a consistent equation set which can be efficiently solved (Samelson and Vallis, 1997b).

Samelson and Vallis numerically solve the planetary geostrophic equations in a closed rectangular hemispheric basin (without bottom bathymetry). The model is forced by a zonal wind-stress that varies with latitude, on a β-plane, and by a surface heat flux that is proportional to the difference between the sea surface temperature and an apparent “equilibrium” temperature. Static instabilities are removed by a convection scheme that vertically mixes grid-point fluid volumes to render the fluid neutrally stable. In this model, the density is defined to have no pressure or salinity dependencies. Hence density, \( \rho \) (kg m\(^{-3}\)), is equivalent to potential density. For convenience the quantity \( \sigma \) is defined as a dimensionless function given by

\[
\sigma = \frac{\rho - 1000 \text{ kg m}^{-3}}{1 \text{ kg m}^{-3}}. \tag{2.1}
\]

The equations of motion are:

- momentum equation, \( \mathbf{k} \times \mathbf{f} \mathbf{v} + \frac{1}{\rho_0} \nabla h \rho = \mathbf{F} \); \hspace{1cm} (2.2)

- continuity, \( \nabla \cdot \mathbf{u} = 0 \); \hspace{1cm} (2.3)

- hydrostatic balance, \( \frac{\partial \rho}{\partial z} + \rho g = 0 \); \hspace{1cm} (2.4)

and the thermodynamic equation, \( \frac{D\sigma}{Dt} = B \), \hspace{1cm} (2.5)
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for the horizontal velocity \( \mathbf{v} \), three-dimensional velocity \( \mathbf{u} \), Coriolis parameter \( f \), density and reference density \( \rho \) and \( \rho_0 \), pressure \( p \), acceleration due to gravity \( g \) and buoyancy forcing \( B \). The effects of salinity are neglected. The friction force, \( \mathbf{F} \), as already described, takes the form of a linear drag term, with coefficient \( \eta \):

\[
\mathbf{F} = -\eta \mathbf{v}.
\]  

(2.6)

The buoyancy forcing term, \( B \), incorporates the biharmonic and Laplacian density diffusion terms,

\[
B = \kappa_v \frac{\partial^2 \sigma}{\partial z^2} + \kappa_h \nabla_h^2 \sigma - \lambda \nabla_h^4 \sigma,
\]  

(2.7)

where the coefficients, \( \lambda \) and \( \kappa_h \), are chosen to be small enough (relative to \( \kappa_v \)) to ensure that vertical diffusion dominates the diapycnal fluxes, but also large enough that the lateral boundary layers can be resolved. At the surface, representing the base of the Ekman layer, the model is mechanically forced by a prescribed Ekman pumping,

\[
w_{ek} = w_0 \cos\left(\frac{2\pi y}{L}\right),
\]  

(2.8)

for constant \( w_0 \) and the meridional domain length \( L \).

The equations of motion are solved by combining (2.2) and (2.3) into an elliptic equation and initially solving for the barotropic pressure field. The technique is described below. Firstly, (2.2) is inverted:

\[
\mathbf{v} = \frac{\gamma}{\rho_0} (f \mathbf{k} \times \nabla p - \eta \nabla p),
\]  

(2.9)

where \( \gamma = 1/(f^2 + \eta^2) \). Then the horizontal velocity, \( \mathbf{v} \), is eliminated from the continuity equation (2.3) to give:

\[
\mathcal{H}(p) = \rho_0 \frac{\partial w}{\partial z} = \eta \gamma \Delta_h p + (f^2 - \eta^2) \gamma^2 \beta \frac{\partial p}{\partial x} - 2\eta f \gamma^2 \beta \frac{\partial p}{\partial y}.
\]  

(2.10)

A diagnostic equation for the barotropic pressure, \( P \), where

\[
P = \int_{\text{sea floor}}^{\text{sea surface}} p \, dz
\]  

(2.11)

is then derived by vertically integrating (2.10) from \( w = 0 \), at the sea floor, to \( w = w_{ek} \), at the surface. The above produces the desired elliptic equation for the barotropic pressure:

\[
\mathcal{H}(P) = w_{ek}.
\]  

(2.12)
Subject to the condition of no net normal flow through the lateral boundaries of the domain, (2.12) is solved. Then, for a known density field at a given time, it is stepped forward in time using the thermodynamic equation (2.5) and integrated vertically using hydrostatic balance (2.4) to give the baroclinic pressure field. The sum of the baroclinic and barotropic pressure fields give the model pressure from which the new horizontal velocity field is deduced via the momentum equation (2.2) and continuity (2.3). The density field is then advected on another timestep, with the new velocity field, and the process is repeated.

In the special case where the linear drag coefficient, $\eta$, and Ekman pumping, $w_{ek}$, are time independent, the solution to the elliptic equation (2.12) need only be calculated once. This is a product of the judicious choice of the form and placement, in (2.2) and (2.5), of the friction and diffusion terms, and makes this formulation of the planetary geostrophic equations an order of magnitude more computationally efficient than other closed domain models (for example: Cox and Bryan, 1984, primitive equation model; Colin de Verdière, 1988, planetary geostrophic model), which have to solve the elliptic equation at each time step. This efficiency makes it feasible to conduct a full steady-state sensitivity analysis on a workstation.

In this thesis, the relevant parameter values are: $\kappa_v = 10^{-5} \text{m}^2\text{s}^{-1}$, $\kappa_h = 10 \text{m}^2\text{s}^{-1}$, $\lambda = 1.25 \times 10^{12} \text{m}^4\text{s}^{-1}$, $w_0 = 10^{-6} \text{m}\text{s}^{-1}$ and $\eta = 4.2 \times 10^{-6} \text{s}^{-1}$. The solution is calculated in an idealised rectangular domain, $65 \times 65 \times 66$ grid points ($5000 \text{km} \times 5000 \text{km} \times 5 \text{km}$) in the $x, y$ and $z$ directions, centred at $35^\circ \text{N}$, and wholly neglects the effects of salinity.

### 2.2 Samelson and Vallis (1997a) overview

Modelling the interactions between large-scale processes, Samelson and Vallis have demonstrated that an adiabatic and a diffusive thermocline can coexist within a subtropical gyre. The basic structure of their closed planetary geostrophic solution is illustrated in Figs. 2.1a–d. Fig. 2.1a shows the amplitude and distribution of the imposed Ekman pumping that varies sinusoidally in the meridional direction such that a subtropical, anticyclonic, recirculation forms in-between two, half gyre, cyclonic circulations. Fig. 2.1b shows the sea surface $\sigma$ field (solid lines) and the
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Figure 2.1: The structure of the planetary geostrophic ocean model solution in a closed domain is illustrated by panels (a)-(d). Panel (a) shows the surface prescribed Ekman pumping velocity, \( w_{ek} \times 10^6 \text{ m s}^{-1} \). Panel (b) shows the surface \( \sigma \) field, with dashed contours representing the apparent equilibrium temperature to which sea surface conditions relax. Panels (c) and (d) give the resulting pressure patterns (Nm\(^{-2}\)) as anomalies relative to the eastern boundary at depths of 90 m (anticyclonic) and 2290 m (cyclonic).
linearly varying apparent equilibrium temperature (dashed lines) to which the sea surface would adjust in the absence of any circulation. In particular, this figure shows that lighter, warmer, fluid is strongly advected northward by the western boundary current. Fig. 2.1c and Fig. 2.1d show pressure field patterns, calculated at depths of 90 m and 2290 m, as anomalies relative to the eastern boundary; the upper ocean flow is similar to the barotropic component, with flow intensification near the western boundary, whereas at depth there is a weak cyclonic recirculation with that which could be described as a deep western boundary current.

Taking a vertical $\sigma$ profile through the centre of the domain (Fig. 2.2a) shows that the density variations are entirely concentrated to the upper 1000 m in the form of two distinct thermoclines that are separated by a region of weak stratification, the mode water. The deeper of the thermoclines coincides with the depth at which the vertical velocity, $w$, tends to zero (Fig. 2.2b). This is in qualitative agreement with the diffusive thermocline theory (Section 1.2) that anticipates a diffusive thermocline to be found at such a depth, below which weak diffusive processes can dominate the diminished surface-forced processes. Cross sectional

![Figure 2.2: (a) $\sigma$ and (b) vertical velocity, $w_{ek} \times 10^6$ m s$^{-1}$, profiles taken at the centre of the domain show that the density variations are restricted to the upper 1000 m and are in the form of two thermoclines. The depth of the lower thermocline coincides with the depth at which the vertical velocity, $w$, first goes to zero.](image-url)
plots of $\sigma$ further support the hypothesis that the internal thermocline is coincident with the $w = 0$ contour (Figs. 2.3a,b) with the zero contour (dashed) adequately 

Figure 2.3: $\sigma$ profiles through the centre of the domain in the (a) meridional and (b) zonal direction are overlayed with the $w = 0$ contour (dashed). The $w = 0$ contour approximately follows the depth of maximum stratification in the subtropical gyre.

capturing the bowling nature of the isopycnals and the general depth of the stratification maximum over most of the subtropical gyre. (The barotropic subtropical gyre boundaries are defined by the latitudes at which the depth integrated flow is entirely zonal. By the linear vorticity balance (1.28), this is where the prescribed surface Ekman pumping velocity is zero).

Other interesting features in Fig. 2.3a include the southward merging of the two thermoclines into a single stratification belt, whereas deep mixed layer convection and the formation of a mode water act to separate the thermoclines in the north. Moreover, a distinction is also seen between outcropping isopycnals; the shallow thermocline and mode water both outcrop in the subtropical gyre whereas the internal thermocline outcrops in the subpolar gyre, north of where $w = 0$. To first approximation, the authors find that the internal thermocline density step is the same as the density step across the surface of the subpolar gyre. Furthermore, Fig. 2.3b show the putative westward deepening of stratification in good agreement with the ventilated thermocline theory of Luyten et al. (1983).
Demonstrating the coexistence of the adiabatic and diffusive thermoclines, Samelson and Vallis numerically test the scalings for the depth and thickness of the internal thermocline as a function of the imposed Ekman pumping and coefficient of vertical mixing, as reviewed in Section 1.2 (see Figs. 2.4a,b). In the limit of weak vertical mixing, small \( \kappa_v \), the depth of the stratification maximum, \( D \), is found to be independent, to first order, of \( \kappa_v \) but varies linearly with \( w_{ek}^{1/2} \) (see Fig. 2.4a), in agreement with the scaling (1.11). Furthermore, the thickness of this thermocline, \( \delta_i \), estimated as the lower half width of the half maximum of the stratification, is found to vary linearly with \( \kappa_v^{1/2} \) (Fig. 2.4b), again consistent with the scaling (1.12).

In summary, for the region where isopycnals outcrop, thermal forcing, Ekman pumping and advection combine to produce a shallow thermocline. The dynamics
of this regime is essentially described by the ventilated thermocline theory of Luyten et al. (1983), whereby advection dominates diapycnal diffusion. Beneath the influence of the wind-driven flow, at the confluence of abyssal upwelling and Ekman pumping, lies an intrinsically diffusive, internal thermocline whose dynamics is essentially described by Stommel and Webster (1962) and Salmon (1990). Between the two thermoclines lies the weakly stratified mode water. This region is ventilated by the densest outcropping isopycnals in the subtropical gyre. The numerical planetary geostrophic vertical structure is especially encouraging as it is in accord with observations in the North Atlantic (Fig. 1.10) and other basins (McCartney, 1982).

2.3 Understanding the density structure using Lagrangian buoyancy budgets

Following fluid trajectories through the thermoclines can help us to develop an understanding of the water-mass transformation processes and water-mass pathways in the subtropical gyre (Cox and Bryan, 1984). In this section, the two steady-state thermocline regimes are investigated further with trajectory analysis, and a common association is made between Lagrangian buoyancy forcing (by convection or diffusion) and parcel displacements from geostrophy. This association is presented in terms of a heuristic steady-state buoyancy balance for trajectory paths that are nearly closed, and motivates the exact balance in potential vorticity fluxes that will be derived in Chapter 3.

2.3.1 Trajectory algorithm

The complex three-dimensional path of a fluid parcel in the planetary geostrophic ocean model is tracked using a trajectory finding routine. The model is spun up to steady state and the velocity field is integrated forward in time using a Runge-Kutta scheme (Press et al., 1999) to find steady-state trajectories. Before the integration can be computed, the velocity field must be interpolated onto the parcel’s position (as the parcel will not necessarily be advected onto a model grid point). A linear interpolation between the eight surrounding Cartesian grid points is chosen for this
operation and also for tracking the parcel’s density.

Finding the Lagrangian parcel position requires the solution to the equation:

\[ \frac{dx}{dt} = u(x). \]  

(2.13)

This is calculated using a fourth order Runge-Kutta routine in each direction, but
in the following is demonstrated only in the zonal direction:

\[ x_{n+1} = x_n + \frac{k_1}{6} + \frac{k_2}{3} + \frac{k_3}{3} + \frac{k_4}{6} + O(\delta t^5), \]  

(2.14)

where

\[ k_1 = u(x_n)\delta t, \quad (2.15) \]
\[ k_2 = u(x_n + \frac{k_1}{2})\delta t, \quad (2.16) \]
\[ k_3 = u(x_n + \frac{k_2}{2})\delta t, \quad (2.17) \]
\[ k_4 = u(x_n + k_3)\delta t, \quad (2.18) \]

and \( x_n \) is the \( x \)-position after \( n \) steps of time, \( \delta t \). At position \( x_n \), the Lagrangian zonal velocity, \( u(x_n) \), is calculated from the Cartesian variables using linear interpolation. Again, only demonstrating the zonal component: where \( x^i < x_n < x^{i+1} \),

\[ u(x_n) = (1 - s)u^i + su^{i+1}, \quad (2.19) \]

for \( s \) in the interval \( 0 - 1 \) and with subscripts denoting the \( n^{th} \) timestep and superscripts denoting the grid point number. Extension to three dimensions is trivial.

### 2.3.2 Fluid trajectory results

The complex, three-dimensional, nature of the ocean circulation is explored by
following the motion of advected water parcels in the subtropical gyre. A water
parcel’s position and properties are tracked by advecting it in the steady state
velocity field. In this section, two trajectories are presented (Fig. 2.5 and Fig. 2.6):
one through the mixed layer and ventilated mode water, and the other through the
mode water into the internal thermocline. The paths of the parcels are those of
a cork-screw, spiralling anticyclonically and outwards as the parcels are pumped
downward.
In contrast to the one-dimensional balance assumed in the internal thermocline theory, variations in the parcels’ densities, measured as $\sigma$, are found to be dependent on both the vertical and horizontal position. In the internal thermocline region (that is, outside the boundary currents) the stratification is nearly vertical, and so $\sigma$ varies principally with depth. On the other hand, in regions such as the convective mixed layer, $\sigma$ only varies with horizontal position. In the western boundary current $\sigma$ also changes with the horizontal position, and in such a way that it interacts with the gyre recirculation (Cox and Bryan, 1984; Spall, 1992).

The first trajectory corresponds to a fluid parcel that cycles through the mixed layer and the ventilated mode water (Fig. 2.5). On each recirculation the buoyancy budget illustrates that the parcel’s $\sigma$ is being reset in the mixed layer and that it is preserved on the subsequent subduction and recirculation into and through the weakly stratified interior mode water. Only on reentry to the convective mixed layer, at a new and deeper location, is $\sigma$ reset. Cross sections of the cork-screw motion in the $x$-$y$ and $y$-$z$ planes (Figs. 2.5a and 2.5b) together with the Lagrangian buoyancy budget (Fig. 2.5c) corroborate the Samelson and Vallis claim that this regime is governed by adiabatic Luyten et al. dynamics, ventilated by the mixed layer. (It is to be noted again that these trajectories are defined by integrating forward the steady-state velocity field and that convective adjustment happens outside of the model’s advective scheme, such that trajectories passing through the mixed layer do not have large vertical velocities associated with convective overturning. These are, in effect, averaged out.) Trajectories in the shallow upper thermocline are similarly ventilated by the mixed layer, but parcels do not complete a whole circuit and are not shown here.

The second trajectory corresponds to a fluid parcel that cycles through the recirculating mode water (no convective buoyancy forcing) before crossing into the internal thermocline (Fig. 2.6). In the absence of convection, changes in $\sigma$ are found to be almost entirely due to vertical diffusion. This is broadly in accord with the internal thermocline theory that assumes a balance between vertical advection and vertical diffusion. However, a more detailed analysis reveals that advection across isopycnals to not only be due to vertical motion but also due to the horizontal...
ageostrophic flow within the western boundary current. Fig. 2.6a, with 20 year markers, shows that the parcel deepens by 200m as it circulates through the mode water during the first 30 years, and then spends the next 90 years penetrating a further 200m into the internal thermocline. Concomitant with the decrease in rate of vertical descent is the increase in dominance of the horizontal, frictional, ageostrophic flow (as manifest in the vertical compression and horizontal expansion of the cork-screw recirculation in Fig. 2.6a).

It is noted that for both trajectories, ventilated and unventilated, changes in the parcels’ densities, measured as $\sigma$, correspond to changes in the parcels’ positions. This seemingly trivial statement forms the essence of a buoyancy balance that will later prove useful in interpreting an integral budget of potential vorticity flux. This buoyancy balance is now explained and is illustrated in Fig 2.7. In a steady-state $\sigma$ field, a parcel that is advected along a closed geostrophic contour will, eventually, return to its start point. At that point, it will have regained its initial $\sigma$ value. Hence the net buoyancy forcing experienced by a fluid parcel recirculating along a closed geostrophic contour is zero. However, as seen in Fig. 2.5 and Fig. 2.6, the actual trajectories have small horizontal and vertical ageostrophic motions. The ageostrophic horizontal flow is due to friction, which acts to retard the motion of the parcel, such that the pressure gradient force exceeds the Coriolis force and there is a flow away from the gyre centre. This effect is most strongly felt in the western boundary current region where the flow is intensified and hence friction is increased (Colin de Verdière, 1989). The vertical flow is forced by the surface wind-stress. The general bowled isopycnic structure of the gyre is such that both the frictional and vertical flows will correspond to the parcel cooling, if steady state is maintained. So, the changes in $\sigma$ attributed to displacements by these flows implies that the net buoyancy forcing along what would have been a closed trajectory can no longer be zero. Instead, the net buoyancy forcing along the trajectory must balance the changes in $\sigma$ attributed to the vertical and frictional displacements. Hence there is a heuristic balance between the vertically advective, frictional and buoyancy processes for contours that are nearly closed. In terms of contour integrals this could be
Chapter 2: The Thermocline in a Closed Planetary Geostrophic Ocean Model

expressed as
\[
\int w \frac{\partial \sigma}{\partial z} dt + \int \mathbf{v}_F \cdot \nabla \sigma \, dt \approx \int \mathcal{B} \, dt, \tag{2.20}
\]

where \( \mathbf{v}_F \), the horizontal ageostrophic frictional velocity, is defined as
\[
\mathbf{v}_F = -\frac{k \times \mathbf{F}}{f}, \tag{2.21}
\]

and the integral is taken along a nearly closed trajectory such that the geostrophic advection of \( \sigma \) cancels. A similar, but exact, balance using potential vorticity fluxes is explored in Chapter 3 to determine which processes are important in setting and maintaining the vertical structure of the ocean. Firstly, Sverdrup balance is interpreted in a new way that makes clear its role in influencing the vertical structure of the thermocline, and from which another balance emerges that involves western boundary current friction.

