An introduction to derived and triangulated categories

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PSSL, Glasgow, 6–7th May 2006

Abelian categories and complexes

Derived categories and functors arise because

1. we want to work with complexes but only up to an equivalence relation which retains cohomological information,

2. many interesting functors between abelian categories are only left (or right) exact.

$\mathbf{A}$ is **abelian** if $\mathbf{A}$ is additive and morphisms in $\mathbf{A}$ have kernels and cokernels.

Examples 1. *1.* $\mathbf{Ab} =$ *Abelian groups,*

2. $\mathbf{R\text{-}Mod} =$ *left modules over a ring* R*,*

3. $\mathbf{R\text{-}Mod}(X) =$ *sheaves of* R*-modules on a topological space* X*.*

Theorem 1 (Freyd–Mitchell)**.** *If* $\mathbf{A}$ *is a small abelian category then* $\mathbf{A}$ *embeds fully faithfully in* $\mathbf{R\text{-}Mod}$ *for some ring* R*.*

Often we can associate a **cochain complex**

$$A^{\cdot} = \cdots \to A^i \xrightarrow{d} A^{i+1} \xrightarrow{d} A^{i+2} \to \cdots$$

in $\mathbf{A}$ to 'some mathematical object'. (Cochain complex means $d^2 = 0$ or $\ker d \supset \operatorname{im} d$.) For example

$$\begin{array}{rcl} \text{Space } X & \longmapsto & C_T^*(X;\mathbb{Z}) \\ \text{Sheaf } \mathcal{F} & \longmapsto & \check{C}^*(\mathcal{U};F) \end{array}$$

In both these cases there is no unique way to do this (we need to choose a triangulation T in the first case and an open cover $\mathcal{U}$ in the second) *but* the **cohomology groups**

$$H^i(A^{\cdot}) = \frac{\ker d : A^i \to A^{i+1}}{\operatorname{im} d : A^{i-1} \to A^i}$$

are well-defined up to isomorphism in $\mathbf{A}$.

The H^i measure the failure of $A^{\cdot}$ to be **exact** i.e. for $\ker d = \operatorname{im} d$ eg.

$$0 \to A \xrightarrow{f} B \to 0$$

has $H^0 = \ker f$ and $H^1 = \operatorname{coker} f$.

First wish: find a good equivalence relation on complexes which retains this cohomological information.

For complexes $A^\cdot$, $B^\cdot$ in $\mathbf{A}$ define

$$\mathrm{Hom}^i(A^\cdot, B^\cdot) = \bigoplus_j \mathrm{Hom}(A^j, A^{i+j})$$

There is a differential

$$\begin{aligned} \delta : \mathrm{Hom}^i(A^\cdot, B^\cdot) &\longrightarrow \mathrm{Hom}^{i+1}(A^\cdot, B^\cdot) \\ f &\longmapsto f d_A + (-1)^{i+1} d_B f \end{aligned}$$

with the property that

$$f \in \mathrm{Hom}^0(A^\cdot, B^\cdot) \text{ is } \begin{cases} \text{a cochain map if } \delta f = 0 \\ \text{null-homotopic if } f = \delta g \end{cases}$$

In particular $H^0\mathrm{Hom}^\cdot(A^\cdot, B^\cdot) =$ homotopy classes of cochain maps.

$\mathbf{Com}(\mathbf{A})$ has objects complexes in $\mathbf{A}$ and morphisms cochain maps (of degree 0).

$\mathbf{K}(\mathbf{A})$ has objects complexes in $\mathbf{A}$ and morphisms homotopy classes of cochain maps.

A cochain map $f : A^\cdot \to B^\cdot$ induces maps

$$H^i(f) : H^i(A^\cdot) \to H^i(B^\cdot).$$

Homotopic cochain maps induce the same map on cohomology.

Say f is a **quasi-isomorphism** (QI) if it induces isomorphisms for all i.