2.4 Understanding the thermocline depth using simple dynamics

It is perhaps surprising how much progress can be made towards understanding the thermocline structure with the simple adiabatic Luyten et al. type model (Section 1.3). So, before tackling the more complex closed gyre problem, it is important to have thoroughly investigated the open gyre dynamics – to establish a framework in which additional processes can be readily incorporated.

In this section, a dynamical reinterpretation of Sverdrup balance is given to interpret the establishment of the thermocline. This is made graphic with a gyre spin-up experiment. The reinterpretation of Sverdrup balance is readily extended to investigate the effect, on the thermocline structure, of closing the gyre with a simple frictional western boundary current. As seen in Section 2.1, the equations for the planetary geostrophic interior are:

geostrophic balance,
\[
f \mathbf{v}_g = \frac{k \times \nabla h p}{\rho_0}; \tag{2.22}
\]
continuity,
\[ \nabla_h \cdot \mathbf{v_g} + \frac{\partial w}{\partial z} = 0; \quad (2.23) \]

and hydrostatic balance,

\[ \frac{\partial p}{\partial z} + \rho g = 0, \quad (2.24) \]

where the vertical velocity is prescribed by wind forcing at the surface, \( w = w_{ek} \), and no normal flow, \( w = 0 \), at the sea floor.

The simplest scenario to consider is the barotropic ocean in which horizontal pressure patterns are independent of depth and hence the horizontal velocity field, \( \mathbf{v_g} \), and its divergence, \( \nabla_h \cdot \mathbf{v_g} \), are also independent of depth. It then follows, by continuity (2.23), that \( \partial w/\partial z \) is independent of depth, such that \( w \) varies linearly between \( w = w_{ek} \), at the surface, and \( w = 0 \), at the sea floor. The interesting feature here is that the barotropic flow has depth independent \( \mathbf{v_g} \) and \( \nabla_h p \) fields but a depth dependent \( w \) field. Of course, \( w \) must vary in order to satisfy the boundary conditions but the important question is: what is the mechanism that controls \( w \)? This is important since knowing, in particular, the depth at which \( w \to 0 \) is to know the depth of the thermocline; the depth to which warm surface waters can be advected, where, to leading order, the density front with the cold abyss lies.

The relationship between the vertical velocity and horizontal velocity fields, for planetary flows, is given by noting that the Coriolis weighted geostrophic velocity is non divergent. Taking the divergence of (2.22) gives

\[ \nabla_h \cdot (f \mathbf{v_g}) = 0, \quad (2.25) \]

and thus, by continuity (2.23),

\[ \beta v_g = f \frac{\partial w}{\partial z}, \quad (2.26) \]

which is the standard linear vorticity balance (that leads to Sverdrup balance when integrated over depth).

At any depth in the geostrophic interior, (2.22) states that the transport between any two adjacent pressure contours is proportional to the pressure difference divided by the Coriolis parameter. So, in the anticyclonic subtropical gyre, the
transport between two adjacent pressure contours will increase moving southwards to compensate for the decrease in Coriolis parameter, $f$. Therefore, as shown in Fig. 2.8, there must be a net mass flux leaving the geostrophic interior into the western boundary current. This divergence of horizontal flow is balanced by a convergence of vertical flow such that fluid pumped downward is entrained into the horizontal circulation, resulting in a decrease in $|\partial w/\partial z|$.

Clearly, the barotropic ocean, where $w$ varies linearly with depth and density does not vary at all, is of limited use for investigating the thermocline! Nonetheless, the above reinterpretation of Sverdrup balance remains useful when the barotropic constraint is relaxed and warm surface water is pumped into a cold interior. Horizontal variations in the depth-integrated mass of the water columns will result in an adjustment to the horizontal pressure gradients, which in turn, by geostrophy (2.22), will result in an adjustment to the flow. This is the essence of thermal wind balance which can be derived by differentiating geostrophic balance (2.22) with respect to $z$ and eliminating the pressure term using hydrostatic balance (2.24):

$$\frac{\partial v_g}{\partial z} = -g \frac{k \times \nabla \sigma}{\rho_0 f}.$$

Since the subtropical gyre is typified by a warm water centre, the thermal wind relation (2.27) requires the anticyclonic flow to decrease with depth. However, integrating the linear vorticity balance (2.26) with respect to $z$, under the prescribed boundary conditions on $w$, implies that the depth-integrated meridional transport must be constant. Consequently, Sverdrup balance (2.26) and thermal wind balance (2.27) combine to produce a surface flow intensification and an abyssal spin-down, relative to the barotropic anticyclonic flow, which in turn adjust $|\partial w/\partial z|$ and hence the depth at which $w \to 0$.

Snapshots of the vertical velocity in the centre of the subtropical gyre, during the spin-up from a cold state of rest, make the interaction between Sverdrup balance and thermal wind balance, and their role in controlling $w$, more graphic (Fig. 2.9). In this planetary geostrophic model barotropic Rossby waves are infinitely fast and both the upper and lower vertical velocities are prescribed by the boundary conditions. The first snapshot, after 600 days, shows the flow to be still barotropic, with $\partial w/\partial z$ independent of depth. But as warm water is pumped into the gyre, surface currents
are intensified by thermal wind balance which, via Sverdrup balance, translates into a change in $\partial w/\partial z$. The upper ocean $w$ profile is then established from the surface downward and is preserved as the ocean spins up such that the depth to which $w \to 0$, in the spun-up gyre, is approximately 600 m.

There is, however, a further subtlety to the combined thermal and wind driven circulation because of the asymmetry in the thermal forcing with its southward bias. Warm water in the southern half of the gyre is rapidly advected, by the horizontal geostrophic flow, into the western boundary current where it is carried northwards. Then, at a certain latitude, the northward warm light fluid finds itself in a region where it is less dense, and hence more buoyant, than the sea surface fluid. Convection ensues and the western boundary current is kept cool by deep convection in the newly formed mixed layer. At the same time, a region of weak stratification forms, a mode water, that divides the thermocline into two. (The mode water is also found to be ventilated by the convective mixed layer in a primitive equation model study, Cox and Bryan, 1984.) This additional mechanism produces a weakly stratified mode water that complicates the vertical structure of density above the internal thermocline. Fig. 2.10 shows the evolution of the potential density, also at the centre of the domain (away from the convection sites where Sverdrup balance does not hold, Colin de Verdière, 1988). Like the $w$ field, the warm surface information propagates downward but after the first snapshot (before 10 years) the convective mixed layer is formed that vigorously cools the deepening warm fluid and leads to the formation of weakly stratified mode water (as shown in the figure). Above the thermocline, as steady state is approached, a thermodynamic balance is established between warming, by Ekman pumping, and cooling, by mixed layer convection. Below the thermocline, as steady state is approached, diffusion of density between the cold abyss and the warmer surface waters becomes important, such that the thermodynamic equation (2.5) scales as

$$w \frac{\partial \sigma}{\partial z} \sim \kappa_v \frac{\partial^2 \sigma}{\partial z^2},$$

and diffusive abyssal warming is balanced by abyssal upwelling (Munk, 1966).

Up to now there has been an implicit assumption, in the dynamics discussed, that the western boundary current does not affect the geostrophic interior. This...
has been an assumption of convenience based on tractability, rather than being grounded on any physical understanding (Ierley and Young, 1983; Young, 1984). The physical interpretation of gyre-scale flow according to Sverdrup balance (illustrated in Fig. 2.8) can be extended to include circulations closed by a western boundary current. As it lies, the problem is that, in the vertical and horizontal plane, there is a net convergence of flow parallel to pressure contours in the western boundary region. This problem can be relieved if friction in the western boundary results in an ageostrophic flow, \( \mathbf{v}_F \), which “leaks” fluid across pressure contours (see Fig. 2.11). In principle, the fluid “leaked” in the western boundary current could exactly balance the mass flux excess, via Sverdrup balance, that flows into the western boundary region. However, this is a very strong constraint to impose on the ageostrophic flow (as is done by Stommel, 1948) which has no physical justification. It is more likely that the ageostrophic flow will contribute an additional component to the horizontal fluid divergence that will further adjust \( \partial w / \partial z \) and hence the thermocline depth. This fuller picture of the western boundary interacting with the circulation, such that there is a close association between vertical velocity, frictional ageostrophic velocity and geostrophic velocity in the western boundary current is consistent with the cork-screw image presented in Section 2.3 and interpreted in terms of a Lagrangian buoyancy budget (c.f. cork-screw trajectories in Fig. 2.6 and Fig. 2.7 with Fig. 2.11).

2.5 Summary of Chapter 2

The Samelson and Vallis planetary geostrophic ocean model has a double thermocline structure in the subtropical gyre. The upper thermocline has been shown to be governed by adiabatic, ventilated, dynamics whereas the internal thermocline is governed by diffusive dynamics. Trajectory analyses through these two regimes has shown recirculating parcels to cork-screw anticyclonically downward, due to Ekman pumping, and outwards, due to frictional effects. This has motivated a steady-state Lagrangian balance for nearly closed trajectories, in which buoyancy-forced \( \sigma \) changes (by convection in the upper thermocline and diffusion in the internal thermocline) are balanced by parcel \( \sigma \) changes that are attributed to vertical advection and advection by a lateral frictional flow.
Finally, a dynamical reinterpretation of Sverdrup balance has been given which has been used to qualitatively interpret the establishment of the thermocline. This dynamical interpretation has been extended to include a western boundary current, that closes the flow, and it has been argued that dissipative western boundary current processes may well affect the vertical structure of the gyre interior. These heuristic Lagrangian buoyancy budgets provide physical motivation for a similar but exact balance, given in Chapter 3, that uses potential vorticity fluxes to determine which processes are important in setting the vertical structure of the ocean.
Figure 2.5: A parcel is followed in the ventilated mode water. Tracking the advected parcel’s position shows $\sigma$ to be set by convection in the mixed layer. Outside this region the density varies relatively little. Panel (a) shows an $x$-$y$ plot of the trajectory with the points where convection occurs marked by crosses. The trajectory is overlayed with the 130m (black dashed) and 350m (red dashed) mixed layer contours. Panel (b) shows a meridional cross section of $\sigma$ through the trajectory centre. This is overlayed with a two-dimensional projection of the trajectory. The crosses along the path mark the points when the trajectory is in the mixed layer. Panel (c) shows the variation of the parcel’s $\sigma$ with time. $\sigma$ is nearly conserved while the fluid parcel lies within the ventilated thermocline but changes abruptly as it passes into the mixed layer.
Figure 2.6: The cork-screwing motion of a parcel is followed as it crosses from the recirculating mode water into the internal thermocline. The advected parcel is followed for 120 years in the subtropical gyre and is found to cross isopycnal surfaces as it cork-screws downward. The velocities are greatly reduced on the transition to the internal thermocline; markers placed at 20 year intervals show that in 30 of the 120 years, the parcel has travelled half its vertical distance and has exited the mode water. Panel (a) shows a meridional $\sigma$ cross section through the centre of the cork-screw trajectory overlayed by a two-dimensional projection of the trajectory with 20 year markers. Panel (b)
Figure 2.7: In the steady state, changes in a parcel’s $\sigma$ value, on a nearly closed trajectory, can be attributed to either vertical advection and an ageostrophic frictional flow through the static $\sigma$ field, or to buoyancy forcing (convective or diffusive) following the parcel.

Figure 2.8: For wind prescribed Ekman pumping velocity, Sverdrup balance, $\beta v_g = f \partial w / \partial z$, controls the depth at which $w$ goes to zero. For geostrophic balance to hold in the interior, that is for the Coriolis force to balance the pressure gradient force, the flow must be faster in the south where the Coriolis parameter, $f$, is smaller. Consequently, in the horizontal plane, and between two pressure contours, more fluid exits the geostrophic interior than enters. The mass balance is closed by a decrease in vertical mass flux, such that the flux into the geostrophic interior is greater than the flux out. Hence $|w|$ decreases downward.
Figure 2.9: This figure shows the evolution of the vertical velocity, $w$ (m s$^{-1}$), as a function of depth at the centre of the domain. The barotropic linear form is gradually eroded as warm water is pumped into the upper layers leading to a surface flow intensification and a change in gradient of $w$ according to the Sverdrup relation. The depth at which $w$ vanishes determines the advection depth of warm surface waters and hence the depth of the thermocline. In the steady state, there is upwelling below the main thermocline that is consistent with diffusive abyssal warming.
Figure 2.10: This figure shows the evolution of the $\sigma$ field as a function of depth at the centre of the domain. The barotropic linear form is gradually eroded as warm water is pumped into the upper layers leading to a surface flow intensification and a change in gradient of $w$ according to the Sverdrup relation. The depth at which $w$ vanishes determines the advection depth of warm surface waters and hence the depth of the thermocline. Note the depth axis is split to more clearly show the upper ocean profile. In steady state, the diffusive abyssal warming is consistent with the abyssal upwelling (Fig. 2.9).

Figure 2.11: The circulation described in Fig. 2.8 can be closed with “leaky” pressure contours (dashed) in the western boundary region (shaded). This is achieved by frictional processes that induce an ageostrophic flow. The mass balance could be closed by just adding on the western boundary region. However, this imposes a very strong, and unphysical, constraint on the ageostrophic flow, which in principle could otherwise further contribute to the depth dependence of $|\partial w/\partial z|$ and hence the thermocline structure.
Is there a precise way of quantifying the role of dissipative western boundary current processes in the control of the vertical structure of the gyre?

In Chapter 2, the vertical structure of the planetary geostrophic ocean was qualitatively discussed in terms of trajectories and it was hinted that friction, manifest in the western boundary current, could possibly play an important role in modulating the vertical structure. In this chapter, a rigorous analysis that is a balance of stratification adjusting processes is presented for a closed gyre. The balance, which can most easily be interpreted as an integral constraint on potential vorticity fluxes, reveals that friction does play an important role in the upper ocean vertical structure and is used to address the following questions:

- Does the existence of the integral constraint have physical implications for the vertical structure?
- Why are there two thermoclines?

In Section 3.1 potential vorticity and its conservation properties are introduced. In Section 3.2 the integral constraint is derived and interpreted in terms of potential vorticity fluxes that can be attributed to vertical advection, buoyancy forcing and frictional forcing processes, which locally adjust the stratification. In Section 3.3 a qualitative association between the evolution of stratification and vertical fluxes of potential vorticity is made. In Section 3.4 the potential vorticity integral constraint is refined wherein a link with the Lagrangian buoyancy budget (Section 2.3) is made.
In Section 3.5 potential vorticity flux components are diagnosed in the planetary geostrophic ocean model and the friction component is found to be essential in closing the balance, although, in Section 3.6, the character of the thermocline structure is shown to be largely insensitive to the choice of friction coefficient. Finally, the chapter is summarised in Section 3.7.

## 3.1 Potential vorticity

Ertel-Rossby potential vorticity (Rossby, 1940; Ertel, 1942), hereafter potential vorticity, is given as

\[
Q = -\frac{q \cdot \nabla \sigma}{\rho},
\]

(3.1)

for absolute vorticity,

\[
q = 2\Omega + \nabla \wedge u,
\]

(3.2)

Earth’s angular velocity, \(\Omega\), actual density, \(\rho\), \(\sigma\) (2.1) and the three-dimensional velocity field, \(u\). The conservation properties associated with potential vorticity have long been recognised as important concepts in geophysical fluid dynamics (for a review, see Hoskins et al., 1985; Holland et al., 1984; and Rhines, 1986). These properties are:

1. In the absence of mechanical or buoyancy forcing, potential vorticity is a materially conserved tracer. That is, it is conserved following the flow;

2. Under appropriate constraints, such as geostrophic balance, the potential vorticity field can be inverted to diagnostically deduce the velocity and static-stability fields for the whole domain. This makes potential vorticity a particularly powerful diagnostic tool.

In particular, the Lagrangian properties and encoded dynamical information make potential vorticity a convenient conceptual tool that can simplify the task of understanding the dynamics and the depth structure of the ocean (Marshall and Nurser, 1992; Marshall, 2000).
Chapter 3: An Integral Constraint on Potential Vorticity Fluxes

In this thesis the flux form of potential vorticity conservation is considered (Truesdell, 1951; Obukhov, 1962; Haynes and McIntyre, 1987, 1990):\footnote{Refer to Appendix A for the following derivation}

$$\frac{\partial}{\partial t}(\rho Q) + \nabla \cdot \mathbf{J} = 0,$$

(3.3)

where the potential vorticity flux vector, \( \mathbf{J} \), is defined as\footnote{In general, \( \nabla a \), for arbitrary \( a \) can be added to \( \mathbf{J} \) but Bretherton and Schär (1993) show that the following physical choice for the potential vorticity flux, that is the sum of a purely advective, local heating and a mechanical forcing term, is unique.}

$$\mathbf{J} = \rho Q \mathbf{u} + q \mathbf{B} + \mathbf{F} \times \nabla \sigma,$$

(3.4)

for arbitrary buoyancy and body forces, \( \mathbf{B} \) and \( \mathbf{F} \).

Precise descriptive terminology of (3.3) requires that \( \mathbf{J} \) is labelled as a flux of \textit{substance}, with mass mixing ratio \( Q \). Henceforth, references to potential vorticity fluxes are thus to be actually understood as fluxes of potential vorticity \textit{substance}.

The flux form of potential vorticity conservation is an exact expression which states that neither diabatic heating nor cooling nor friction, nor any other external force, can induce a net transport of potential vorticity across isopycnals (though of course there can be a net diapycnal transport of mass or chemical tracer). This result is another fundamental property associated with potential vorticity – named the “impermeability theorem” (Haynes and McIntyre, 1987) – whereby isopycnals are impermeable to the potential vorticity flux vector, \( \mathbf{J} \). Consequently, potential vorticity substance can neither be created nor destroyed within a layer bounded by isopycnal surfaces; within the layer it can only be concentrated or diluted. The potential vorticity substance can, however, be fluxed into or out of the layer at places where the isopycnals intersect a boundary (for example, the sea floor, walls or surface. See Fig. 3.1). These boundary fluxes are induced by diabatic heating or cooling or by a non zero curl of an external force, \( \nabla \wedge \mathbf{F} \), acting into the layer.

The “impermeability” principle has been applied to the ocean to investigate the ventilation of the main thermocline (Marshall and Nurser, 1992; Marshall \textit{et al.}, 2001). Potential vorticity fluxes are particularly well suited to the problem of subduction from the mixed layer into the thermocline since potential vorticity,
buoyancy forcing and mechanical forcing are all discontinuous across this interface whereas potential vorticity flux, owing to impermeability, must be continuous. This affords a simplification and a mass subduction rate can be derived in terms of the potential vorticity flux that is favourably compared, in model data, with an equivalent kinematic calculation.

Furthermore, in a steady state, the intersection of surfaces of constant density and constant Bernoulli potential, $\Pi$, are aligned with the flux of potential vorticity – even in the presence of arbitrary forcing and diabatic heating (Schär, 1993) – such that

\[
\mathbf{J} = \nabla \Pi \times \nabla \sigma,
\]  

(3.5)

where

\[
\Pi = p/\rho_0 + \mathbf{u}^2/2 + \phi,
\]  

(3.6)

and $\phi$ is the gravitational potential. This implies that in a steady state, potential vorticity flux vectors cross neither $\sigma$ nor $\Pi$ surfaces.
3.2 The basic potential vorticity flux integral constraint

At the heart of the integral constraint lies the impermeability theorem; that steady-state isopycnals and Bernoulli surfaces are impermeable to fluxes of potential vorticity. Hence, for a volume, $V$, bounded by a Bernoulli potential and a surface of constant depth, the potential vorticity flux vector, $J$, only breaches the bounding surface across the plane of constant depth (as in Fig. 3.2). Therefore, in the steady state, the net flux of potential vorticity across an area, at constant depth, bounded by a Bernoulli contour is zero. Marshall (2000) derived this integral constraint, evaluating $J$ in terms of its potential vorticity flux components (3.4), to conceptually distinguish between different regimes found in the steady-state vertical structure of the ocean. Here the potential vorticity flux constraint is derived with the planetary geostrophic approximation, including time dependence, before diagnosing the components in the Samelson and Vallis planetary geostrophic ocean.

In the planetary geostrophic ocean, where the Rossby number is small, absolute vorticity is determined by the local Coriolis parameter, $f$, such that

$$q = fk. \quad (3.7)$$
Density, \( \rho \), where not differentiated, is approximated by a constant reference density, \( \rho_0 \), and the Bernoulli potential at constant depth reduces to a constant multiple of pressure. Therefore, fluxes, \( J \), of potential vorticity, \( Q \), where

\[
Q = \frac{f}{\rho_0} \frac{\partial \sigma}{\partial z},
\]

are evaluated over areas bounded by closed pressure contours. In Chapter 4, this will be extended to include the remaining neglected processes.

Preempting the association between the potential vorticity integral constraint and the Lagrangian buoyancy budget, the constraint is derived by decomposing the velocity field into components and substituting them into the Lagrangian buoyancy budget. Firstly, the velocity field, \( \mathbf{u} \), is split, using (2.2), into three components – horizontal geostrophic, horizontal ageostrophic (frictional) and a vertical term – as follows:

\[
\mathbf{v}_g = \frac{1}{\rho_0 f} \mathbf{k} \times \nabla p,
\]

\[
\mathbf{v}_F = -\frac{1}{f} \mathbf{k} \times \mathbf{F},
\]

\[
\mathbf{u} = \mathbf{v}_g + \mathbf{v}_F + w \mathbf{k}.
\]

These are substituted into the buoyancy balance (2.5),

\[
\mathcal{B} = \frac{\partial \sigma}{\partial t} + \mathbf{u} \cdot \nabla \sigma,
\]

to get

\[
\mathcal{B} = \frac{\partial \sigma}{\partial t} + \frac{1}{\rho_0 f} \mathbf{k} \times \nabla p \cdot \nabla \sigma + \mathbf{v}_F \cdot \nabla \sigma + w \frac{\partial \sigma}{\partial z}.
\]

At constant depth, multiplying by \( f \), integrating over an area enclosed by a closed pressure contour, and invoking Stokes’ theorem to remove the geostrophic term gives

\[
\iint_p \{ J_{\text{adv}} + J_{\text{buoy}} + J_{\text{fric}} \} dA = \iint_p f \frac{\partial \sigma}{\partial t} dA.
\]

Here \( J_{\text{adv}}, J_{\text{fric}} \) and \( J_{\text{buoy}} \) are the vertical potential vorticity fluxes associated with advection, buoyancy forcing and friction as follows:

\[
J_{\text{adv}} = -f \frac{\partial \sigma}{\partial z} w,
\]

\[
J_{\text{buoy}} = f \mathcal{B},
\]

\[
J_{\text{fric}} = -f \mathbf{v}_F \cdot \nabla \sigma.
\]
Hence, at constant depth and in steady state, the three components of potential vorticity flux due to advection, buoyancy and friction must balance when integrated within any closed pressure contour (3.14).

The advective and buoyancy terms, $J_{\text{adv}}$ and $J_{\text{buoy}}$, can be conveniently interpreted in terms of stratification. $J_{\text{adv}}$ can be visualised in terms of the vertical advection of stratified fluid (Fig. 3.3a) and $J_{\text{buoy}}$ can be visualised in terms of a stratification adjustment by the raising or lowering of isopycnals by the action of a buoyancy forced cooling or heating, $B$ (Fig. 3.3b). For the interpretation of the frictional component of potential vorticity flux, it is instructive to recall that friction was employed in the momentum equation (2.2) as an energy sink required to balance the action of the wind-stress. The resultant ageostrophic frictional velocity (3.10) then adiabatically slumps the isopycnals (naturally bowled in the subtropical gyre) at the gyre’s edge, removing available potential energy (Fig. 3.3ci). This is seen as an increase in stratification at the gyre edge that is achieved by an isopycnic influx of potential vorticity which has a downward component (Fig. 3.3cii).

In the anticyclonic subtropical gyre the advective and frictional vertical potential vorticity fluxes are both negative, so in order for the integral constraint (3.14) to hold it is anticipated (Marshall, 2000) that the upward fluxes are provided by the buoyancy term. In the ventilated thermocline this will be a convective buoyancy flux and in the internal thermocline (below the mixed layer) it will be a diffusive buoyancy flux. Furthermore, in the absence of friction the integral constraint at the internal thermocline will reduce to a balance between advection and diapycnal diffusion, as shown below: neglecting the effects of horizontal diffusion,

$$B = \kappa_w \frac{\partial^2 \sigma}{\partial z^2},$$

such that

$$\int \int_p \left\{ J_{\text{adv}} + J_{\text{buoy}} \right\} \, dA = 0,$$  \hspace{1cm} (3.19)

or

$$\int \int_p f w \frac{\partial \sigma}{\partial z} \, dA = \int \int_p f \kappa_w \frac{\partial^2 \sigma}{\partial z^2} \, dA.$$  \hspace{1cm} (3.20)

Hence, the internal thermocline balance (reviewed in section 1.2), which has hitherto been adopted in an ad hoc manner, has a fundamental theoretical justification in
Figure 3.3: Physical interpretations for the (a) advective, (b) buoyancy forced and (c) frictional vertical components of potential vorticity flux. Vertical velocity will act to advect stratified fluid downward (ai), this corresponds to (a(ii)) a vertical advective flux of potential vorticity, $Q$. Buoyancy forced cooling, $B$, (bi) raises the isopycnal and adjusts the stratification. This corresponds to an upward flux of potential vorticity, $Q$, (bii). Frictional ageostrophic velocity with strength diminishing with depth (ci) slumps the isopycnals and corresponds to an adjustment of potential vorticity achieved by an inward isopycnic potential vorticity flux (cii). This has a downward vertical component.
terms of an integral constraint on the structure of a closed gyre. In practice, frictional processes are not negligible and modify the internal thermocline scalings.

### 3.3 Thermocline genesis and development

To further motivate an intuitive understanding of the nature of the vertical potential vorticity fluxes, and before looking at integral budgets, it is helpful to consider how a thermocline develops in an initially unstratified ocean. In the following, the potential vorticity flux vectors are not integrated over closed pressure contours, as in (3.14), but are plotted locally, simply to give a qualitative understanding of the role of potential vorticity fluxes in the setting of the complex ocean stratification.

In Fig. 3.4 snap-shots of the thermocline, evolving from a uniformly cold ocean, are plotted after 2 years, 6 years, 16 years and 29 years. The plots are meridional $\sigma$ cross sections one quarter of the way across the domain, and the overlying vectors are the sum of the vertical components of the advective, buoyancy and frictional potential vorticity fluxes at that meridian.

After 2 years (Fig. 3.4a), the vertical flux of potential vorticity is downward, leading to the establishment of a shallow thermocline. This flux is dominated by vertical advection that pumps the “stratification” downward. In terms of the integral constraint (3.14) the balance (not shown) is between $J_{\text{adv}} < 0$ and $f \partial \rho / \partial t < 0$, the latter being equivalent to a local warming. This pattern is continued after 6 years with further deepening of the immature thermocline (Fig. 3.4b).

After 16 years (Fig. 3.4c), upward fluxes of potential vorticity, due to buoyancy forcing, arrest the rapid growth of the thermocline by partly balancing the downward vertical advection flux (3.14). The buoyancy flux, $J_{\text{buoy}} > 0$, is associated with convective cooling where, in the northern part of the gyre, warm water has penetrated to around 250 m and has become convectively unstable.

After 29 years (Fig. 3.4d), the gyre is approaching a steady-state equilibrium with a double thermocline structure. It is, however, not immediately clear from Fig. 3.4 why, during the formation of the double thermocline, the $\sigma = 28.0$ isopycnal should shoal (Fig. 3.4c-3.4d). Cross sections through the domain are simply not sufficient to understand the complex structure of the three-dimensional, nonlinear,
Chapter 3: An Integral Constraint on Potential Vorticity Fluxes

a) 2 years

b) 6 years
Figure 3.4: Meridional $\sigma$ cross sections through the subtropical gyre after approximately (a) 2 years, (b) 6 years, (c) 16 years and (d) 29 years. The section is a quarter of the way across the gyre. The vectors show the magnitude of the vertical potential vorticity flux for the three depths (located at the tail of the arrows). Panels (a) and (b) show “stratification” being pumped into the interior, predominantly by Ekman pumping. The thermocline deepening is partly arrested in (c) by the onset of convection, which is synonymous with the upward potential vorticity flux. Panel (d) shows the formation of the double thermocline as a product of the shoaling $\sigma=28.0$ isopycnal. By this stage the thermocline is approaching steady state and so the net vertical flux of potential vorticity through any closed pressure contour must be zero.
subtropical gyre. This is why it is useful to diagnose the *integrated* vertical fluxes of potential vorticity. Though it amalgamates horizontal features, it does impose a strong constraint on the dynamics.

### 3.4 Refining the potential vorticity flux integral constraint

As alluded to in the previous section, taking integral quantities over an area, whilst determining which processes are important in the gyre-scale balance, will lose information associated with the geographic distribution of the processes. This is illustrated in Fig. 3.5 where differently shaded regions show the distribution of the potential vorticity flux vertical components. It is anticipated that $J_{\text{buoy}}$ will be dominant in the convective mixed layer (where there is strong buoyancy forced cooling), that $J_{\text{fric}}$ will be manifest in the western boundary current region (where the flow velocity is greatest) and that $J_{\text{adv}}$ will be approximately zonally symmetric about the centre of the gyre (characteristic of the wind prescribed Ekman pumping velocity). Mid-gyre balances (near where $\nabla_h p = 0$) are separated from edge-gyre

![Figure 3.5: Schematic showing the geographic distribution of vertical potential vorticity fluxes over the gyre.](image-url)
balances by taking integrals over annular regions enclosed by two adjacent closed pressure contours, \( p \) and \( p + \Delta p \). The integral balance, which states that there will be no net flux through the area of integration, \( A \), will still hold since this new area integral is equivalent to the difference of two zero integrals over the areas bounded by \( p \) and \( p + \Delta p \). In the limit as \( \Delta p \to 0 \), the average vertical flux of potential vorticity through the annular region bounded by \( p \) and \( p + \Delta p \) is defined as

\[
\mathcal{J}^p = \lim_{\Delta p \to 0} \frac{\iint J \, dA}{\iint dA}.
\] (3.21)

In terms of \( \mathcal{J}^p \) components, the steady-state integral constraint (3.14) becomes

\[
\iint_p \{ \mathcal{J}^p_{\text{adv}} + \mathcal{J}^p_{\text{buoy}} + \mathcal{J}^p_{\text{fric}} \} \, dA = 0.
\] (3.22)

This diagnostic, by virtue of the normalisation, allows direct comparisons to be made between the magnitudes of the fluxes at different depths and pressure values.

Furthermore, by making this choice of area integral, \( \mathcal{J}^p \) is an area-weighted average vertical potential vorticity flux and the area weighting is a function of the geostrophic velocity around the pressure contour, such that \( \mathcal{J}^p \) can be conveniently interpreted in terms of a trajectory buoyancy budget. To demonstrate this interpretation, natural coordinates are adopted whereby \( s \) and \( n \) are the unit vectors following and normal to the geostrophic flow (see Fig. 3.6). The geostrophic velocity is then given as

\[
v_g = \frac{ds}{dt}.
\] (3.23)

and geostrophic balance can be written as

\[
\rho_0 f v_g = \frac{dp}{dn} s,
\] (3.24)

such that for small and constant \( \Delta p \),

\[
dn = \frac{\Delta p}{\rho_0 f v_g}.
\] (3.25)

Then, using (3.23), the area element of integration (as shaded in Fig. 3.6) is given by

\[
ds \, dn = \frac{\Delta p \, dt}{\rho_0 f}.
\] (3.26)

Recalling (3.5) and the form for \( v_g \), the vertical flux of potential vorticity can be

\[\text{In the limit as } \Delta p \to 0, \int \frac{\Delta p}{\rho_0 f} \mathbf{k} \times \nabla p \cdot \nabla \sigma \, dA / \iint \, dA = 0. \text{ See Appendix B.}\]
Figure 3.6: The vertical flux of potential vorticity through an annulus bounded by closed pressure contours can be interpreted in terms of a buoyancy budget in the limit as $\Delta p \to 0$. The shaded area is an area element of integration in natural coordinates.

written as

$$J = \frac{\nabla p}{\rho_0} \times \nabla \sigma \cdot \mathbf{k} = f \mathbf{v}_g \cdot \nabla \sigma. \quad (3.27)$$

The net flux, $J$, through an area element, $dsdn$, (as in Fig. 3.6) is then

$$\frac{\Delta p}{\rho_0 f} J dt = \frac{\Delta p}{\rho_0 f} (J_{adv} + J_{buoy} + J_{fric}) dt. \quad (3.28)$$

If this is integrated around part of the closed pressure contour (to avoid cancelling terms before seeing their meaning), then

$$\int_p \frac{\Delta p}{\rho_0 f} J dt = \int_p \frac{\Delta p}{\rho_0 f} (J_{adv} + J_{buoy} + J_{fric}) dt. \quad (3.29)$$

Substituting for $J$ and its components using (3.27), (3.15), (3.16), (3.17) and adopting term labels to keep track of their transformations,

$$\frac{\Delta p}{\rho_0} \int_p \mathbf{v}_g \cdot \nabla \sigma \ dt = \frac{\Delta p}{\rho_0} \int_p (-w \frac{\partial \sigma}{\partial z} + B - \mathbf{v}_F \cdot \nabla \sigma) \ dt, \quad (3.30)$$

where the label $geo$ is adopted for the net flux term since it is equivalent to a geostrophic advection of $\sigma$ along the geostrophic contour. Dividing by the constant

\[58\]
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\[ \Delta p/\rho_0 \] and rearranging,

\[
\int_p \left( \mathbf{v}_g \cdot \nabla \sigma + w \frac{\partial \sigma}{\partial z} + \mathbf{v}_F \cdot \nabla \sigma \right) dt = \int_p B \, dt.
\]

That is, the buoyancy forced change in \( \sigma \) for a fluid parcel following a horizontal geostrophic trajectory can be decomposed into a geostrophic advection, a vertical advection and an ageostrophic, frictional, advection of \( \sigma \). In particular, when the geostrophic contour is closed then using (3.23),

\[
\oint_p \mathbf{v}_g \cdot \nabla \sigma \, dt = \oint_p \nabla \sigma \cdot d\mathbf{s} = 0,
\]

the geostrophic term cancels and the buoyancy forced change in \( \sigma \) along a geostrophic trajectory is entirely due to ageostrophic vertical and frictional advection,

\[
\oint_p w \frac{\partial \sigma}{\partial z} \, dt + \oint_p \mathbf{v}_F \cdot \nabla \sigma \, dt = \oint_p B \, dt.
\]

This can be compared with the heuristic trajectory buoyancy balance (2.20) proposed in section 2.3 (illustrated by Fig. 2.7), which, though taken along real trajectories, was not exact. Hence, there is a direct relationship between vertical fluxes of potential vorticity, \( \overline{J^p} \), and physically intuitive buoyancy forcing along geostrophic trajectories\(^4\). Before applying the integral budget in the planetary geostrophic ocean, the buoyancy component of the potential vorticity flux is further decomposed into two separate components: \( J_{\text{mix}} \) and \( J_{\text{conv}} \), respectively associated with changes in the \( \sigma \) field attributed to vertical mixing (that is, model diffusion) and convective buoyancy forcing.

### 3.5 Potential vorticity flux analysis in the planetary geostrophic ocean model

#### 3.5.1 Evaluating the potential vorticity integral constraint

The potential vorticity flux components (3.15)–(3.17) are calculated from outputted \( \sigma, \rho \) and \( \mathbf{u} \) data fields after a 490 000 model-year spin-up. Before calculating the

\(^4\)A similar technique, described in Appendix C, is used to investigate the sensitivity of the vorticity distribution, in a shallow water gyre, to changes in wind forcing.
potential vorticity flux components these fields are interpolated from the model 
(65×65 point) horizontal grid onto a finer resolution (500×500 point) horizontal grid. 
Up to this resolution the accuracy of the closure increases. All spatial derivatives 
are calculated as centred differences and the Coriolis parameter, $f$, is calculated 
according to the β-plane assumption, centred about 35°N. Particular details, specific 
to individual potential vorticity flux components, are described below.

The buoyancy forcing, $\mathcal{B}$, in $J_{\text{buoy}}$ (3.16), is calculated as

$$
\mathcal{B} = \frac{\partial \sigma}{\partial t} + \mathbf{u} \cdot \nabla \sigma,
$$

(3.34)

for the full, three-dimensional, velocity field, where the transient term is calculated 
as a finite difference between two consecutive $\sigma$ data sets with a time-step separation 
of $10^5 \text{s} \approx 28 \text{ hours}$.

The split of $J_{\text{buoy}}$ into two further components is executed by a point-wise 
comparison between the $\sigma$ field at the surface ($\sigma = \sigma_s$) and the $\sigma$ field at depth 
($\sigma = \sigma(z)$) according to:

$$
J_{\text{buoy}} = \begin{cases} 
J_{\text{conv}} & \text{if } \sigma(z) = \sigma_s \\
J_{\text{mix}} & \text{if } \sigma(z) \neq \sigma_s.
\end{cases}
$$

This simplistic dichotomisation is justified despite the fact that mixing doubtlessly 
occurs in convective regions, as shown in the trajectory analysis in Section 2.3 
(Fig. 2.5c and Fig. 2.6b), because convection will dominate mixing where the two 
processes coexist.

The form of $\mathbf{v}_F$ (3.10), in $J_{\text{fric}}$ (3.17), is obtained according to the formulation 
for $F$ as a linear drag term (2.6).

Finally, a new term is defined:

$$
J_{\text{tot}} = J_{\text{adv}} + J_{\text{buoy}} + J_{\text{fric}},
$$

(3.35)

which is evaluated alongside the other components as a measure of the integral 
closure.

At each depth, the integrals of the components $J_{\text{adv}}$, $J_{\text{buoy}}$, $J_{\text{fric}}$ and $J_{\text{tot}}$ are 
evaluated over the areas enclosed by 20 pressure contours, ranging from the largest
(longest) to the smallest (shortest) possible. Then, the difference between consecutive integrations is taken and divided by the difference in integral areas. This gives an approximation for the $\mathcal{J}^p$ components defined in (3.21).