Examples 2. *1. homotopy equivalences are QIs*

2. not all QIs are homotopy equivalences eg.

$$\begin{array}{rccccccc} A^\cdot = & 0 & \longrightarrow & \mathbb{Z} & \longrightarrow & \mathbb{Z} & \longrightarrow & 0 \\ & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ B^\cdot = & 0 & \longrightarrow & 0 & \longrightarrow & \mathbb{Z}/2 & \longrightarrow & 0 \end{array}$$

QIs generate an equivalence relation on the objects of $\mathbf{Com}(\mathbf{A})$ which is stronger than homotopy equivalence.

The **derived category** $\mathbf{D}(\mathbf{A})$ is the localisation of $\mathbf{Com}(\mathbf{A})$ at the class of QIs: it has the same objects as $\mathbf{Com}(\mathbf{A})$ and morphisms given by diagrams

$$A^{\cdot} \leftarrow C_0^{\cdot} \rightarrow \cdots \rightarrow C_n^{\cdot} \leftarrow B^{\cdot}$$

where the wrong-way arrows are QIs.

For technical reasons we usually want to consider the full subcategory of bounded below, bounded above or bounded complexes and we write

$$\mathbf{D}^{+}(\mathbf{A}) \qquad \mathbf{D}^{-}(\mathbf{A}) \qquad \mathbf{D}^{\mathbf{b}}(\mathbf{A})$$

accordingly.

Fact: $\mathbf{D}^{+}(\mathbf{A})$ is the localisation of $\mathbf{K}^{+}(\mathbf{A})$ at QIs i.e. inverting QIs automatically identifies homotopic cochain maps!

The class of QIs in $\mathbf{K}(\mathbf{A})$ is **localising** i.e.

1. $1_X \in QI$ and $s, t \in QI$ implies $st \in QI$,

2. We can complete diagrams as below

$$\begin{array}{ccc} W & \dashrightarrow & Y \\ {\scriptstyle t}\downarrow & & \downarrow{\scriptstyle s} \\ X & \longrightarrow & Z \end{array} \qquad\qquad \begin{array}{ccc} W & \dashleftarrow & Y \\ {\scriptstyle t}\uparrow & & \uparrow{\scriptstyle s} \\ X & \longleftarrow & Z \end{array}$$

3. $sf = sg$ for $s \in QI \implies ft = gt$ for some $t \in QI$.

Follows that morphisms in $\mathbf{D}^+(\mathbf{A})$ can be represented by 'roofs'

$$\begin{array}{ccccc} & & C^{\cdot} & & \\ & \overset{\sim}{\swarrow} & & \searrow & \\ A^{\cdot} & & & & B^{\cdot} \end{array}$$

with composition given by 2.

Functors between abelian categories

Most of the functors in which we are interested are **additive** i.e.

$$\text{Hom}(A, B) \to \text{Hom}(FA, FB)$$

is a homomorphism of abelian groups eg.

$$\begin{aligned} - \otimes A : \mathbf{Ab} &\to \mathbf{Ab} \\ \text{Hom}(A, -) : \mathbf{Ab} &\to \mathbf{Ab} \\ \text{Hom}(-, A) : \mathbf{Ab}^{op} &\to \mathbf{Ab} \end{aligned}$$

Say an additive functor F is **exact** if it preserves kernels and cokernels, equivalently

$$\begin{aligned} & 0 \to A \to B \to C \to 0 \quad \text{exact} \\ \implies & 0 \to FA \to FB \to FC \to 0 \quad \text{exact} \end{aligned}$$

Say it is **left exact** if $0 \to FA \to FB \to FC$ is exact and **right exact** if $0 \to FA \to FB \to FC$ is exact.

Examples 3. *Consider the complex of abelian groups*

$$A^{\cdot} = 0 \to \mathbb{Z} \xrightarrow{2} \mathbb{Z} \to \mathbb{Z}/2 \to 0.$$

1. $A^{\cdot} \otimes \mathbb{Z}/2 = 0 \to \mathbb{Z}/2 \xrightarrow{0} \mathbb{Z}/2 \xrightarrow{1} \mathbb{Z}/2 \to 0$

2. $\mathrm{Hom}(\mathbb{Z}/2, A^{\cdot}) = 0 \to 0 \to 0 \to \mathbb{Z}/2 \to 0$

3. $\mathrm{Hom}(A^{\cdot}, \mathbb{Z}) = 0 \leftarrow \mathbb{Z} \xleftarrow{2} \mathbb{Z} \leftarrow 0 \leftarrow 0$

In fact $- \otimes A$ is right exact and $\mathrm{Hom}(A, -)$ and $\mathrm{Hom}(-, A)$ are left exact (for any A and in any abelian category).