3.5.2 Analysis using the potential vorticity integral constraint

To illustrate the different regimes, analyses at 90 m, 400 m, 580 m and 700 m are presented. These correspond to the ventilated thermocline, mode water, internal thermocline and the abyss, respectively (see Fig. 1.9). At each of these levels, shown in Figs. 3.7ai, 3.7bi, 3.7ci and Fig. 3.7di, components of $\mathcal{J}^p$ are calculated and plotted against both the fractional area of the domain enclosed by the corresponding $p$ contour (on the lower $x$-axis) and the $p$ contour itself (on the upper $x$-axis). The pressure contours are shown in the adjacent plots in Figs. 3.7aii, 3.7bi, 3.7ci and Fig. 3.7dii. Then, in Fig. 3.8–Fig. 3.11, the horizontal structure of the individual potential vorticity flux components at each of the integration depths is shown. As already discussed, the planetary geostrophic assumptions allow a convenient interpretation of the integral constraint in terms of buoyancy forcing such that, in particular, the non integrated potential vorticity fluxes (Fig. 3.8–Fig. 3.11), in a planetary geostrophic ocean, will capture similar characteristics to a planetary geostrophic ocean buoyancy balance (for example, Colin de Verdière, 1988).

Note that $\mathcal{J}^p_{tot}$ is small for all the depths; that is, the integral closure is good. This justifies our steady-state assumption and the integrity of the numerical integration.

Ventilated regime: 90 m

In the upper 200 m the dominant integral balance is between the downward flux of potential vorticity due to strong vertical advection, $\mathcal{J}^p_{adv}$, and an upward flux of potential vorticity due to convective buoyancy loss (Fig. 3.7ai). The flux of potential vorticity due to friction is smaller but becomes significant away from the centre of the gyre, where it reinforces the downward advective flux (as anticipated by Marshall, 2000).

The horizontal structure of the potential vorticity flux components is shown in
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ai) 90m

Pressure (Pa)

Fractional area

adi) Pressure (Pa)

M"eridional position (km)

Fractional area

adv frc conv mix tot

Pressure (Pa)

Zonal position (km)

aii) 400m

Pressure (Pa)

Fractional area

adv frc conv mix tot

Pressure (Pa)

Zonal position (km)

bi) 400m

Pressure (Pa)

Fractional area

adv frc conv mix tot

Pressure (Pa)

Zonal position (km)

bi) 580m

Pressure (Pa)

Fractional area

adv frc conv mix tot

Pressure (Pa)

Zonal position (km)

ii) 580m

Pressure (Pa)

Fractional area

adv frc conv mix tot
Figure 3.7: On the left, the $\mathcal{J}^p$ components are plotted against the fractional area of the domain enclosed by the integration pressure contour (on the lower $x$-axis) and also against the value of the integration pressure contour (on the upper $x$-axis). On the right the range of closed pressure contours, at each depth, from the longest to the shortest, is shown. Pressure (Pa) is given relative to the value at the eastern boundary. The analyses are conducted at depths (a) 90 m, (b) 400 m, (c) 580 m and (d) 700 m. These depths correspond to the ventilated thermocline, mode water, internal thermocline and abyss respectively (see Fig. 1.9).

Fig. 3.8. The advective flux, $\mathcal{J}_{adv}^p$, vanishes within the mixed layer where there is no stratification and is downward within the ventilated thermocline. There is a small region within the western boundary current where the advective flux is upward, but this is not significant in terms of the integral budget. The frictional flux, $\mathcal{J}_{fric}^p$, is largest within the western boundary current and is mainly directed downward. These downward advective and frictional fluxes are balanced by an upward convective flux, $\mathcal{J}_{conv}^p$, within the mixed layer. This is in accord with the ventilated thermocline theory of Luyten et al. (1983), in which warm fluid is pumped adiabatically down into the gyre interior and buoyancy forcing is confined to the mixed layer.
Mode water: 400 m

Analysis at a depth of around 400 m includes both the recirculating and ventilated mode waters identified by Samelson and Vallis (1997a). There is a complicated four way balance in the integral constraint (Fig. 3.7bi). Note that each of the terms are an order of magnitude smaller than in the ventilated regime.

Near the centre of the gyre, corresponding to the recirculating mode water, the convective flux is negligible as the pressure contours do not thread the mixed layer. The balance is between the downward advective flux, \( \mathcal{J}_{\text{adv}}^p \), and an upward flux due to mixing, \( \mathcal{J}_{\text{mix}}^p \). The frictional component vanishes at the centre of the gyre because this is a stagnation point and hence the frictional force (here a linear drag) vanishes. This result necessarily holds at all depths. Thus at the centre of the gyre, the integral constraint reduces to precisely the same balance as assumed in the internal boundary layer thermocline theories (which is also the classical balance of Munk (1966)); that is (3.14) reduces to

\[
\frac{w}{\partial \sigma} - \kappa_v \frac{\partial^2 \sigma}{\partial z^2} = 0. \tag{3.36}
\]

(Strictly this balance should include a lateral diffusion term, but this turns out to be small.) Since at the depth of the recirculating mode water, above the internal thermocline, \( w \) is large relative to \( \kappa_v \), (3.36) can only be satisfied if the stratification, \( |\partial \rho/\partial z| \), is small; hence the existence of the mode water.

Further away from the centre of the gyre, corresponding to the ventilated mode water, \( \mathcal{J}_{\text{conv}}^p \) becomes significant as the pressure contours thread the mixed layer (Fig. 3.9). The advective flux, \( \mathcal{J}_{\text{adv}}^p \), changes sign due to upwelling on the northern flank of the gyre, nevertheless, this is compensated by a downward frictional flux, \( \mathcal{J}_{\text{fric}}^p \). In contrast to the recirculating mode water above, here the dominant balance is between lateral frictional advection across sloping isopycnals and mixing. This differs from the classical thermocline balance (3.36) which neglects the contribution of friction but corroborates the picture of the internal thermocline, from the trajectory analysis in Section 2.3 (Fig. 2.6).
Internal thermocline: 580 m

Below the influence of deep convection, at a depth of 580 m, the potential vorticity flux integral constraint can only be between $J_{adv}^p$, $J_{mix}^p$, and $J_{fric}^p$. In the centre of the gyre (where friction tends to zero) the competing downward wind-driven and abyssal upwelling components of the vertical velocity must balance. Hence, there exists an internal boundary layer at the depth where $w$, and $J_{adv}^p$, vanish. Also at the centre (where $w = 0$), as $J_{fric}^p$ is zero then $J_{mix}^p$ ($\sim \partial^2 \rho / \partial z^2$) must also be zero (3.14). This is consistent with the internal boundary layer thermocline theory whereby the thermocline, that is the region of most intense stratification ($\partial^2 \rho / \partial z^2 = 0$), is at the confluence of the downwelling and upwelling flows, as $w = 0$ implies $\partial^2 \rho / \partial z^2 = 0$ (3.36).

In isopycnic coordinates, moving away from the centre of the gyre is akin to moving upward. This is due to the bowled shape of the isopycnals (Fig. 1.9) and the flow being isopycnic to first order approximation. Similarly, moving away from the centre of the gyre at a fixed depth is akin to moving downward. This explains the increase in $J_{adv}^p$ towards the edge of the gyre (Fig. 3.10), as the strength of the abyssal upwelling would similarly increase with depth away from the thermocline. Moreover, further away from the centre of the gyre $J_{fric}^p$ becomes a significant component in the integral constraint and so only when $J_{fric}^p$ and $J_{mix}^p$ are combined is the strong upward $J_{adv}^p$ flux balanced (Fig. 3.7ci). As in the mode water region, this differs from the classical thermocline balance (3.36) which wholly neglects the contribution of friction.

Abyss: 700 m

In the abyssal region, below the internal thermocline, the potential vorticity flux integral balance is between a basin-scale upward $J_{adv}^p$ and a basin-scale downward $J_{mix}^p$ (Fig. 3.11). This is reminiscent of the classical Stommel and Arons (1960b) picture in which the abyssal ocean is globally upwelling. Steady state is then maintained by globally warming the upwelling fluid by vertical mixing. The frictional flux, $J_{fric}^p$, is not found to be significant at these greater depths partly because the closed, integration, pressure contours have changed relative position;
they no longer extend to within 500 km of the western boundary, where the flow is enhanced and hence, where friction is greater (c.f. Figs. 3.10 and 3.11).

**Summary of regimes**

Integrating the potential vorticity fluxes over areas between closed pressure contours helps us to understand the balance of processes that are important in maintaining the steady-state subtropical gyre. In particular, at the centre of the gyre we recover the diffusive thermocline balance between vertical advection and diffusion.

In the upper 100 m there exists a ventilated thermocline which can be viewed in the following way: the Ekman pumping acts to flux potential vorticity downward into the interior and is balanced by the convective buoyancy loss, which acts to flux potential vorticity up to the surface. Hence, the steady state can be maintained.

Below the ventilated thermocline, convective buoyancy loss is unable to flux potential vorticity upward. This explains the weak stratification of the mode water (at 400 m in the gyre centre); the only upward potential vorticity flux is due to diffusion and is characteristically small, hence the advective flux (3.15) must also be small. Then, as the vertical velocity is set by the wind forcing, it is the stratification that is constrained to be weak to maintain the balance. Hence there exists a mode water.

Below the mode water there is an inflection point in the density field (at 580 m in the gyre centre) which is the internal thermocline. Above the inflection point the advective flux is downward due to the Ekman pumping; below the inflection point the advective flux is upward due to abyssal upwelling. These are balanced at the centre of the gyre by the diffusive flux, which is upward above the inflection point and downward below. This is equivalent to the classical diffusive thermocline balance.

Away from the gyre centre, the frictional flux plays a significant role in the integrated potential vorticity flux balance. The frictional flux is particularly important around the mode water and the internal thermocline where it is the dominant negative term in the integral balance. Hence, away from the centre of the gyre friction will modify the internal thermocline structure as compared with the
classical similarity theory.

Having seen that friction plays an essential role in closing the integral constraint, it is appropriate to ask how robust this dependence is. In the following section, a sensitivity analysis is conducted in which the requirement for friction to close the integral constraint is found to be insensitive to the choice of friction coefficient.

3.6 Sensitivity to coefficients

Thermocline sensitivity to vertical diffusivity, $\kappa_v$, and wind prescribed Ekman pumping, $w_{ek}$, has already been studied by Samelson and Vallis (1997a). Here it is shown, through an illustrative example, that doubling the friction coefficient, from $\eta = 4.2 \times 10^{-6}$ s$^{-1}$ (denoted $\eta_0$) to $\eta = 8.4 \times 10^{-6}$ s$^{-1}$ (denoted $2\eta_0$), does not significantly affect the characteristics of the integral constraint or the magnitudes of its components. It does, however, affect the depths at which the different integral flux regimes are found.

The left hand panels in Fig. 3.12 ($\eta = 2\eta_0$) show the integrated potential vorticity fluxes calculated at the same depths as the analysis in Fig. 3.7 ($\eta = \eta_0$), namely at 90 m, 400 m, 580 m and 700 m. The right hand panels show the integrated potential vorticity fluxes calculated at depths where the trends in the components are the same as those shown in Fig. 3.7. Due to the depth level discretisation, the corresponding depth to match the patterns at 90 m in the $\eta = \eta_0$ case is still found at 90 m when $\eta = 2\eta_0$. Instead of duplicating plot Fig. 3.12ai, panel Fig. 3.12aii shows the depths at which the analyses are conducted. Comparisons between regime depths show that the whole depth structure contracts by a factor of 0.8 on doubling the friction coefficient (for example, the integral constraint characteristics found at 400 m for $\eta = \eta_0$ are found at 320 m when $\eta = 2\eta_0$).

In understanding the sensitivity of the integral constraint, and in particular the robustness of the result that $\overline{J}_{fric}$ is important in the gyre-scale balance, it is instructive to investigate the dependence on $\eta$ of two key parameters: the western boundary current velocity, $v_{wbc}$, and the depth of the internal thermocline, $D$. The western boundary current velocity, $v_{wbc}$, defined as the maximum velocity at a depth of 90 m along the meridional data line nearest to the western boundary, is found to
vary with the friction coefficient, $\eta$, according to
\[ v_{wbc} \propto \eta^{-1}, \quad (3.37) \]
for a range of vertical diffusivities, $\kappa_v$ (see Fig. 3.13). The depth of the internal thermocline, defined as the depth at which $|\partial \rho/\partial z|$ reaches a maximum in the centre of the domain, is found to vary with the friction coefficient, $\eta$, according to
\[ D - D_0 \propto \eta, \quad (3.38) \]
for $D_0$ independent of $\eta$ but a function of the vertical diffusivity, $\kappa_v$, (see Fig. 3.14). Both of these trends can be qualitatively accounted for. Firstly, to understand (3.37) it is instructive to approximate the stratified ocean as a shallow water system with an interfacial depth being the thermocline depth, $D$. For the range of friction coefficients tested, the streamlines do not significantly change and the variations in $D$ are small compared to variations in $v_{wbc}$. Following Niiler (1966)\textsuperscript{5}, the steady state circulation budget along a closed streamline in a shallow water layer can be written as
\[ \oint_{\psi} \tau_0 - \rho_0 D \cdot dl = \oint_{\psi} \eta v \cdot dl. \quad (3.39) \]
In doubling $\eta$, the left hand side of (3.39) will remain approximately constant, suggesting that the western boundary current velocity, $v_{wbc}$, should vary inversely with the friction coefficient, $\eta$ (3.37).

Secondly, adopting the Stommel (1948) scaling for the western boundary current thickness,
\[ \delta_{wbc} \sim \eta/\beta, \quad (3.40) \]
the northward mass transport, $D_0 v_{wbc} \delta_{wbc}$, should be approximately independent of the frictional coefficient, $\eta$. Consequently, the interior geostrophic velocity will also be largely unaltered. However, the interior frictional ageostrophic flow (3.10) will decrease and so by continuity $|\partial w/\partial z|$ will also decrease. Therefore, because the vertical velocity is prescribed at the surface as wind-induced Ekman pumping,
\textsuperscript{5}Another sensitivity adaption of the Niiler (1966) circulation budget is presented in Appendix C. There, the regions of a gyre where vorticity is most sensitive to changes in wind forcing are diagnosed.
the depth at which $w$ tends to zero will deepen and so the depth of the internal thermocline, $D$, will deepen.

These relationships can then be used to interpret the nature of the $\mathcal{J}_\text{fric}^p$ trend, by showing that the fractional change in $\mathcal{J}_\text{fric}^p$ varies linearly with fractional changes in the dependent variable $\eta$. Recalling the form of $J_{\text{fric}}$ (3.17) and rearranging to give

$$J_{\text{fric}} = \nabla \sigma \times \mathbf{k} \cdot \mathbf{F},$$

(3.41)

thermal wind balance (2.27) can be used to eliminate $\sigma$. Then, for friction that is given by a linear drag, $J_{\text{fric}}$ scales as

$$J_{\text{fric}} \sim \eta U \frac{\rho_0 U}{gD} \propto \eta U^2 D. \quad (3.42)$$

The quadratic dependence of $J_{\text{fric}}$ on the velocity scale, $U$, might be anticipated to be a problem but because of the inverse relation (3.37), this becomes

$$J_{\text{fric}} \sim \frac{1}{\eta D}, \quad (3.43)$$

such that the fractional variation in $J_{\text{fric}}$ is

$$\frac{\delta J_{\text{fric}}}{J_{\text{fric}}} \sim -\frac{\delta \eta}{\eta} - \frac{\delta D}{D}. \quad (3.44)$$

Recalling that $D$ decreases with increasing $\eta$ (3.38) there will be a measure of cancellation further reducing the fractional change in $J_{\text{fric}}$. Indeed comparison between $\mathcal{J}_{\text{fric}}^p(\eta_0)$ and $\mathcal{J}_{\text{fric}}^p(2\eta_0)$ does indicate $|\delta J_{\text{fric}}/J_{\text{fric}}| < |\delta \eta/\eta|$ in the upper ocean (c.f. first columns of Fig. 3.7 and Fig. 3.12). Hence $J_{\text{fric}}$ varies slowly with changing friction coefficients.

### 3.7 Summary of Chapter 3

Closing the gyre with a western boundary current exerts a strong constraint on the vertical structure (recall Section 2.4). This restriction can be expressed as an integral constraint such that, in a steady state and at constant depth, the net flux of potential vorticity through a closed pressure contour must be zero. In particular, the net flux between two closed pressure contours is also zero and in the limit as the
pressure contour separation tends to zero, the integral constraint can be interpreted as a buoyancy budget that is directly comparable with the Lagrangian buoyancy budget in Chapter 2.

The integral constraint serves as a way of quantifying the relative importance of stratification adjustment processes at any depth and along any closed pressure contour. In particular, the integral of the potential vorticity flux component attributed to friction is shown to be important in closing the integral balance.

At the pressure extremum, in the “middle” of the gyre, the integral constraint reduces to a point-wise balance between advective and buoyancy fluxes of potential vorticity. Here, where the friction force and its associated flux are, by construction, identically zero, the internal thermocline arises as a necessary inflection point in the vertical density profile at the depth where $w \to 0$. Also in the middle of the gyre, the mode water is shown to be a consequence of the potential vorticity flux balance; by inference, the double thermocline structure which sandwiches the mode water is also a consequence of the potential vorticity flux integral constraint.

Away from the middle of the gyre, pressure contours pass through the western boundary current and frictional processes become important. At mode water depths the convective mixed layer is present in the northern part of the domain. For pressure contours that pass through the mixed layer, the additional (positive) buoyancy flux of potential vorticity can only be balanced by a (negative) frictional flux. Friction is also found to be important in the neighbourhood of the internal thermocline where the mixing flux is insufficient to balance the advective flux.

The Luyten et al. (1983) genre of adiabatic thermocline and the Stommel and Arons (1960b) picture of the abyssal ocean are both recovered by this analysis.

In summary, the integral constraint quantifies the role of friction in maintaining the steady-state subtropical thermocline in terms of a potential vorticity flux budget. The constraint shows that friction, manifest in the western boundary current, is important for gyre-scale integral closure. The integral constraint also provides a fundamental justification for the adiabatic and diffusive thermocline theories, reviewed in Chapter 1, in terms of an exact balance for a closed gyre.

Though it is abundantly clear that for a closed gyre there must be some kind
of energy sink to balance the wind-stress energy source, it is not at all obvious that linear drag provides a “realistic” parameterisation of this process, especially since the western boundary current flow is, to leading order, inviscid (Hughes and de Cuevas, 2001). In the next chapter this difficulty is addressed by diagnosing potential vorticity fluxes in an eddy-permitting ocean general circulation model to investigate the possible role that eddies play in maintaining the mean vertical structure of the ocean.
Figure 3.8: The ventilated thermocline. The vertical components of the potential vorticity flux vector are shown at a depth of 90m. Contours are shown for values with a modulus less than $250 \times 10^{-15} \text{ kg m}^{-3} \text{ s}^{-2}$, using a contour interval of $50 \times 10^{-15} \text{ kg m}^{-3} \text{ s}^{-2}$. Dashed contours denote negative values. A heavy black contour denotes zero. The figure is shaded in two tones, with an intensity corresponding to the magnitude of the modulus (light grey: $50 - 250 \times 10^{-15} \text{ kg m}^{-3} \text{ s}^{-2}$ and dark grey: $> 250 \times 10^{-15} \text{ kg m}^{-3} \text{ s}^{-2}$). The dominant balance is between a negative advective term and a positive convection term. Friction is also significant towards the edge of the gyre.
Figure 3.9: The mode water. The vertical components of the potential vorticity flux vector are shown at a depth of 400 m. Contours are shown for values with a modulus less than $20 \times 10^{-15}$ kg m$^{-3}$ s$^{-2}$, using a contour interval of $4 \times 10^{-15}$ kg m$^{-3}$ s$^{-2}$. Dashed contours denote negative values. A heavy black contour denotes zero. The figure is shaded in two tones, with an intensity corresponding to the magnitude of the modulus (light grey: $8-20 \times 10^{-15}$ kg m$^{-3}$ s$^{-2}$ and dark grey: $>20 \times 10^{-15}$ kg m$^{-3}$ s$^{-2}$). Convection now occurs in a restricted area in the northern part of the gyre. The dominant balance must include friction, the only negative flux.
Figure 3.10: The internal thermocline. The vertical components of the potential vorticity flux vector are shown at a depth of 580 m. Contours are shown for values with a modulus less than $10 \times 10^{-15}$ kg m$^{-3}$ s$^{-2}$, using a contour interval of $2 \times 10^{-15}$ kg m$^{-3}$ s$^{-2}$. Dashed contours to denote negative values. A heavy black contour denotes zero. The figure is shaded in two tones, with an intensity corresponding to the magnitude of the modulus (light grey: $4 - 10 \times 10^{-15}$ kg m$^{-3}$ s$^{-2}$ and dark grey: $> 10 \times 10^{-15}$ kg m$^{-3}$ s$^{-2}$). There is no convection at this depth. The dominant balance, as shown by the shading, is three way and between $J_{\text{adv}}$, $J_{\text{mix}}$ and $J_{\text{fric}}$. Some smoothing has been applied to the fields to aid visualisation.
Figure 3.11: The abyss. The vertical components of the potential vorticity flux vector are shown at a depth of 700 m. Contours are shown for values with a modulus less than $14 \times 10^{-15} \text{ kg m}^{-3} \text{ s}^{-2}$, using a contour interval of $10^{-15} \text{ kg m}^{-3} \text{ s}^{-2}$. Dashed contours to denote negative values. A heavy black contour denotes zero. The figure is shaded in two tones, with an intensity corresponding to the magnitude of the modulus (light grey: $2 - 14 \times 10^{-15} \text{ kg m}^{-3} \text{ s}^{-2}$ and dark grey: $> 14 \times 10^{-15} \text{ kg m}^{-3} \text{ s}^{-2}$). There is no convection at this depth. The dominant balance is between a positive advective term and a negative mixing term. Friction is not significant in this balance. Some smoothing has been applied to the fields to aid visualisation.
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ai) 90 m

bi) 400 m

ci) 580 m

aii) analysis depths

bii) 320 m

cii) 460 m
Figure 3.12: Average vertical potential vorticity flux components integrated over thin annular regions, $\mathcal{J}^p$, plotted against the fractional area of the domain enclosed by the pressure contour. These integrations are for a frictional coefficient, $\eta = 8.4 \times 10^{-6} \text{ s}^{-1}$ (twice that of Fig. 3.7). On the left $\mathcal{J}^p$ are at the same analysis depths as in Fig. 3.7. On the right the depths are found that give the same $\mathcal{J}^p$ trends as shown in Fig. 3.7. It is found that doubling the friction parameter contracts the depth scale of regimes by a factor of 0.8.
Figure 3.13: Variations in the western boundary current maximum velocity, \( v_{wbc} \), is shown to vary with the friction coefficient, \( \eta \), on (a) linear and (b) logarithmic axes, such that \( v_{wbc} \) varies as \( 1/\eta \).
Figure 3.14: The depth, $D$, of the thermocline varies linearly with variations in the friction coefficient, $\eta$. 

\[ \kappa_v = \begin{cases} 4 \times 10^{-5} \text{ m}^2 \text{s}^{-1} \\ 2 \times 10^{-5} \text{ m}^2 \text{s}^{-1} \\ 1 \times 10^{-5} \text{ m}^2 \text{s}^{-1} \end{cases} \]

Depth (m) vs. $\eta$ (x $8.4 \times 10^{-5}$ s$^{-1}$)
Chapter Four

Do Eddies Affect the Subtropical Gyre Stratification?

In the previous chapter, we saw that friction in the western boundary current provides a non negligible contribution to the net vertical flux of potential vorticity within a steady-state gyre and can thus modify the vertical structure of the gyre. In reality, however, the ocean is not steady but populated by a vigorous field of geostrophic eddies, of scales comparable to the Rossby deformation radius, which is of order 50 km in mid-latitudes. Indeed “friction” in a planetary geostrophic model is no more than a crude parameterisation for some of the effects of eddies. So the question arises: do mesoscale eddies play an important role in setting the ocean’s thermocline structure?

Certainly in a homogeneous ocean, the net vorticity budget within a gyre can be significantly altered by the inclusion of eddies. Following Niiler (1966), the wind-induced source of vorticity, within a steady-state gyre, is balanced by a frictional sink of vorticity. In the presence of an active eddy field, however, Marshall (1984) showed that the wind-induced source of vorticity can alternatively be balanced by a horizontal eddy flux of vorticity out of the gyre, for example across an intergyre jet separating two adjacent gyres. Then the wind-induced source of vorticity in one gyre is balanced by the wind-induced sink of vorticity in the other, counter-rotating gyre.

The Rhines and Young (1982a) mechanism, described in Section 1.4, further explores the role eddies may have in modulating a preexisting background stratification. In particular, they argue that eddies flux potential vorticity anomalies down-
gradient along isopycnals, possibly leading to regions in which potential vorticity is homogenised within the closed streamlines. This exerts a strong constraint on the variations of stratification along isopycnals, though because of the a priori assumption of a preexisting background stratification this mechanism can not modulate the gyre-mean stratification.

Furthermore, eddies may also play a role in maintaining the mean vertical structure of a gyre. For example, in the subtropical gyre, where Ekman pumping acts to steepen the bowled isopycnals and warm the gyre, Marshall et al. (2002) argue that lateral shedding of warm core eddies through baroclinic instability may act to arrest this steepening process and thus maintain the vertical structure of the gyre.

In this chapter, the potential vorticity flux integral constraint is extended to include an eddy term, thus allowing the investigation into the role that eddies play in setting the vertical structure of the thermocline. The new constraint is then applied to diagnostics from an eddy-permitting ocean general circulation model.

In Section 4.1 the anticipated role of eddies is presented in terms of eddy fluxes of potential vorticity. In Section 4.2 the ocean general circulation model is introduced with brief model formulation details. In Section 4.3 the integral constraint is derived in terms of time-mean and eddy flux model diagnostics. In Section 4.4 the methodology of the integral constraint application is detailed and results are given at four distinct depths characterising the vertical structure of the model’s subtropical gyre. It is found that, in some cases, eddy fluxes of potential vorticity oppose the buoyancy and, or, advective fluxes. The chapter is concluded with a summary in Section 4.5.

### 4.1 Conceptual eddy paradigms

In further discussion of the role of eddies it is convenient to adopt the Transformed Eulerian Mean flow formulation (Andrews and McIntyre, 1976; Andrews et al., 1987; Gent et al., 1995; Treguier et al., 1997), whereby an eddy-induced isopycnic flux of potential vorticity (or isopycnic thickness) can be represented in terms of an advection by an eddy-induced transport velocity, $u^*$, in addition to
the Eulerian mean velocity, $\mathbf{u}$. This notation naturally leads into the potential vorticity flux language that is adopted for subsequent sections. Eddy processes can be summarised in terms of their interaction with the vertical structure by considering the resultant vertical component of the eddy flux of potential vorticity, $J_{\text{eddy}}$, defined in Section 4.3; that is, the effect of eddies on stratification (as shown in Fig. 3.3 for advective, buoyancy and frictional fluxes). Two scenarios arise (represented in Figs. 4.1a and 4.1b):

**a) Buoyancy flux cancellation.** The presence of strong convective buoyancy forced cooling, which results in the formation of the mixed layer, leads to the shedding of baroclinic eddies that attempt to restratify the mixed layer by the

![Diagram of eddy flux scenarios](image)

Figure 4.1: There are two eddy flux scenarios arising from (a) buoyancy forced and (b) wind-driven processes. (a) Cooling in the mixed layer removes stratification and results in an upward buoyancy flux of potential vorticity. Baroclinic instability along the mixed layer density fronts produces an eddy flux that acts to warm and restratify the mixed layer. Hence, the eddy and buoyancy fluxes oppose each other. (b) Across a wind-forced jet, an ageostrophic Eulerian mean flow acts to steepen the isopycnals and the ensuing baroclinic instability gives rise to an eddy flux that slumps them. Hence, the eddy and advective fluxes oppose each other.
eddy-induced slumping of isopycnals (Visbeck et al., 1997; Marshall, 1997). As shown in Fig. 3.3b, the cooling represents an upward flux of potential vorticity and so the eddy-induced warming results in a downward flux of potential vorticity. In this scenario, there is a point-wise cancellation between the buoyancy flux of potential vorticity, $J_{buoy}$, and the eddy flux of potential vorticity, $J_{eddy}$.

b) **Advective flux cancellation.** The role of $u^*$ has been studied in some detail in the Southern Ocean where, in a zonally averaged sense, it is found to oppose the wind-driven Deacon Cell at leading order. For example, Danabasoglu et al. (1994), using the Gent and McWilliams (1990) eddy closure, find a complete cancellation. The addition of surface buoyancy forcing leads to an incomplete cancellation (Marshall, 1997) and can help reconcile the structure of the meridional circulation with tracer observations (Callahan, 1972). As in Fig. 3.3a, the vertical Eulerian mean velocity, $\overline{w}$, advects potential vorticity and has the effect of steepening isopycnals; the vertical component of the eddy flux of potential vorticity must therefore be in the opposite direction, causing a corresponding slumping of the isopycnals. In this scenario, there is a point-wise cancellation between the vertical advective flux of potential vorticity, $J_{adv}$, and the vertical eddy flux of potential vorticity, $J_{eddy}$.

Eddy processes are widely accepted as important for the Southern Ocean dynamics (Rintoul et al., 2001), however, there has been little attempt to extend these ideas to a closed gyre (Marshall, 2000; Karsten et al., 2002), despite gyre recirculations being dynamically similar to the Antarctic Circumpolar Current (see Fig. 4.2). Approximating the Antarctic Circumpolar Current as a purely zonal flow and interpreting the potential vorticity flux integral constraint an ageostrophic buoyancy budget following geostrophic flow (Section 3.4), the zonally-averaged eddy-adveective cancellation of the Deacon Cell (Danabasoglu et al., 1994) can be understood in terms of the potential vorticity flux constraint. Therefore, the integral constraint links eddy mechanisms, accepted as important in the Southern Ocean, with subtropical gyre dynamics.
In the next section the $1/4^\circ \times 1/4^\circ$ eddy-permitting Ocean Circulation and Climate Advanced Modelling (OCCAM)$^1$ project is introduced. This model is then used to investigate the possible role of eddies in controlling the vertical structure of the subtropical thermocline.

### 4.2 The OCCAM model

OCCAM is a high resolution eddy-permitting global ocean model, based on the Bryan-Cox-Semtner ocean model, designed for running on a Cray T3D (for further details consult Webb et al. 1998b). It includes a free surface and an improved advection scheme. It was initially run for 14 model years, with a 7 year spin-up phase followed by a 7 year analysis phase. Comparing the climatology taken from years 8-12 with Pacific Ocean observations, OCCAM reproduces the large-scale distribution of sea surface height (Fig. 4.3) and near surface currents well. It is, however, generally found that the variability is underestimated by the model climatology (Saunders et al., 1999). Fig. 4.4 compares the model sea level variability with that observed by the TOPEX/POSEIDON altimeter. Spatial agreement is

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$^1$Henceforth this model is referred to as OCCAM.
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Figure 4.3: The Ocean Circulation and Climate Advanced Modelling (OCCAM) project is a $1/4^\circ \times 1/4^\circ$ eddy-permitting model. Large-scale features seen in the sea surface height field, such as the subtropical gyres, western boundary currents and the Antarctic Circumpolar Current, are rich with eddy activity. Taken from Webb et al. (1998a).

good, but away from regions of high eddy activity the model shows significantly less variability than the satellite observations. OCCAM was one of the first global ocean models to be run at such high resolution, enabling numerical investigation of the interaction between fine-scale processes and the global circulation. The model’s pedigree has been established through a wide range of studies including issues as diverse as data assimilation (Fox et al., 2000), thermohaline upwelling (Coward, 1998) and the discovery of mid-ocean jets that are induced by shallow topography (Webb, 2000).

While OCCAM does not fully resolve the geostrophic eddy field, it provides an ideal opportunity to explore the potential vorticity flux integral constraint with an eddy term and therefore to assess the likely importance, or otherwise, of eddies in setting the vertical structure of the thermocline. Clearly results obtained from a $1/4^\circ$ model must be regarded as tentative. Further diagnostics will be required, for example from the planned $1/12^\circ$ OCCAM integrations, to confirm the impact of a fully resolved eddy field on the thermocline structure.
Figure 4.4: Comparing the root mean square sea level variability (cm) of (a) OCCAM with (b) sea level anomalies for 1993, as measured by the TOPEX/POSEIDON satellite. Agreement between spatial patterns is good, but the magnitude is underestimated especially in mid-ocean regions where the mesoscale variability is weaker. Taken from de Cuevas and Webb (2001).
4.2.1 OCCAM model description

Equations

The evolution of the model ocean is specified using the horizontal momentum equations and the potential temperature and salinity equations. In formulating these equations four important assumptions (that are generally made for all ocean models) need to be borne in mind:

1. The ocean is incompressible;
2. The material acceleration and Coriolis terms can be neglected in the vertical momentum equation. This gives hydrostatic balance;
3. Variations in density can be neglected in the horizontal momentum equations, except where it appears in conjunction with horizontal pressure gradients;
4. Components of the Coriolis force involving the vertical velocity, \( w \), are neglected in the horizontal momentum equations.

These lead to the following equations:

the horizontal momentum equations,
\[
\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla_h) \mathbf{v} + w \frac{\partial \mathbf{v}}{\partial z} + f \mathbf{k} \times \mathbf{v} = -\frac{1}{\rho_0} \nabla_h p + \mathbf{D_v} + \mathbf{G_v},
\] (4.1)

where \( \mathbf{v} \) is the horizontal velocity, \( \mathbf{D_v} \) is the model diffusion (representing dissipation) and \( \mathbf{G_v} \) represents the forcing (both detailed in the following discussion);

the potential temperature and salinity equations,
\[
\frac{\partial T}{\partial t} + (\mathbf{v} \cdot \nabla_h) T + w \frac{\partial T}{\partial z} = B_T,
\] (4.2)
\[
\frac{\partial S}{\partial t} + (\mathbf{v} \cdot \nabla_h) S + w \frac{\partial S}{\partial z} = B_S,
\] (4.3)

where \( B_T \) and \( B_S \) are the total buoyancy forcing terms, as set by model dissipation and forcing, in the advection of the potential temperature and salinity dynamical tracers;

hydrostatic balance,
\[
\frac{\partial p}{\partial z} + \rho g = 0;
\] (4.4)
continuity,
\[ \nabla_h \cdot \mathbf{v} + \frac{\partial w}{\partial z} = 0; \tag{4.5} \]
and an equation of state,
\[ \rho_2 = \rho(T, S, p), \tag{4.6} \]
where potential density, \( \rho_2 \), is calculated using a 3rd order polynomial fit to the UNESCO equation of state (see Webb et al., 1998b) and is referenced to 2000 m. Then the thermodynamic equation is obtained from the potential temperature and salinity equations (4.2) and (4.3) by using the equation of state (4.6):
\[ \frac{\partial \sigma_2}{\partial t} + (\mathbf{v} \cdot \nabla_h) \sigma_2 + w \frac{\partial \sigma_2}{\partial z} = B, \tag{4.7} \]
where \( B = \frac{D\sigma_2}{Dt} \) is the total buoyancy forcing and, for convenience, the dimensionless quantity \( \sigma_2 \) is defined as
\[ \sigma_2 = \frac{\rho_2 - 1000 \text{ kg m}^{-3}}{1 \text{ kg m}^{-3}}. \tag{4.8} \]

The Grid

The ocean is divided up into boxes by surfaces of constant longitude, latitude and depth. The tracer fields (potential temperature and salinity) are defined at the centre of these boxes and the velocity fields are defined at the corners. This, a horizontal Arakawa-B grid, is convenient because it defines the velocity on the coastline boundaries, making the velocity boundary conditions much easier to implement. 36 levels are chosen in the vertical ranging from 20 m thickness near the surface to 255 m at a depth of 5500 m. These values are chosen to provide an “adequate” model of the mixed layer, thermocline and bottom bathymetry of the deep ocean. To maintain stability the timestep must be chosen to be less than the time for the fastest wave to propagate a grid-box length. In order to prevent a singularity at the North Pole, an orthogonal grid is used for the North Atlantic and Arctic Oceans that meshes at the equator and is also connected via a simple channel model, at the Bering Strait, to the rest of the domain.
**Diffusion and viscosity**

The model uses Laplacian operators to represent the effect of horizontal mixing in the momentum equations. The coefficient for the horizontal mixing of tracers is set at $10^2 \text{ m}^2 \text{s}^{-1}$ and the coefficient for horizontal viscosity in the velocity field is $2 \times 10^2 \text{ m}^2 \text{s}^{-1}$. In the vertical, a Richardson number dependent mixing scheme for tracers (Pacanowski and Philander, 1981) is employed that improves the equatorial depth structure. In the vertical, Laplacian mixing with a coefficient of $10^{-4} \text{ m}^2 \text{s}^{-1}$ is used for the velocity fields. Further diffusion is introduced by the implementation of the computationally efficient split-QUICK advection scheme (see Webb *et al.*, 1998b, for details).

**Equation of state**

Density is calculated from potential temperature and salinity using a 3rd order polynomial fit to the UNESCO equation of state.

**Time stepping**

The horizontal velocity can be broken into a barotropic component (that is, a depth-averaged velocity) and a baroclinic component (the residual). These are computed independently and are recombined after each baroclinic timestep. Instead of using a rigid-lid, as in the Bryan-Cox-Semtner scheme, OCCAM has a free surface. The resulting barotropic component is forced using a simple tidal model with a timestep of 18s.

**Forcing**

The potential temperature and salinity fields are initialised from the Levitus (1982) global average dataset and are subsequently forced by monthly average ECMWF wind-stresses and by surface heat and fresh water fluxes that are calculated to return the surface layer back to the Levitus monthly average values.
The potential vorticity flux analysis is applied to the OCCAM eddy-permitting model by combining the time averages of the horizontal momentum equations (4.1) with the thermodynamic equation (4.7). In the following section the integral constraint for the time-mean potential vorticity flux is derived before applying it to the OCCAM data.