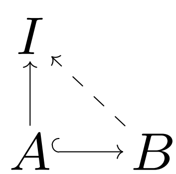
Proposition 2. *Left adjoints are right exact (and right adjoints are left exact).*

For example, $-\otimes A$ is left adjoint to $\mathrm{Hom}(A, -)$.

Second wish: measure the failure of a left or right exact functor to be exact.

Step 1: Find complexes on which left exact functors are exact.

Definition 1. $I \in \mathbf{A}$ *is* **injective** $\iff$ Hom$(-, I)$ *is exact* $\iff$



For example $\mathbb{Q}$ is injective in $\mathbf{Ab}$.

Injectives are well-behaved for any left exact functor F, in the sense that if

$$0 \to I^r \to I^{r+1} \to \cdots$$

is a bounded below exact complex of injectives then

$$0 \to FI^r \to FI^{r+1} \to \cdots$$

is bounded below and exact (but the FI^i need not be injective).

Step 2: Replace objects by complexes of injectives.

An **injective resolution** is a QI

$$\begin{array}{ccccccccccc} A = & \cdots \longrightarrow & 0 & \longrightarrow & A & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow \cdots \\ & & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\ I^{\cdot} = & \cdots \longrightarrow & 0 & \longrightarrow & I^0 & \longrightarrow & I^1 & \longrightarrow & I^2 & \longrightarrow \cdots \end{array}$$

Note that

$$H^i(I^{\cdot}) = \begin{cases} A & i = 0 \\ 0 & i \neq 0 \end{cases}$$

Example 4. *The diagram*

$$\begin{array}{ccccccccccc} \cdots \longrightarrow & 0 & \longrightarrow & \mathbb{Z} & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow \cdots \\ & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\ \cdots \longrightarrow & 0 & \longrightarrow & \mathbb{Q} & \longrightarrow & \mathbb{Q}/\mathbb{Z} & \longrightarrow & 0 & \longrightarrow \cdots \end{array}$$

defines an injective resolution of $\mathbb{Z}$ *in* $\mathbf{Ab}$.

If $\mathbf{A}$ has **enough injectives**, i.e. every object injects into an injective, then every object A has an injective resolution (defined inductively by embedding A into an injective, then embedding the cokernel of this injection into and injective and so on).

Theorem 3. $\mathbf{R}\text{-}\mathbf{Mod}(X)$ *has enough injectives.*

In fact, we can show that every bounded below complex is QI to a bounded below complex of injectives which is unique up to (non-canonical) isomorphism in $\mathbf{K}^{+}(\mathbf{A})$. (Uniqueness requires bounded belowness!)

Hence, in $\mathbf{D}^{+}(\mathbf{A})$, every object is isomorphic to a complex of injectives.

Step 3: Given left exact $F : \mathbf{A} \to \mathbf{A}'$ define an 'exact' functor

$$RF : \mathbf{D}^+(\mathbf{A}) \to \mathbf{D}^+(\mathbf{A}).$$

Need to make sense of 'exactness' because $\mathbf{D}^+(\mathbf{A})$ is additive but not, in general, abelian. (In fact $\mathbf{D}^+(\mathbf{A})$ is abelian if and only if

$$\mathbf{D}^+(\mathbf{A}) \to \mathbf{Com_0}(\mathbf{A}) : A^\cdot \mapsto H^\cdot(A)$$

is an equivalence.)

Want a replacement for kernels and cokernels.