4.3 Extending the potential vorticity flux integral constraint to include an eddy term

The integral constraint in Chapter 3 demonstrated that friction is non negligible in the gyre-scale balance of potential vorticity flux, which in turn determines the thermocline stratification. To apply a similar integral constraint to the vertical structure of the OCCAM subtropical gyre requires a reformulation of the constraint as a function of statistically-steady and mean-eddy fluxes of obtainable model diagnostics. Before deriving the time-mean form of the integral constraint, the time-mean momentum and thermodynamic equations are first obtained.

Using the standard notation, where over-bars denote time-averages and primes denote deviations from the mean, for an arbitrary variable \( \varphi \),

\[
\varphi = \overline{\varphi} + \varphi',
\]  

the following identities can be deduced:

\[
\overline{\varphi' \vartheta} = \overline{\varphi' \vartheta'} = 0
\]

and

\[
\overline{\varphi \vartheta} = (\overline{\varphi} + \varphi')(\overline{\vartheta} + \vartheta') = \overline{\varphi \vartheta} + \varphi' \vartheta'.
\]

The only complication in obtaining the time-averaged momentum and thermodynamic equations results from the need to derive an appropriate form for the Lagrangian derivative operator, which contains a product of variables.
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For an arbitrary variable, $\varphi$, the Lagrangian operator, $\frac{D\varphi}{Dt}$, is manipulated into flux form by using the continuity equation, (4.5):

$$\frac{D\varphi}{Dt} = \frac{\partial \varphi}{\partial t} + u \cdot \nabla \varphi + \varphi \cdot \nabla u$$

(4.12)

and hence,

$$\frac{D\varphi}{Dt} = \frac{\partial \varphi}{\partial t} + \nabla \cdot (\varphi u).$$

(4.13)

Then, separating $\varphi$ into $\varphi + \varphi'$ and $u$ into $u + u'$, taking the time average and using (4.11), gives the desired expression for the Lagrangian operator:

$$\frac{\overline{D\varphi}}{Dt} = \frac{\partial \overline{\varphi}}{\partial t} + \nabla \cdot (\overline{\varphi u})$$

(4.14)

$$= \frac{\overline{D}}{Dt} \varphi + \nabla \cdot (\overline{\varphi' u'}).$$

(4.15)

where

$$\frac{\overline{D}}{Dt} = \frac{\partial}{\partial t} + \overline{u} \cdot \nabla$$

(4.16)

is the Lagrangian derivative following the time-mean flow. Taking the time average of the horizontal momentum equations (4.1), without invoking the full complexity that led to (4.15), gives

$$\frac{\partial \overline{v}}{\partial t} + (\overline{u} \cdot \nabla) \overline{v} + f k \times \overline{v} = -\frac{1}{\rho_0} \nabla h p + \overline{D}_v + \overline{C}_v + \left\{ -u' \cdot \nabla v' \right\},$$

(4.17)

where $\overline{F}_R$ is the Reynolds stress form of the momentum eddy fluxes. Grouping the forcing terms together,

$$\overline{F} = \overline{D}_v + \overline{C}_v + \overline{F}_R,$$

(4.18)

the time-mean horizontal momentum equations can be expressed in terms of diagnosable time-mean variables,

$$\frac{\partial \overline{v}}{\partial t} + (\overline{u} \cdot \nabla) \overline{v} + f k \times \overline{v} = -\frac{1}{\rho_0} \nabla h p + \overline{F}.$$  

(4.19)

By similar application of the time-mean Lagrangian derivative, the time average of the thermodynamic equation (4.7) gives

$$\frac{\partial \overline{\sigma_2}}{\partial t} + \overline{u} \cdot \nabla \overline{\sigma_2} = \overline{B} - \nabla \cdot (\overline{u' \sigma_2'}).$$

(4.20)
An exact integral constraint can be derived for the time-mean equations (4.19) and (4.20) as follows. Using the standard vector identity (D.4) the velocity advection term in (4.19) becomes

\[
(\mathbf{u} \cdot \nabla)\mathbf{v} = \nabla \times \mathbf{u} \times \mathbf{u} + \nabla \left( \frac{\mathbf{u} \cdot \mathbf{u}}{2} \right) - \mathbf{u} \cdot \nabla \mathbf{w} \mathbf{k} \tag{4.21}
\]

\[
= \mathbf{\xi} \times \mathbf{u} + \nabla \left( \frac{\mathbf{u} \cdot \mathbf{u}}{2} \right) - \mathbf{u} \cdot \nabla \mathbf{w} \mathbf{k}, \tag{4.22}
\]

for relative vorticity, \( \mathbf{\xi} \). Defining the time-mean Bernoulli potential in terms of time-mean variables,

\[
\Pi = \frac{\mathbf{u} \cdot \mathbf{u}}{2} + \frac{p}{\rho} + \phi, \tag{4.23}
\]

the time-mean momentum equation (4.19) can be rewritten as

\[
\frac{\partial \mathbf{v}}{\partial t} + (f + \mathbf{\xi}) \times \mathbf{v} = -\nabla \Pi + \mathbf{F} + \mathbf{u} \cdot \nabla \mathbf{w} \mathbf{k} + \nabla \phi. \tag{4.24}
\]

Both vertical terms on the right hand side of (4.24) disappear on taking the vertical component after a cross product with \( \nabla \sigma_2 \), such that

\[
\frac{\partial \mathbf{v}}{\partial t} \times \nabla \sigma_2 \cdot \mathbf{k} - \rho \mathbf{Q} \mathbf{w} - \mathbf{q} (\mathbf{u} \cdot \nabla \sigma_2) \cdot \mathbf{k} = -\nabla \Pi \times \nabla \sigma_2 \cdot \mathbf{k} + \mathbf{F} \times \nabla \sigma_2 \cdot \mathbf{k}, \tag{4.25}
\]

where mean potential vorticity, \( \overline{\mathbf{Q}} \), is given by

\[
\overline{\mathbf{Q}} = -\frac{\mathbf{q} \cdot \nabla \sigma_2}{\rho}; \tag{4.26}
\]

and absolute vorticity, \( \mathbf{q} \), is given by

\[
\mathbf{q} = f \mathbf{k} + \mathbf{\xi}. \tag{4.27}
\]

Substituting for \( \mathbf{u} \cdot \nabla \sigma_2 \) using (4.20) and then rearranging gives

\[
\rho \mathbf{Q} \mathbf{w} + (f + \mathbf{\xi}) \overline{\mathbf{B}} + \mathbf{F} \times \nabla \sigma_2 \cdot \mathbf{k} - (f + \mathbf{\xi}) \nabla \cdot (\mathbf{u} \sigma_2^2) \]

\[
= \nabla \Pi \times \nabla \sigma_2 \cdot \mathbf{k} + \frac{\partial \mathbf{v}}{\partial t} \times \nabla \sigma_2 \cdot \mathbf{k} + (f + \mathbf{\xi}) \frac{\partial \sigma_2}{\partial t}, \tag{4.28}
\]

where \( \mathbf{\xi} = \mathbf{\xi} \cdot \mathbf{k} \). Finally, integrating over the area at constant depth enclosed by a Bernoulli contour, and invoking Stokes’ theorem to remove the \( \nabla \Pi \) term, leads to

\[
\iint_{\Pi} \left\{ J_{\text{adv}} + J_{\text{buoy}} + J_{\text{fric}} + J_{\text{eddy}} \right\} dA = \iint_{\Pi} \left\{ J_{\text{drift}(\mathbf{v})} + J_{\text{drift}(\sigma_2)} \right\} dA, \tag{4.29}
\]
where the potential vorticity flux components are defined as:

\[
\begin{align*}
J_{\text{adv}} &= \rho \overline{\nabla v \\ w}, \\
J_{\text{buoy}} &= (f + \xi) \overline{\mathbf{B}}, \\
J_{\text{fric}} &= \mathbf{F} \times \nabla \sigma_2 \cdot \mathbf{k}, \\
J_{\text{eddy}} &= -(f + \xi) \nabla \cdot \left( \mathbf{u} \sigma_2' \right), \\
J_{\text{drift}(\mathbf{v})} &= \frac{\partial \mathbf{v}}{\partial t} \times \nabla \sigma_2 \cdot \mathbf{k}, \\
J_{\text{drift}(\sigma_2)} &= (f + \xi) \frac{\partial \sigma_2}{\partial t}.
\end{align*}
\]

This is the potential vorticity flux integral constraint for the time-mean case, and written in terms of time-mean variables. If the time averaging is taken over a long enough period relative to the ocean’s internal variability, and if the model is sufficiently spun up, then the time-mean variables will be in steady state. In this case, the drift fluxes (4.34) and (4.35), which are associated with an evolving ocean, will vanish. The resulting balance will then be between \( J_{\text{adv}} \), \( J_{\text{buoy}} \), \( J_{\text{fric}} \) and \( J_{\text{eddy}} \) such that the net flux at constant depth through a closed Bernoulli contour is zero (a direct comparison can be made with (3.14), the integral constraint for the planetary geostrophic ocean).

### 4.4 Potential vorticity flux analysis in OCCAM

#### 4.4.1 Evaluating the potential vorticity flux components

A propitious quality of the time-mean integral constraint (4.29) is the ease with which the components can be calculated from relatively standard model output. For the following analysis all the diagnostics are calculated from 3 year running means (to remove all mesoscale variability) of \( u, v, \rho_2, u \rho_2, v \rho_2 \), potential temperature, salinity and sea surface height, and from 5 day averages (to remove gravity wave signals) at the start and finish of the 3 year averaging period of \( u, v \) and \( \rho_2 \).

The mean vertical velocity, \( \overline{w} \), is calculated using an OCCAM subroutine that calculates the horizontal mass flux into each grid box and then integrates this upward from the sea floor, where \( \overline{w} = 0 \).
The transient terms in (4.34) and (4.35) are calculated as time centred differences over the 3 year period using the start and finish 5 day average quantities. For example,

$$\frac{\partial \sigma_2}{\partial t} = \frac{\sigma_{\text{finish}}^2 - \sigma_{\text{start}}^2}{T},$$

(4.36)

where $T = 94176000$ s (3 years). The transient term, $\partial \sigma / \partial t$, in (4.34) is similarly defined.

Actual density, $\overline{\rho}$, is calculated as the potential density referenced to its own level by processing the mean fields of potential temperature and salinity through the OCCAM equation of state (4.6). Pressure can then be obtained from density and sea surface height using the hydrostatic equation (4.4).

The net forcing term, $\overline{F}$, is calculated as the residual of the momentum equation (4.19), such that the Reynolds stresses are not included in the potential vorticity eddy flux term. The Reynolds stresses are not specially dealt with, both because the data was not available, but also on the grounds that they are anticipated to be a small component in the integral constraint compared with the eddy buoyancy flux component.

The divergence of the eddy $\sigma$ flux,

$$\nabla \cdot \overline{u'\sigma_2^2} = \nabla_h \cdot \overline{v'\sigma_2^2} + \frac{\partial}{\partial z} (\overline{w'\sigma_2^2}),$$

(4.37)

in (4.33), is approximated as

$$\nabla \cdot \overline{u'\sigma_2^2} \approx \nabla_h \cdot \overline{v'\sigma_2^2},$$

(4.38)

on the assumption that the vertical component is small (Treguier et al., 1997; Roberts and Marshall, 2000), and also because $\overline{w'\sigma_2^2}$ data were not directly available. In practice, since we have three year means of $u\rho_2$ and $v\rho_2$, the required eddy flux is calculated as

$$\overline{v'\sigma_2^2} = \overline{v\sigma_2^2} - \overline{\overline{v\sigma_2}},$$

(4.39)

which recalling (4.8) can be expressed as

$$\overline{v'\sigma_2^2} = \frac{v\rho_2 - \overline{v\rho_2}}{1 \text{ kg m}^{-3}}.$$ 

(4.40)

The buoyancy forcing term, $\overline{B}$, is calculated as the residual of the time-mean thermodynamic equation (4.20).
Mean Ertel-Rossby potential vorticity (4.26) is approximated, on scaling grounds, to include only the vertical component of relative vorticity, such that

$$Q = -\frac{(f + \xi_k \cdot \nabla \sigma_2)}{\bar{\rho}}.$$  \hspace{1cm} (4.41)

Finally, since only horizontal variations in the Bernoulli potential are necessary to define the integration areas, it is convenient to redefine a dimensionless $\Pi$, which varies with depth, relative to its maximum in the Gulf of California, $C(z)$:

$$\Pi = \frac{\nabla \Pi}{1 \text{ m}^4 \text{s}^{-4}} + C(z),$$  \hspace{1cm} (4.42)

where the vertical velocity contribution to $\Pi$ has been dropped on scaling grounds.

All the variables are interpolated onto a $1/16^\circ$ resolution grid. All the potential vorticity fluxes (4.30)–(4.35) can be calculated from these variables.

The assumptions made that might contribute to errors in the integral constraint are summarised as follows:

1. Discarding the vertical derivative component in the divergence of eddy $\sigma$ flux occurring in $J_{\text{eddy}}$ (4.33);

2. Ignoring the non linearity of the equation of state in calculating the mean actual density, $\bar{\rho}$, from the mean fields of potential temperature and salinity. This mean density is then used to calculate the pressure in $\Pi$ and also appears in the horizontal momentum equations that are used to obtain $\bar{F}$;

3. Assuming a 3 year time-scale separation between steady large-scale flow patterns and mesoscale variability;

4. Ignoring non vertical components in the expression for relative vorticity, $\xi$, in the calculation of $Q$;

5. Discarding $w^2$ in the calculation of Bernoulli potential, $\Pi$;


In terms of potential vorticity fluxes, these can be categorised into two types of error: (i) contributions that are erroneously assigned to the wrong potential vorticity
flux component and (ii) contributions to potential vorticity flux components that are not evaluated and lead to a failure in closure of the integral constraint. Clearly the most “dangerous” errors are type (i) since these will be fallaciously manifest in the potential vorticity flux components that are calculated as residuals from the time-mean equations. This gives errors in $\mathcal{B}$ and $\mathcal{F}$ which lead to errors in $\mathcal{J}_{\text{buoy}}$ and $\mathcal{J}_{\text{fric}}$. In the context of the integral constraint, however, the errors from the vertical divergence of the eddy $\sigma$ flux that appears in $\mathcal{B}$ is small (Roberts and Marshall, 2000), and the error that accumulates in $\mathcal{F}$ is also insignificant (as $\mathcal{J}_{\text{fric}}$ is a non dominant term in the integral constraint). On the other hand, type (ii) errors lead to imperfect integral closure. This error class serves to quantify the accuracy of the integral closure. It is found that the sum of the potential vorticity flux components,

$$\mathcal{J}_{\text{tot}} = \mathcal{J}_{\text{adv}} + \mathcal{J}_{\text{buoy}} + \mathcal{J}_{\text{fric}} + \mathcal{J}_{\text{eddy}} + \mathcal{J}_{\text{drift}(\sigma)} + \mathcal{J}_{\text{drift}(\sigma^2)},$$

is small when integrating over a closed $\Pi$ contour, thus confirming that type (ii) errors are small.

### 4.4.2 Diagnostics implementation

The integral constraint on the flux of potential vorticity is expressed entirely in terms of mean variables and an eddy $\sigma$ flux. The different components are diagnosed to ascertain the role of the additional eddy term in controlling the oceanic vertical structure. The analysis is done in the Pacific subtropical gyre because in the absence of a strong meridional overturning circulation (as could be found in the Atlantic) there are large closed Bernoulli potentials. Following the same methodology as in Section 3.4, vertical fluxes of potential vorticity are calculated over areas bounded by closed Bernoulli contours for a range of depths. In the following presentation, four depths have been chosen that best characterise the different regimes found in the eddy-permitting ocean general circulation model. These depths are 52 m, 295 m, 717 m and 1516 m (as shown in Fig. 4.5). Unlike the planetary geostrophic ocean model, there is no clear double thermocline or mixed layer structure in the profiles of $\sigma_2$. This is due to the seasonally varying nature of convection, not modelled by the steady-state planetary geostrophic ocean, that in
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Figure 4.5: A cross section of 3 year mean $\sigma_2$ through the model subtropical gyre at 160E. The horizontal lines mark the analysis depths that characterise the gyre depth structure. These are at model levels 3 (52 m), 10 (295 m), 15 (717 m) and 20 (1516 m). The total model depth extends to 5.5 km.

The OCCAM running mean output precludes a separation of the buoyancy forcing term into a convective and a diffusive component.

At each depth, 20 integrations are calculated over areas enclosed by the largest to the smallest Bernoulli contours. Areas outside the largest contour are masked to prevent dislocated closed Bernoulli anomalies entering the integrations. Figs. 4.6aii,biicii,diidii show the closed $\Pi$ contours at the analysis depths. Above 1000 m the circulation is anticyclonic ($\Pi > 0$) and below 1000 m the flow is cyclonic ($\Pi < 0$). As in Section 3.4, average vertical fluxes of potential vorticity, $\mathcal{J}^\Pi$, are calculated by differencing the flux integrals of two adjacent closed contours and dividing by the corresponding area (recall (3.21) and Fig. 3.6) such that

$$\mathcal{J}^\Pi = \lim_{\Delta\Pi \to 0} \frac{\iint \mathcal{J} dA}{\iint dA}.$$ (4.44)
Components of $\mathcal{J}^{\Pi}$ are plotted against the corresponding $\Pi$ (on the upper $x$-axis) and the area enclosed by $\Pi$ (on the lower $x$-axis) in Figs. 4.6ai,bi,ci,di (the data are processed using a three-point binomial smoother to enhance visual clarity). Figs. 4.7–4.10 show the vertical flux of potential vorticity plotted for each component. Each figure corresponds to a different analysis depth and has 6 panels, one for each of the vertical potential vorticity flux components, each shaded with an intensity according to its magnitude and a colour (blue or red) according to its sign (positive or negative). Each panel also has the Bernoulli potential, $\Pi$, overlayed. The same log colour scale is used for all the panels on all the plots.

### 4.4.3 Analysis

For all integrations, at each depth, the transient potential vorticity fluxes $\mathcal{J}_{\text{drift}(\nabla)}$ and $\mathcal{J}_{\text{drift}(\sigma^2)}$ are small (this is also reflected by the non integrated, $\mathcal{J}$, diagnostics in Figs. 4.7–4.10). This means that the model is sufficiently spun up for the variables to be statistically steady over a 3 year period (and that the model mesoscale variability has a shorter than 3 year time scale).

The unresolved error amalgam, $\mathcal{J}^{\Pi}_{\text{tot}}$, is also small and for most integrations is not a significant term in the integral constraint. Therefore, the type (ii) errors do not accumulate and adversely affect the integral constraint closure. Further, the $\mathcal{J}^{\Pi}_{\text{fric}}$ term is seen to be small, as anticipated, such that the eddy momentum fluxes are not significant terms in the integral constraint and type (i) errors are verified as small. In contrast with the planetary geostrophic analysis, where $\mathcal{J}_{\text{fric}}^{\Pi}$ is important, the role of friction is, therefore, broadly assumed by the eddy flux of potential vorticity, $\mathcal{J}^{\Pi}_{\text{eddy}}$.

Inspection of the distribution of the potential vorticity flux components (Figs. 4.7-4.10) reveals that all the components are largest near the western boundary recirculation region and that the shape of the Bernoulli potentials is convoluted, despite the 3 year averaging, in the mid-gyre regions. These standing Rossby waves can be compared to similar features described by the barotropic models of Moore (1963), Veronis (1966) and Marshall (1984), where streamlines in the region of a wind-stress source of vorticity must meander, and lengthen, to balance the vorticity sink due to
friction (Veronis, 1966) and intergyre eddy fluxes (Marshall, 1984). The final point to notice is the strongly zonal $J_{\text{buoy}}$ distribution at 50 m. This pattern arises as a consequence of latitudinally and seasonally varying mixed layer convective cooling and atmospheric forcing (not shown).

In the following, a detailed analysis is given at each depth in terms of the integrated potential vorticity flux components, to investigate the possible role of eddies in the maintenance of the thermocline structure.

### 52 m

Both at the gyre edge and at the centre, the dominant balance is between a negative $J_{\text{adv}}$ and a positive $J_{\text{buoy}}$ (Fig. 4.6ai). This result is reminiscent of the ventilated thermocline theory of Section 1.3, which is also recovered in the planetary geostrophic model analysis detailed in Section 3.5. The downward advective flux of potential vorticity is approximately constant across the whole range of integrals, whereas the cooling buoyancy potential vorticity flux is significantly altered in the vicinity of the tight recirculations. There, the integral constraint is closed by an additional eddy flux of potential vorticity, which acts to oppose the enhanced buoyancy forced cooling. This scenario can be compared directly with the eddy paradigm shown in Fig. 4.1a, where convective buoyancy forcing steepens isopycnals and makes them unstable and where the subsequent shedding of eddies attempts to restratify the mixed layer. Hence, the upward flux of potential vorticity associated with buoyancy forced cooling is balanced by a downward, restratifying, flux of potential vorticity associated with eddy shedding. This integral cancellation is also reflected in a point-wise cancellation between $J_{\text{buoy}}$ and $J_{\text{eddy}}$ in Fig. 4.7, which further supports the mixed layer slumping hypothesis.

### 295 m

Again the dominant balance is between a downward advective flux of potential vorticity and a positive (cooling) buoyancy flux. Though this is below the vertical extent of the large-scale convective zone, convection still occurs in the vicinity of the tight inertial recirculations. The closure is less accurate along the longest contours
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ai) $\Pi$ at 52 m

![Graph showing $\Pi$ at 52 m](image)

bii) $\Pi$ at 295 m

![Graph showing $\Pi$ at 295 m](image)

a) $\Pi$ at 52 m

![Image showing $\Pi$ at 52 m](image)

b) $\Pi$ at 295 m

![Image showing $\Pi$ at 295 m](image)
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ci) $\mathcal{J}^{\Pi}$ at 717 m
di) $\mathcal{J}^{\Pi}$ at 1516 m

cii) $\mathcal{J}$ at 717 m
dii) $\mathcal{J}$ at 1516 m

Figure 4.6: The potential vorticity flux integral constraint is evaluated at 4 depths: (a) 52 m; (b) 295 m; (c) 717 m; and (d) 1516 m. Panels (i) show the decomposition of the balance of potential vorticity fluxes into components of $\mathcal{J}^{\Pi}$ for a range of $\Pi$. Here the $\mathcal{J}^{\Pi}$ are plotted against $\Pi$ and the area enclosed by the $\Pi$ contour. The integration contours are shown in panels (ii). The general balance is between $\mathcal{J}^{\Pi}_{\text{adv}}$ and $\mathcal{J}^{\Pi}_{\text{buoy}}$, but in the eddy rich centre of the gyre $\mathcal{J}^{\Pi}_{\text{eddy}}$ opposes $\mathcal{J}^{\Pi}_{\text{adv}}$ and $\mathcal{J}^{\Pi}_{\text{buoy}}$. $\mathcal{J}_{\text{drift}(\sigma)}$ and $\mathcal{J}_{\text{drift}(\pi)}$ are not plotted because their deviations from zero cannot be resolved in the figure.
Figure 4.7: The decomposition, at 52 m, of the vertical potential vorticity flux (kg m$^{-3}$ s$^{-2}$) into its components.
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Figure 4.8: The decomposition, at 295 m, of the vertical potential vorticity flux (kg m$^{-3}$s$^{-2}$) into its components.
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(However, $J_{\text{tot}}^{\Pi}$ seems to vary with $J_{\text{fric}}^{\Pi}$ and $J_{\text{eddy}}^{\Pi}$ rather than altering the balance between $J_{\text{buoy}}^{\Pi}$ and $J_{\text{adv}}^{\Pi}$). Near the centre of the gyre the $J_{\text{eddy}}^{\Pi}$ term becomes large and opposes the advective flux. This is similar to the anticipated role of eddies portrayed in Fig. 4.1b where, in the vicinity of a strong jet, the ageostrophic circulation acts to steepen isopycnals and subsequent eddy shedding attempts to slump the isopycnals (Marshall et al., 2002). Though a broad cancellation can be seen in Fig. 4.8 between $J_{\text{buoy}}^{\Pi}$ and $J_{\text{adv}}^{\Pi}$, a point-wise cancellation can also be seen between $J_{\text{eddy}}^{\Pi}$ and $J_{\text{adv}}^{\Pi}$ along the eddy rich meandering inter-gyre jet (along $\Pi = 2$, at 37N). This lends support to the isopycnal slumping hypothesis.

717 m

As shown in Fig. 4.6ci, for over 80% of the closed gyre (up to $\Pi = 4$) all the $J^{\Pi}$ components are relatively small compared to the values in the tight recirculation region. This marks the internal thermocline depth, characterised by the changing sign of $J_{\text{adv}}^{\Pi}$, where there is upwelling below and downwelling above the thermocline. Also at this depth, the relative magnitude of $J_{\text{eddy}}^{\Pi}$ is diminished. Similar to the analysis at 295 m, variations in $J_{\text{eddy}}^{\Pi}$ oppose variations in $J_{\text{adv}}^{\Pi}$. However, here this cancellation is seen outside the tight recirculations and even where $J_{\text{adv}}^{\Pi}$ changes sign. This further supports the supposition that the mechanism in Fig. 4.1b (where eddies act to slump isopycnals otherwise steepened by the wind-driven circulation) can explain the observed advective-eddy flux cancellation, as the mechanism has no predilection for fluxes of a particular sign. Within the tight recirculation, the balance of terms is more complicated. Convection occurs, with an associated positive $J_{\text{buoy}}^{\Pi}$, and is principally balanced by a downward $J_{\text{adv}}^{\Pi}$. Entering the tight recirculation region with decreasing contour length (that is, moving from right to left in Fig. 4.6ci), however, reveals that $J_{\text{eddy}}^{\Pi}$ switches from an eddy-advective to an eddy-buoyancy cancellation of potential vorticity flux.

Fig. 4.9 shows the broad pattern of cancellation between $J_{\text{adv}}^{\Pi}$ and $J_{\text{buoy}}^{\Pi}$ in accordance with the advective diffusive balance, where upwelling and downwelling are balanced by diffusive warming and cooling. Fig. 4.9 also shows a very distinct zonal symmetry, away from the boundary current, in $J_{\text{fric}}^{\Pi}$. This is a consequence of a
Figure 4.9: The decomposition, at 717 m, of the vertical potential vorticity flux (kg m\(^{-3}\) s\(^{-2}\)) into its components.
Figure 4.10: The decomposition, at 1516 m, of the vertical potential vorticity flux ($\text{kg m}^{-3} \text{s}^{-2}$) into its components.
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weak, but broad, northward flow that is clearest at this depth where other processes are weaker.

1516 m

At 1516 m the circulation is cyclonic (Fig. 4.6dii). This marks the diffusive abyssal region found below the internal thermocline where upwelling and downwelling are balanced by diffusive warming and cooling (Munk, 1966). The integral constraint of potential vorticity fluxes reflects the classical diffusive balance that is, to first order approximation, a balance between \( \overline{J^{\Pi}_{\text{adv}}} \) and \( \overline{J^{\Pi}_{\text{buoy}}} \). Furthermore, Fig. 4.10 reflects the point-wise cancellation between \( \overline{J^{\Pi}_{\text{adv}}} \) and \( \overline{J^{\Pi}_{\text{buoy}}} \), but without the integral constraint analysis it would not be clear that frictional and eddy fluxes of potential vorticity are only important in the second order approximation closure, whereby \( \overline{J^{\Pi}_{\text{fric}}} \) balances \( \overline{J^{\Pi}_{\text{eddy}}} \).

4.5 Summary of Chapter 4

In Chapter 3 it was shown that friction in the western boundary current plays a non negligible role in the integral constraint of potential vorticity fluxes and could therefore modify the vertical structure of the thermocline. As friction is really a crude parameterisation of some of the effects of eddies, it was therefore natural to extend the ideas to an ocean model that includes eddies. However, relative to their atmospheric counterpart, oceanographic eddies are small and therefore numerical simulations require a high resolution if a significant portion of the eddy energy spectrum is to be captured. Currently global ocean simulations are only “eddy permitting”, in that their grid-box spacing is of the same order of magnitude as the Rossby deformation radius (the length scale for the most energetic eddies). Hence, the model “permits” some eddies rather than “resolves” the whole eddy field. This deficiency is seen in the OCCAM sea surface height variance when viewed in comparison with the equivalent satellite observations (Fig. 4.4). OCCAM only really models eddies in the vicinity of intense eddy activity (for example, in the western boundary currents) and not in mid-gyre regions. Hence, with this caveat, an
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OCCAM investigation into the role that eddies play in the structure of the thermocline can, at most, qualitatively show that, in this eddy-permitting ocean, the role of eddies is important. A more quantitative analysis will only be possible once higher resolution data is available (for example, the 1/12° OCCAM simulation). In this chapter, an exact integral constraint on the vertical fluxes of potential vorticity was derived for a statistically-steady ocean, expressed in terms of obtainable model diagnostics. On application to the eddy-permitting ocean general circulation model, it was found that the integral balance closed, such that the assumptions made in Section 4.4 were justified, and that to first order, the balance was broadly between advective fluxes and buoyancy fluxes of potential vorticity. In the upper ocean the buoyancy fluxes were convective and the balance was similar to the ventilated thermocline theory, reviewed in Section 1.3 (the same result was found in the planetary geostrophic model). Below the internal thermocline there was some indication of the Stommel and Arons (1960a) and Munk (1966) picture whereby abyssal upwelling balances abyssal diffusive warming.

It is shown that, in close proximity to the western boundary regions (characterised by vigorous eddy activity, seen in Fig. 4.4a) the eddy flux of potential vorticity acts to oppose the buoyancy flux of potential vorticity at 52 m (Fig. 4.6ai) and the advective flux of potential vorticity at 295 m (Fig. 4.6bi), in accordance with the conceptual paradigms proposed in Section 4.1. In the eddy-buoyancy potential vorticity flux cancellation (Fig. 4.1a), convective cooling destratifies the upper ocean and steepens isopycnals. Ensuing baroclinic instability is removed by the lateral shedding of eddies that advect warm water and attempt to restratify the region. In the eddy-advective potential vorticity flux cancellation at 295 m (Fig. 4.6bi) an ageostrophic overturning circulation is established by the action of wind-forcing, which acts to increase isopycnic slopes but is arrested by the lateral shedding of eddies that attempt to slump isopycnals. Though this latter mechanism has been observed in the Southern Ocean model studies (Danabasoglu et al., 1994) it has not been previously diagnosed as a mechanism operating in a closed gyre. In further support of the eddy cancellation mechanisms, which are themselves point-wise balances, it was also seen that the potential vorticity fluxes cancel in a point-wise manner.
(Figs. 4.7–4.10).

In conclusion, it has been shown that eddies provide a non negligible contribution to the net vertical flux of potential vorticity within closed statistically-steady gyres and can thus modify its vertical structure. It could be anticipated that if a more complete eddy field could be analysed, then the role of eddies would correspondingly increase, but this will require further investigation.
Chapter Five

Conclusions

Current understanding of the vertical structure of the subtropical thermocline is broadly based on one of three competing theories: the diffusive thermocline theory, the ventilated thermocline theory, or the eddy homogenisation theory. None of these theories include an active western boundary current, but instead assume that the western boundary current passively closes the circulation without significantly influencing the thermocline stratification. Closing the gyre with a western boundary current, however, imposes a strong constraint on the vertical structure. This constraint is most conveniently interpreted as an integral constraint on potential vorticity fluxes and highlights the need for careful modelling of eddies and other dissipative processes in the western boundary current, which can, in principle, affect the stratification. Furthermore, the diagnostics of the integral constraint developed in this thesis could be used, in the next generation of high resolution eddy-resolving ocean models, to determine more fully how eddies affect the stratification.

In this chapter, the key results of the thesis are summarised in Section 5.1. Future work extensions and concluding remarks are given in Section 5.2.

5.1 Summary of key results

In Chapter 2, the Samelson and Vallis planetary geostrophic ocean model was used to investigate the thermocline structure in a closed gyre. The model’s thermocline was shown to have a realistic double thermocline structure, wherein the upper thermocline is governed by adiabatic dynamics and the lower thermocline is controlled by diffusive dynamics. (It could be argued that the eddy mixing of potential vorticity
mechanism was excluded \textit{a priori} by the choice of a small horizontal mixing coefficient, $\kappa_h$.) The subtropical gyre structure was investigated using a trajectory finding algorithm that tracks a parcel’s position through the steady-state background $\sigma$ field. Applying this trajectory analysis in ventilated waters and in the internal thermocline gave rise to a heuristic buoyancy balance where $\sigma$ changes can be accounted for either by along-path buoyancy forcing (convective or diffusive) or by diapycnal flows attributed to vertical advection and horizontal frictional advection. The observed importance of frictional $\sigma$ changes is especially significant, as this western boundary current process is excluded from the standard diffusive and adiabatic thermocline theories. A simple model consisting of a Sverdrupian interior and dissipative western boundary current was also used to demonstrate that, in general, the western boundary current will affect the thermocline structure.

In Chapter 3, it was shown that closing a gyre circulation with a western boundary current gives rise to a strong and exact constraint on the dynamics. This constraint can be most conveniently interpreted using the language of potential vorticity fluxes. These potential vorticity fluxes act to adjust the stratification, but can also be expressed in terms of an exact buoyancy budget, akin to the heuristic Lagrangian buoyancy budget presented in Chapter 2. The potential vorticity flux integral constraint states, in its simplest form, that the net flux of potential vorticity through an area, bounded by a closed pressure contour, at constant depth and in steady state, is exactly zero. This balance was diagnosed in the Samelson and Vallis planetary geostrophic ocean model and, in the absence of friction, the ventilated thermocline was recovered. At greater depths, it was shown that a weakly stratified mode water is the natural consequence of combined wind-driven and convective buoyancy effects. Deeper still, the internal thermocline was shown to occur where diffusive processes dominate the dynamics. Meanwhile, away from the centre of the gyre, in regions where friction is not zero, the friction flux of potential vorticity plays a non negligible role in the integral constraint. In particular, in the mode water and internal thermocline regions, the frictional flux is the only downward term and therefore plays a crucial role in closing the integral constraint, thereby influencing the stratification in the subtropical gyre. This dependence on friction is a robust
result, rather than a consequence of a particular choice of friction coefficient.

In Chapter 4, the potential vorticity flux integral constraint was extended for application to a more “realistic” ocean general circulation model (OCCAM). Though OCCAM is only eddy-permitting, exhibiting less mesoscale variability than the observed ocean, it was shown that, in regions of intense eddy activity, eddy fluxes of potential vorticity play a non negligible role in closing the integral constraint and could, therefore, affect the stratification around a given Bernoulli contour. The action of the eddy fluxes conforms to two proposed paradigms. In the upper ocean convective mixed layer, there is an eddy-buoyancy flux cancellation whereby buoyancy forcing acts to remove stratification by steepening isopycnals, and the subsequent eddy shedding restratifies the fluid by flattening the isopycnals. At greater depths, in the absence of strong convective buoyancy forcing, there is an eddy-advective flux cancellation whereby isopycnic steepening, forced by Ekman pumping, is opposed by eddy shedding that acts to slump the isopycnals. Deeper down the eddies play a negligible role. This investigation of OCCAM data, then, has suggested that eddies can, indeed, affect the subtropical gyre stratification in regions of simulated intense mesoscale activity.

5.2 Extensions and concluding remarks

This thesis has been focused on diagnostic analyses in the subtropical gyre. The developed framework, however, also lends itself to wider applications that are not necessarily either diagnostic or subtropical. For example, closed circulations in the subpolar gyre regions permits the application of the same potential vorticity flux integral constraint. Here, wind-stress acts to drive an upward Ekman pumping velocity such that the associated flux of potential vorticity is also upward\(^1\). Moreover, in the presence of convective buoyancy forcing, \(J_{\text{conv}}\) will again be positive and, for the integral constraint to hold, there must thus be a downward flux of potential vorticity. It is likely that friction and eddies provide this flux and therefore play a quantifiable role in maintaining the subpolar gyre stratification. Similarly,

\(^1\)In the southern hemisphere, where the Coriolis parameter is of opposite sign, it is convenient to reverse the sign of all potential vorticity flux components to preserve their intuitive directions.
in the Southern Ocean, Ekman pumping and convective buoyancy forcing induce upward fluxes of $J_{\text{adv}}$ and $J_{\text{conv}}$, making it likely that $J_{\text{fric}}$ and $J_{\text{eddy}}$ close the integral constraint. Analysis in this region would divulge the nature of the balance, while explicit diagnosis of the eddy flux of potential vorticity could perhaps reveal details of a dual cancellation of eddy-buoyancy and eddy-advective mechanisms. Furthermore, the integral constraint might even shed new light on the global ocean stratification (c.f. Gnanadesikan, 1999) as the Southern Ocean is at the hub of the global ocean circulation.

In particular, for the planetary geostrophic ocean, it has been shown that aspects of the vertical structure of the subtropical thermocline arise as consequences of the potential vorticity flux integral constraint where friction is not important. It has also been shown that, where friction is important, the thermocline structure is largely independent of the choice of friction coefficient. It was not, however, considered whether different friction parameterisations (other than linear drag) could influence this structure (although it is true that any friction parameterisation that is somewhere zero within the gyre will recover a similar result). For example, with Fickian diffusion, $\mathbf{F} = \nu \nabla^2_h \mathbf{v}$, for friction coefficient $\nu$, the friction will only be zero where the pressure field is such that $\nabla^3 p = 0$. In this way, numerical stability schemes less transparent than linear drag may inadvertently affect the vertical structure.

So far, all the work has been strictly diagnostic and has shown that the western boundary current is active in closing the gyre circulation. It would, therefore, be interesting to see how the gyre responds to changes in the western boundary current. For example, can changes in the gyre-scale circulation be understood in terms of teleconnections via potential vorticity flux tubes? Potential vorticity substance, being neither created nor destroyed, is constrained to flow along potential density surfaces and, in steady state, along Bernoulli surfaces. The intersection of these surfaces form potential vorticity flux tubes in three-dimensions that link the western boundary current with the interior of the gyre. Hence, changes in western boundary current conditions could have remote effects on the mid-gyre dynamics, possibly leading to novel ocean monitoring techniques.
Chapter 5: Conclusions

The focus here has been on the wind-driven problem; alternatively, new understanding of the thermohaline circulation might be gained from diagnosing the potential vorticity flux integral constraint in a region where the thermohaline circulation is particularly strong. For example, in such a region the integral constraint might have a characteristic balance that could shed light on dynamics of the thermohaline circulation, or could perhaps even be useful for monitoring its circulation.