Given $f : A^{\cdot} \to B^{\cdot}$ in $\mathbf{Com(A)}$ define the **mapping cone** $\mathrm{Cone}^{\cdot}(f)$ by

$$\begin{array}{ccc} \mathrm{Cone}^i(f) & = & A^{i+1} \oplus B^i \\ \downarrow & & \downarrow \begin{pmatrix} -d_A & 0 \\ f^{i+1} & d_B \end{pmatrix} \\ \mathrm{Cone}^{i+1}(f) & = & A^{i+2} \oplus B^{i+1} \end{array}$$

There are maps

$$A^{\cdot} \xrightarrow{f} B^{\cdot} \xrightarrow{(01)^t} \mathrm{Cone}^{\cdot}(f) \xrightarrow{(10)} \Sigma A^{\cdot} \qquad (1)$$

where Σ is the **left shift**:

$$\Sigma A^{\cdot} = \cdots \to A^{i+1} \xrightarrow{-d} A^{i+2} \to \cdots$$

with A^i in degree $i-1$.

Theorem 4. *Applying H^0 to (1) gives an exact complex*

$$\cdots H^0 A^\cdot \to H^0 B^\cdot \to H^0 \text{Cone}^\cdot(F) \to H^1 A^\cdot \to \cdots$$

Example 5. *Let $f : A \to B$ be a morphism in $\mathbf{A}$ considered as a cochain map of complexes which are zero except in degree 0. Then*

$$\text{Cone}^\cdot(f) = \cdots \to 0 \to A \xrightarrow{f} B \to 0 \to \cdots$$

where B is in degree 0 and the associated exact complex of cohomology groups is

$$0 \to \ker f \to A \xrightarrow{f} B \to \operatorname{coker} f \to 0$$

Note that

$$\begin{aligned} f \text{ is a QI} &\iff H^i f \text{ an isomorphism } \forall i \\ &\iff H^i \text{Cone}^\cdot(f) = 0 \quad \forall i \\ &\iff \text{Cone}^\cdot(f) \cong 0 \text{ in } \mathbf{D}^+(\mathbf{A}) \end{aligned}$$

Say a diagram

$$A^{\cdot} \to B^{\cdot} \to C^{\cdot} \to \Sigma A^{\cdot}$$

in $\mathbf{D}^{+}(\mathbf{A})$ is an **exact triangle** if it is isomorphic to a diagram of the form (1).

Example 6. *An exact sequence*

$$0 \to A \to B \to C \to 0$$

in $\mathbf{A}$ *determines an exact triangle*

$$A \to B \to C \to \Sigma A$$

because $\mathrm{Cone}^{\cdot}(A \to B)$ *is QI to* $\mathrm{coker}\,(A \to B)$. *(Here we think of objects of* $\mathbf{A}$ *as complexes which are zero except in degree* 0.*)*

Any morphism $f : A^{\cdot} \to B^{\cdot}$ can be completed to an exact triangle (but not uniquely).

Say an additive functor $\mathbf{D}^{+}(\mathbf{A}) \to \mathbf{D}^{+}(\mathbf{A}')$ is **triangulated** if it commutes with Σ and preserves exact triangles.

If F is left exact then naively applying F to complexes term-by-term is a bad idea because

1. $A^{\cdot} \cong 0 \iff A^{\cdot}$ exact $\not\Longrightarrow FA^{\cdot}$ exact,

2. f a QI $\not\Longrightarrow Ff$ a QI.

Solution: apply F to complexes of injectives!

If we have functorial injective resolutions (for instance for sheaves of vector spaces)

$$I : \mathbf{Com}^{+}(\mathbf{A}) \to \mathbf{Com}^{+}(\mathbf{Inj}\,\mathbf{A})$$

then define the **right derived functor** of F by

$$RF = F \circ I.$$

More generally use fact that there is an equivalence

$$\mathbf{K}^{+}(\mathbf{Inj}\,\mathbf{A}) \to \mathbf{D}^{+}(\mathbf{A})$$

(follows because QIs of complexes of injectives are homotopy equivalences).

Eyes on the prize! We've turned a left exact functor $F : \mathbf{A} \to \mathbf{A}'$ into a triangulated functor $RF : \mathbf{D}^+(\mathbf{A}) \to \mathbf{D}^+(\mathbf{A}')$. Furthermore, direct computation shows that for $A \in \mathbf{A}$

$$H^0(RFA) \cong FA.$$

Now we can measure the failure of F to be exact as follows: an exact sequence

$$0 \to A \to B \to C \to 0$$

in $\mathbf{A}$ becomes an exact triangle

$$A \to B \to C \to \Sigma A$$

in $\mathbf{D}^+(\mathbf{A})$ becomes an exact triangle

$$RFA \to RFB \to RFC \to \Sigma RFA$$

in $\mathbf{D}^+(\mathbf{A}')$ becomes an exact sequence

$$0 \to FA \to FB \to FC \to H^1(RFA) \to \cdots$$

in $\mathbf{A}'$.

Examples 7. *1.* $H^i \circ R\mathsf{Hom}(A, -) \cong \mathsf{Ext}^i(A, -)$ *is the classical Ext functor (and Tor is the cohomology of the left derived functor of tensor product).*

2. Global sections $\Gamma_X : \mathbf{Ab(X)} \to \mathbf{Ab}$ *is left exact and*

$$H^i \circ R\Gamma_X(-) \cong H^i(X; -)$$

is sheaf cohomology.

3. Group cohomology, Lie algebra cohomology, Hochschild cohomology...

Typically higher cohomology groups of a derived functor can be interpreted as obstruction groups to some problem.

We can define the left derived functor LF of a right derived functor F in a similar way but using projective resolutions, provided of course that there are enough projectives i.e. that every object is a quotient of a projective.

We say an object P in $\mathbf{A}$ is **projective** if $\mathrm{Hom}(P, -)$ is exact or equivalently

$$\begin{array}{ccc} & & P \\ & \swarrow & \downarrow \\ A & \twoheadrightarrow & B \end{array}$$

and that a QI

$$\begin{array}{lccccccccccc} P^{\cdot} = & \cdots & \longrightarrow & P^2 & \longrightarrow & P^1 & \longrightarrow & P^0 & \longrightarrow & 0 & \longrightarrow & \cdots \\ & & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\ A = & \cdots & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & A & \longrightarrow & 0 & \longrightarrow & \cdots \end{array}$$

is a **projective resolution**. Since projective resolutions go to the left we need to work with bounded above complexes and define

$$LF : \mathbf{D}^{-}(\mathbf{A}) \to \mathbf{D}^{-}(\mathbf{A}').$$

What do we do if there are not enough injectives or projectives?

We can abstract the properties of projectives (and of course similarly for injectives) as follows. Say a class $\mathcal{C}$ of objects in $\mathbf{A}$ is **adapted** to a right exact functor F if

1. $\mathcal{C}$ is closed under $\oplus$,

2. F takes exact complexes in $\mathbf{Com}^{-}(\mathcal{C})$ to exact complexes,

3. any object is a quotient of an object in $\mathcal{C}$.

Examples 8. *If there are enough projectives then the class of projectives is adapted to any right exact functor. The class of flat modules is adapted to* $-\otimes A$ *in* R-$\mathbf{Mod}$.

Theorem 5. *If* $\mathcal{C}$ *is adapted to* F *then there is an equivalence*

$$\mathbf{K}^{-}(\mathcal{C})_{QI} \xrightarrow{\sim} \mathbf{D}^{-}(\mathbf{A})$$

from the localisation of the homotopy category of complexes in $\mathcal{C}$ *at QIs to the derived category.*

We can define $LF = F \circ \Phi$ where Φ is an inverse to the equivalence.

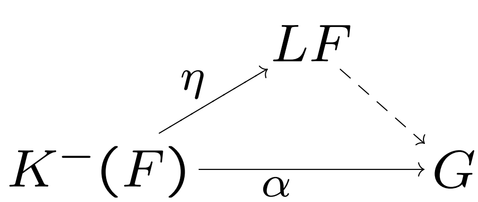
Since there may be many adapted classes and the inverse to the equivalence is not unique we need to characterise derived functors more precisely. Formally a left derived functor of a right exact functor $F\mathbf{A} \to \mathbf{A}'$ is a pair (LF, η) where

$$LF : \mathbf{D}^-(\mathbf{A}) \to \mathbf{D}^-(\mathbf{A}')$$

is triangulated and $\eta : K^-(F) \to LF$ is a natural transformation such that for any triangulated

$$G : \mathbf{D}^-(\mathbf{A}) \to \mathbf{D}^-(\mathbf{A}')$$

and $\alpha : K^-(F) \to G$ we have a unique factorisation



Easy mistakes to be avoided!