Owing to the disparate heat capacities of water and air, even slight changes to the oceanic stratification will significantly alter the oceanic heat content relative to the atmosphere. Understanding the processes that control and adjust the ocean stratification is therefore an important part of understanding climate change. The intimate association between heat fluxes and vertical fluxes of potential vorticity and the common principle of diagnosing area-integrated fluxes in both heat budgets and in the integral constraint are indicative that the integral constraint could be well suited to tackle aspects of the climate change problem.

Other possible avenues of research include using the integral constraint as a balance condition to develop a simplified two-dimensional gyre model, analogous to zonally-averaged models of the Antarctic Circumpolar Current.

Understanding the role of eddies in ocean general circulation models and coupled ocean atmosphere simulations, because of their prevalence in the real ocean, will remain an active area of research for some time. It remains to be shown whether the diagnostics developed in this thesis would reveal that eddy activity in a more comprehensive eddy field is proportionally more important than that of an eddy-permitting model. This analysis will have to wait until higher resolution data is available (for example, OCCAM 1/12°).
Appendix A

Derivation of the flux form of the potential vorticity equation

Ertel-Rossby potential vorticity is defined as

\[ Q = -\frac{q \cdot \nabla \sigma}{\rho}, \]  

(A.1)

for relative vorticity, given by

\[ q = 2\Omega + \nabla \wedge u, \]  

(A.2)

de three-dimensional velocity, \( u \), a dimensionless linear function of potential density, \( \sigma \), actual density, \( \rho \) and angular velocity of the Earth, \( \Omega \). Subject to an arbitrary diabatic heating rate, expressed as the material derivative of \( \sigma \),

\[ B = \frac{D\sigma}{Dt}, \]  

(A.3)

and, \( F \), an arbitrary body forcing per unit mass, the flux form of the potential vorticity equation can be derived from the three-dimensional momentum equations,

\[ \frac{\partial u}{\partial t} + (u \cdot \nabla) \cdot u + 2\Omega \times u = -\frac{1}{\rho} \nabla p - \nabla \phi + F, \]  

(A.4)

for geopotential, \( \phi \), and pressure, \( p \). The vorticity equation is first obtained by taking the curl of (A.4) and using the vector identities (D.3) and (D.4) to give

\[ \frac{\partial q}{\partial t} + \nabla \wedge (q \times u) = \frac{1}{\rho^2} \nabla \rho \times \nabla p + \nabla \wedge F. \]  

(A.5)

In essence, the desired potential vorticity equation is obtained by rearranging the dot product of the vorticity equation (A.5) and \( \nabla \sigma \). Assuming the triple product \( \nabla \rho \times \nabla \sigma \cdot \nabla p = 0 \), (A.5)\cdot \nabla \sigma \) gives

\[ \frac{\partial q}{\partial t} \cdot \nabla \sigma + [\nabla \wedge (q \times u)] \cdot \nabla \sigma = (\nabla \wedge F) \cdot \nabla \sigma. \]  

(A.6)
Appendix A: Derivation of the flux form of the potential vorticity equation

Then, using \((D.5)\), this can be written as

\[
\frac{\partial \mathbf{q}}{\partial t} \cdot \nabla \sigma - \nabla \cdot [\nabla \sigma \times (\mathbf{q} \times \mathbf{u})] = -\nabla \cdot [\nabla \sigma \times \mathbf{F}], \tag{A.7}
\]

or

\[
\frac{\partial \mathbf{q}}{\partial t} \cdot \nabla \sigma + \nabla \cdot [(\mathbf{q} \times \mathbf{u}) \times \nabla \sigma - \mathbf{F} \times \nabla \sigma] = 0. \tag{A.8}
\]

Observing

\[
\nabla \cdot (q \frac{\partial \sigma}{\partial t}) = \frac{\partial \sigma}{\partial t} \nabla \cdot q + q \cdot \nabla \frac{\partial \sigma}{\partial t} = q \cdot \frac{\partial}{\partial t} \nabla \sigma, \tag{A.9}
\]

this expression is added to \((A.8)\), in which the triple product has been expanded,

\[
\frac{\partial \mathbf{q}}{\partial t} \cdot \nabla \sigma + q \cdot \frac{\partial}{\partial t} (\nabla \sigma) + \nabla \cdot [(q \cdot \nabla \sigma) \mathbf{u} - q \frac{\partial \sigma}{\partial t} - (\mathbf{u} \cdot \nabla \sigma) \mathbf{q} - \mathbf{F} \times \nabla \sigma] = 0. \tag{A.10}
\]

Hence,

\[
\frac{\partial}{\partial t} (\rho Q) + \nabla \cdot \mathbf{J} = 0, \tag{A.11}
\]

where

\[
\mathbf{J} = \rho Q \mathbf{u} + q \frac{D \sigma}{Dt} + \mathbf{F} \times \nabla \sigma. \tag{A.12}
\]
Appendix B

Proof that the advection term cancels in the integral constraint

The following is a proof that, in the limit as $\Delta p \to 0$,

$$
\mathcal{J}_{\nu g}^p \equiv \lim_{\Delta p \to 0} \frac{\iint \frac{1}{\rho_0} \mathbf{k} \times \nabla p \cdot \nabla \sigma \, dA}{\iint dA} = 0,
$$

(B.1)

for an area bound by adjacent closed pressure contours, as shown in Fig. 3.6.

Defining the integration limits in terms of a constant, $p_0$, such that $p + \Delta p = p_0$, then $\mathcal{J}_{\nu g}^{p_0}$ is given by

$$
\mathcal{J}_{\nu g}^{p_0} = \lim_{p \to p_0} \frac{\iint_{p_0}^{p} \frac{1}{\rho_0} \mathbf{k} \times \nabla p \cdot \nabla \sigma \, dA}{\iint_{p}^{p_0} dA}.
$$

(B.2)

Recalling L'Hôpital's Rule:

$$
\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{f'(a)}{g'(a)},
$$

(B.3)

where $f(a) = g(a) = 0$ and the derivative $g'(a) \neq 0$, the following functions are defined:

$$
f(p) = \iint_{p}^{p_0} \frac{1}{\rho_0} \mathbf{k} \times \nabla p \cdot \nabla \sigma \, dA
$$

(B.4)

and

$$
g(p) = \iint_{p}^{p_0} \mathbf{k} \cdot dA.
$$

(B.5)

Then,

$$
f'(p) = \frac{\partial}{\partial p} \left( \iint_{p_0}^{p} \frac{1}{\rho_0} \nabla \cdot (p \nabla \sigma) \cdot \mathbf{k} \, dA - \iint_{p_0}^{p} \frac{1}{\rho_0} \nabla \cdot (p \nabla \sigma) \cdot \mathbf{k} \, dA \right),
$$

(B.7)
Appendix B: Proof that the advection term cancels in the integral constraint

but by Stokes’ Theorem, for areas bound by pressure contours,

\[
\int \int \frac{1}{\rho_0} \nabla \wedge (p \nabla \sigma) \cdot \mathbf{k} \, dA = \oint \frac{1}{\rho_0} p \nabla \sigma \cdot d\mathbf{l} = \frac{p}{\rho_0} \oint \nabla \sigma \cdot d\mathbf{l} = 0.
\]  

(B.8)  

(B.9)  

(B.10)

Hence,

\[ f'(p) = \frac{\partial}{\partial p} (0 - 0) = 0, \]  

(B.11)

whereas, for the pressure maximum in the centre of the gyre given as \( p_{\text{max}} \),

\[
g'(p) = \frac{\partial}{\partial p} \left( \int \int_{p_0}^p dA - \int \int_p^p dA \right) = - \frac{\partial}{\partial p} \int \int_{p_0}^p dA \neq 0, \ \forall p \neq p_{\text{max}}.
\]  

(B.12)  

(B.13)

So, for \( p_0 \neq p_{\text{max}} \),

\[
\frac{f'(p_0)}{g'(p_0)} = 0.
\]  

(B.14)

Therefore, (B.1) is true.
Appendix C

A vorticity impulse balance

In Section 3.4, the quantity $\mathcal{J}^p$ is defined as a normalised difference between two area integrals that are bounded by pressure contours (Fig. 3.6). The components of $\mathcal{J}^p$ are then interpreted as terms in a buoyancy-forcing trajectory budget. Using a similar technique, a subtropical gyre vorticity budget can be interpreted as a “vorticity impulse” balance and reveal where the steady-state gyre is most sensitive to changes in wind-forcing. This is demonstrated in the classical homogeneous Stommel (1948) gyre. The equations for the beta-plane homogeneous ocean, with depth $H$, density $\rho_0$, surface wind-stress $\tau_0$, linear drag coefficient $\eta$, and horizontal velocity $v$, are:

\[
\frac{\partial v}{\partial t} + v \cdot \nabla v + f k \times v = -\frac{1}{\rho_0} \nabla p + \frac{\tau_0}{\rho_0 H} - \eta v, \tag{C.1}
\]

and

\[
\nabla_h \cdot v = 0. \tag{C.2}
\]

Cancellation of the pressure term gives the vorticity equation, as shown below. Rewriting the velocity advection term using a vector identity (D.4),

\[
v \cdot \nabla v = \frac{1}{2} \nabla (v \cdot v) + \nabla \wedge v \times v, \tag{C.3}
\]

and taking the curl of (C.1), gives

\[
\frac{\partial q}{\partial t} + \nabla \wedge (q \times v) = \frac{\nabla \wedge \tau_0}{\rho_0 H} - \eta \nabla \wedge v, \tag{C.4}
\]

where $q$ is the absolute vorticity,

\[
q = f k + \nabla \wedge v = q k. \tag{C.5}
\]
Recalling the vector identities (D.6), (D.8) and continuity (C.2) leads to

$$\nabla \wedge (q \times v) = -\frac{\partial f}{\partial y} v + (v \cdot \nabla) q k - (q \frac{\partial}{\partial z}) v. \quad (C.6)$$

Hence, taking the vertical component gives the vorticity equation:

$$\frac{\partial q}{\partial t} + v \cdot \nabla q = \frac{1}{\rho_0 H} \nabla \wedge \tau_0 \cdot k - \eta \nabla \wedge v \cdot k. \quad (C.7)$$

At constant depth and steady state, integrating (C.7) over an annulus bounded by closed streamlines, $\psi$ and $\psi + \Delta \psi$, (instead of pressure contours, as in Fig. 3.6) gives, in natural coordinates,

$$\int \int_{\psi}^{\psi + \Delta \psi} v \cdot \nabla q \, ds \, dn = \int \int_{\psi}^{\psi + \Delta \psi} \frac{\nabla \wedge \tau_0 \cdot k}{\rho_0 H} \, ds \, dn - \int \int_{\psi}^{\psi + \Delta \psi} \eta \nabla \wedge v \cdot k \, ds \, dn, \quad (C.8)$$

where the streamfunction, $\psi$, is defined by

$$v = \frac{\partial \psi}{\partial n} s. \quad (C.9)$$

Noting that the velocity, $v$, can also be expressed in natural coordinates,

$$v = \frac{D_s}{Dt} s, \quad (C.10)$$

the area element of integration, $ds \, dn$, can be rewritten as

$$ds \, dn = dt \, d\psi. \quad (C.11)$$

Hence, dividing by constant $\Delta \psi$, (C.8) can be written as a contour integral along a streamline, where, by construction (that is, integrating around a closed steady-state contour) the advection term goes to zero\(^1\), giving a new balance:

$$\oint_{\psi} \frac{\nabla \wedge \tau_0 \cdot k}{\rho_0 H} \, dt = \oint_{\psi} \eta \nabla \wedge v \cdot k \, dt. \quad (C.12)$$

Following the flow, along closed steady-state streamlines, the input of vorticity, $dq$, by the wind-stress curl is equal to the sink of vorticity by the curl of the body force. This balance is distinct from that derived by Niiler (1966):

$$\oint_{\psi} \frac{\tau_0}{\rho_0 H} \cdot dt = \oint_{\psi} \eta v \cdot dt, \quad (C.13)$$

\(^1\)It can be shown that in the limit as $\Delta \psi \to 0$, the term attributed to the advection of vorticity, $\iint v \cdot \nabla q \, ds \, dn / \iint d\psi$, is identically zero by application of the proof given in Appendix B, having noted $v = k \cdot \nabla \psi$. 
which is a circulation balance, or alternatively a balance between “work done” by wind-stress and a retarding friction force, around a closed streamline. This alternative is stated as a means for motivating (C.12) not as a “force × distance = work done” balance but as a “vorticity × time = vorticity impulse” balance. The subtlety in the vorticity impulse balance (C.12) lies in the integration variable, \( dt \), which weights the vorticity integrand with a \( 1/v \) scaling. This feature arises from the choice of integration area – the area between two adjacent closed streamlines – as when the velocity is slower, the separation of the streamlines increases and hence the local integration element \( ds \, dn \) becomes larger. The utility of the balance (C.12) is apparent when thinking in terms of sensitivity to changes in forcing and in ascribing a ranking to whichever regions are most influential in setting a parcel’s vorticity. As this is a Lagrangian balance, the regions of weaker flow will have longer parcel residence time, giving the local forcing longer to change the parcel’s vorticity. Hence, \( dt \) is a proxy for the regions where small changes in the curl of the forcing produces the greatest change in parcel vorticity. This, in conjunction with the spatial distribution of the forcing curl, will highlight which regions dictate parcel vorticity. The diagnostic \( \nabla \wedge \mathbf{\tau}_0 \, dt \), then, grades regions according to how influential they are in setting the parcel’s vorticity.

The diagnostics are evaluated for the classic Stommel (1948) subtropical gyre that is a steady-state solution to (C.7). In a rectangular domain, length of \( L \), depth \( H \), for small Rossby number \( (f \gg \nabla \wedge \mathbf{v} \cdot \mathbf{k}) \), a purely zonal wind-stress, \( \mathbf{\tau}_0 \), antisymmetric about its middle latitude, where

\[
\mathbf{\tau}_0 = -\tau_0 \cos\left(\frac{\pi y}{L}\right) \mathbf{j},
\]

(C.14)

for constant \( \tau_0 \), and for a depth integrated stream function, \( \Psi \), defined as

\[
\int_{-H}^{0} \mathbf{v} \, dz = H \mathbf{v} = \mathbf{k} \times \nabla \Psi,
\]

(C.15)

Mellor (1996) gives the solution:

\[
\Psi = \frac{\pi \tau_0}{\beta \rho_0 H} \sin\left(\frac{\pi y}{L}\right) \left(1 - \frac{x}{L} - e^{-\frac{\rho_0 y}{L}}\right).
\]

(C.16)

In this study, \( L = H = \tau_0 = \rho_0 = \beta = 1 \) and \( \eta = 0.1 \). Fig. C.1a shows unlabelled streamlines overlaying a shaded field of undimensional \( dt \). Thus comparisons can
Figure C.1: Components from the vorticity impulse balance overlayed with streamfunction contours (unlabelled but with constant interval separation) for the classic steady-state Stommel gyre. Panel (a) is a plot of $dt$, the residence time for a parcel at any point. This is a proxy that shows which areas are most sensitive to changes in wind-stress curl. The fields in panels (b) and (c) denote the change in a parcel’s vorticity as it is geostrophically advected through that area forced by wind-stress curl and the curl of a frictional body force (here linear drag). These will, by construction, balance if integrated along a stream line. In particular, it can be seen that the wind-stress curl in the eastern part of the domain plays a more important role in setting a parcel’s vorticity than that in the western part of the domain. Also, that changes in wind-stress curl will be more acutely felt, in terms of parcel vorticity, in the eastern part of the domain. The units of the shaded field are dimensionless but the magnitudes are given.
be made, following a trajectory, to determine which regions have the longest parcel residence time and are therefore most sensitive to changes in the wind-stress curl. The figure shows the eastern side of the gyre to be most sensitive because the flow is slower there reducing the action time of the wind-stress curl at any point. It is noted that $dt$ is actually smallest in the centre of the gyre, but this is not of interest for the sensitivity problem as parcels in the centre are on short streamlines that do not transport information (surface forced vorticity) into other regions of the gyre. As $dt$ is a proxy for the areas, following a trajectory, that would be most sensitive to changes in external forcing, $\nabla \wedge \tau_0 dt$ is a proxy for regions, following a trajectory, most influential in setting the steady-state vorticity. Fig. C.1b shows unlabelled streamlines overlaying a shaded field of undimensional $\nabla \wedge \tau_0 dt$. The important thing to look for is a change in $\nabla \wedge \tau_0 dt$ following a streamline, as this illustrates a trajectory that threads regions that are more influential in setting the parcel’s vorticity that others. Furthermore, if a region is dominant in setting the parcel vorticity, then changes in the wind-stress curl in that region will have a greater overall impact on the parcel vorticity than if the wind-stress curl was to vary elsewhere along the trajectory. The figure shows the eastern mid-latitudes to be the most dominant region.

Fig. C.1c gives the balancing sink of vorticity arising from the curl of the frictional forcing which, as would be expected, is strongest in the western boundary current region where the flow is fastest.

This simple, steady-state, model is therefore used to draw the conclusion that the vorticity structure of this simple gyre is most sensitive to changes in wind-stress curl forcing in the mid-latitudes and on the gyre’s eastern side.
Appendix C: A vorticity impulse balance
Appendix D

Vector calculus identities

The following are standard vector calculus identities for arbitrary scalars, $\psi$ and $\phi$, and arbitrary vectors, $A$ and $B$.

\[
\nabla \cdot (\psi \phi) = \psi \nabla \phi + \phi \nabla \psi \quad (D.1)
\]

\[
\nabla \cdot (\psi A) = \psi \nabla \cdot A + \nabla \psi \cdot A \quad (D.2)
\]

\[
\nabla \wedge (\psi A) = \psi \nabla \wedge A + \nabla \psi \times A \quad (D.3)
\]

\[
\nabla \cdot (A \cdot B) = (A \cdot \nabla)B + (B \cdot \nabla)A + A \times (\nabla \wedge B) + B \times (\nabla \wedge A) \quad (D.4)
\]

\[
\nabla \cdot (A \times B) = B \cdot (\nabla \wedge A) - A \cdot (\nabla \wedge B) \quad (D.5)
\]

\[
\nabla \wedge (A \times B) = (\nabla \cdot B)A - (\nabla \cdot A)B + (B \cdot \nabla)A - (A \cdot \nabla)B \quad (D.6)
\]

\[
\nabla \wedge (\nabla \wedge A) = \nabla (\nabla \cdot A) - \nabla^2 A \quad (D.7)
\]

\[
\nabla \cdot (\nabla \wedge A) = 0 \quad (D.8)
\]

\[
\nabla \wedge (\nabla \psi) = 0 \quad (D.9)
\]
Bibliography


Bibliography


Unfortunately, Fish mysteriously vanished before telling Eel the secret.