1. $A^{\cdot}$ is not in general QI to $H^{\cdot}(A)$ eg.

$$\mathbb{C}[x,y]^2 \xrightarrow{(x\ \ y)} \mathbb{C}[x,y]$$

and

$$\mathbb{C}[x,y] \xrightarrow{0} \mathbb{C}$$

have the same cohomology groups but are not QI as complexes of $\mathbb{C}[x,y]$-modules.

2. There are non-zero maps f in $\mathbf{D}^+(\mathbf{A})$ which induce the zero map on cohomology, i.e.

$$H^i f = 0 \qquad \forall i.$$

For example, the connecting map δ in the exact triangle associated to a non-split short exact sequence.

Why bother with the derived category?

The exact sequence

$$0 \to FA \to FB \to FC \to H^1(RFA) \to \cdots$$

can be constructed without this machinery (using the snake lemma and a little homological algebra) *but* RF contains more information than just its cohomology groups. For example we retain higher order information such as Massey products.

Secondly, derived functors are often 'better behaved'. For example Hom$(-, \mathbb{Z})$ doesn't induce a duality on $\mathbf{Ab}$ but the right derived functor

$$R\text{Hom}(-, \mathbb{Z}) : \mathbf{D}^+_{\mathbf{fg}}(\mathbf{Ab})^{op} \to \mathbf{D}^+_{\mathbf{fg}}(\mathbf{Ab})$$

does square to the identity.

Thirdly...

Triangulated categories

We can abstract the structure of the derived category as follows.

An additive category $\mathbf{D}$ is **triangulated** if there are

1. an additive automorphism Σ,

2. a class of diagrams, closed under isomorphism and called exact triangles,

   $$A \to B \to C \to \Sigma A$$

   satisfying four axioms as below.

TR1 $A \xrightarrow{1} A \to 0 \to \Sigma A$ is exact and any morphism can be completed to an exact triangle (not necessarily uniquely);

TR2 we can 'rotate' triangles:

$$\begin{array}{rll} & A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} \Sigma A & \text{exact} \\ \iff & B \xrightarrow{g} C \xrightarrow{h} \Sigma A \xrightarrow{-\Sigma f} \Sigma B & \text{exact} \end{array}$$

TR3 we can complete commuting diagrams of maps between triangles as below (but not necessarily uniquely)

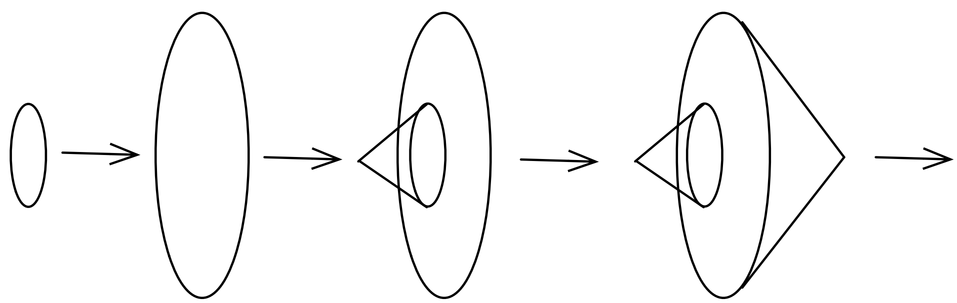
$$\begin{array}{ccccccc} A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & \Sigma A \\ {\scriptstyle a}\downarrow & & {\scriptstyle b}\downarrow & & {\scriptstyle \exists}\downarrow & & {\scriptstyle \Sigma a}\downarrow \\ A' & \longrightarrow & B' & \longrightarrow & C' & \longrightarrow & \Sigma A' \end{array}$$

TR4 the octahedron axiom (beyond my TEXskills)!

Warning: there are categories which admit several different triangulated structures.

Examples 9. $\mathbf{K}^*(\mathbf{A}), \mathbf{D}^*(\mathbf{A})$ *where* $* = \pm, b$.

There are other important examples, in particular the stable homotopy category: consider the homotopy category $\mathbf{K}(\mathbf{CW})$ of pointed CW complexes and homotopy classes of pointed maps between them. Up to homotopy any map is a cofibration, essentially an inclusion. Puppe sequences



$$X \xrightarrow{f} Y \xrightarrow{g} \mathrm{Cone}(f) \xrightarrow{h} \mathrm{Cone}(g) \to$$

give a *potential* class of triangles because the mapping cone Cone(g) is homotopic to the suspension ΣX.

$\mathbf{K(CW)}$ is not triangulated because

1. there is no additive structure on the morphisms $[X, Y]$ and,

2. Σ is not invertible.

However, suspension does have a right adjoint, the loop space functor,

$$[\Sigma X, Y] \cong [X, \Omega Y].$$

Furthermore $[X, \Omega Y]$ is a group and $[X, \Omega^2 Y]$ an abelian group. By replacing CW complexes by CW-spectra, very roughly infinite loop spaces with explicit deloopings, we obtain a triangulated category

$$\mathbf{K(CW - spectra)}$$

called the **stable homotopy category**.

A t-**structure** is a pair of full subcategories

$$\mathbf{D}_{\leq 0} \qquad \mathbf{D}_{\geq 0}$$

satisfying

1. $\mathbf{\Sigma}\mathbf{D}_{\leq 0} \subset \mathbf{D}_{\leq 0}$ and $\mathbf{\Sigma}\mathbf{D}_{\geq 0} \supset \mathbf{D}_{\geq 0}$

2. $\mathrm{Hom}(A, B) = 0$ for any $A \in \mathbf{D}_{\leq 0}$ and $B \in \mathbf{\Sigma}^{-1}\mathbf{D}_{\geq 0}$

3. any B is in an exact triangle

$$A \to B \to C \to \mathbf{\Sigma}A$$

   with $A \in\in \mathbf{D}_{\leq 0}$ and $B \in \mathbf{\Sigma}^{-1}\mathbf{D}_{\geq 0}$.

Example 10. *Derived categories come with a natural choice of t-structure given by the full subcategories of complexes which are zero in strictly positive and in strictly negative degrees.*

Theorem 6. *The* **heart** $\mathbf{D}_{\leq 0} \cap \mathbf{D}_{\geq 0}$ *of a t-structure is an abelian category.*

There can be different t-structures on the same triangulated category. Hence equivalences of triangulated categories can provide interesting relationships between different abelian categories. This is a very fruitful point of view and we end with some examples from geometry.

A. Birational geometry:

Conjecture 7 (Bondal–Orlov)**.** *Smooth complex projective Calabi–Yau varieties X and Y are birational $\iff$ there is a triangulated equivalence*

$$\mathbf{D}^{\mathbf{b}}_{coh}(\mathbf{X}) \xrightarrow{\sim} \mathbf{D}^{\mathbf{b}}_{coh}(\mathbf{Y})$$

between the bounded derived categories of coherent sheaves.

This is known to hold in dimension 3.

B. Riemann–Hilbert correspondence: for any complex projective variety X there is a triangulated equivalence

$$\mathbf{D}^{\mathbf{b}}_{\mathsf{rh}}(\mathcal{D}_{\mathbf{X}} - \mathbf{Mod}) \ \stackrel{\sim}{\longrightarrow}\ \mathbf{D}^{\mathbf{b}}_{\mathsf{alg\text{-}c}}(\mathbf{X})$$

relating regular holonomic D-modules, certain systems of analytic differential equations on X, to algebraically constructible sheaves, which encode the topology of the subvarieties of X.

C. Homological mirror symmetry:
Conjecture 8 (Kontsevitch)**.** *Smooth projective Calabi–Yau varieties come in mirror pairs X and Y so that there is a triangulated equivalence*

$$\mathbf{D}^{\mathbf{b}}_{coh}(\mathbf{X}) \stackrel{\sim}{\longrightarrow} \mathbf{D}^{\mathbf{b}}\mathbf{Fuk_0}(\mathbf{Y}$$

between the dounded derived category of coherent sheaves (algebraic geometry) of X and the bounded derived Fukaya category (symplectic geometry) of Y, and vice versa.

This is known to hold, for instance, for elliptic curves.