Witt groups and Witt spaces

1. Geometric interpretations of the Witt groups $W(\mathbb{Z})$ and $W(\mathbb{Q})$
2. Witt space bordism and Balmer–Witt groups of PL-constructible sheaves
3. Witt bordism proof of Cappell–Shaneson's L-class formula
4. Remarks on Balmer–Witt groups of algebraically constructible sheaves

1. Witt and IP spaces

Theorem 1 (Siegel)**.**

$$\Omega_*^{\textit{Witt}} \cong \begin{cases} \mathbb{Z} & * = 0 \\ W(\mathbb{Q}) & * = 4k, k > 0 \\ 0 & \textit{otherwise} \end{cases}$$

Theorem 2 (Pardon)**.**

$$\Omega_*^{\textit{IP}} \cong \begin{cases} W(\mathbb{Z}) & * = 4k, k > 0 \\ \mathbb{Z}_2 & * = 4k+1, k > 0 \\ 0 & \textit{otherwise} \end{cases}$$

The Witt group $W(R)$ of a commutative ring R is $M(R)/\sim$ where

- $M(R)$ is the monoid of isomorphism classes of inner products on finitely-generated projective R-modules with direct sum,

- $A \sim B \iff A \oplus P \cong B \oplus Q$ where P, Q possess Lagrangians.

Examples

1. $W(\mathbb{Z}) \cong \mathbb{Z}$ (the signature)

2. For prime p we have

$$W(\mathbb{Z}_p) \cong \begin{cases} \mathbb{Z}_2 & p = 2 \\ \mathbb{Z}_2 \oplus \mathbb{Z}_2 & p = 1 \bmod 4 \\ \mathbb{Z}_4 & p = 3 \bmod 4 \end{cases}$$

3. There is a split exact sequence

$$\begin{array}{ccccccccc} 0 & \longrightarrow & W(\mathbb{Z}) & \longrightarrow & W(\mathbb{Q}) & \longrightarrow & W(\mathbb{Q}/\mathbb{Z}) & \longrightarrow & 0 \\ & & \| & & & & \| & & \\ & & \mathbb{Z} & & & & \bigoplus_p W(\mathbb{Z}_p) & & \end{array}$$

A **stratified space** X is a compact Hausdorff space with a locally finite decomposition

$$X = \bigsqcup_{S \in I} S$$

into locally closed manifolds (the **strata**). Each stratum S has a neighbourhood homeomorphic to a locally trivial bundle over S with fibre

$$\text{Cone}(L_S) = \frac{L_S \times [0,1)}{L_S \times \{0\}}$$

where the **link** L_S is a stratified space of dim $\text{codim}\, S - 1$. This homeomorphism preserves the respective stratifications.

X an **n-pseudomanifold** $\iff$ no codimension 1 strata and X the closure of the n-dim strata.

[Goresky–MacPherson]

The **intersection homology groups** $IH_*(X)$ of a pseudomanifold are the homology groups of a subcomplex of the simplicial chains.

1. Intersection homology is a homeomorphism, but not a homotopy, invariant.

2. If X is a manifold $IH_*(X) \cong H_*(X)$.

3. If $\dim X = 2n$ and X is compact then

$$IH_*(\text{Cone}(X)) \cong \begin{cases} IH_*(X) & * \leq n \\ 0 & * > n \end{cases}$$

$$IH_*^{\text{cl}}(\text{Cone}(X)) \cong \begin{cases} 0 & * \leq n+1 \\ IH_{*-1}(X) & * > n+1 \end{cases}$$

Examples 1. If X is a $(2n-1)$-dim pseudo-manifold then

$$IH_i(\mathsf{Susp}\,(X)) \cong \begin{cases} IH_i(X) & i < n \\ 0 & i = n \\ IH_{i-1}(X) & i > n. \end{cases}$$

2. If $(M, \partial M)$ is a $2n$-dim manifold with boundary then

$$IH_i(M/\partial M) \cong \begin{cases} H_i(M) & i < n \\ \mathrm{Im} : H_i(M) \to H_i(M, \partial M) & i = n \\ H_i(M, \partial M) & i > n. \end{cases}$$

[Siegel] A pseudomanifold W is a **Witt space** $\iff$ for each $(2k+1)$-codim stratum S

$$IH_k(L_S; \mathbb{Q}) = 0.$$

Examples: manifolds, complex varieties, suspensions of odd dim Witt spaces but **not** eg. Susp(T^2):

[Goresky–Siegel] A Witt space W is an **intersection Poincaré (IP) space** $\iff$ for each $2k$-codim stratum S

$$IH_{k-1}^{\text{tor}}(L_S; \mathbb{Z}) = 0.$$

Examples: manifolds, some complex varieties but **not** eg. $\mathbb{C}^{2n}/\mathbb{Z}_m$.

Theorem 3 (Pardon, Siegel)**.** *The intersection form*

$$I_X : IH_{2k}(X;R) \to IH^{2k}(X;R)$$

is symmetric and is an isomorphism when

- $R = \mathbb{Q}$ *and* X *is Witt*

  $\Rightarrow [I_X] \in W(\mathbb{Q})$

- $R = \mathbb{Z}$ *and* X *is IP*

  $\Rightarrow [I_X] \in W(\mathbb{Z})$

In either case $X = \partial Y \Rightarrow [I_X] = 0$.

(In particular signature is a bordism invariant of both Witt and IP spaces.)

Recall that

$$0 \to W(\mathbb{Z}) \to W(\mathbb{Q}) \to \bigoplus_p W(\mathbb{Z}_p) \to 0.$$

To realise an obstruction $\beta \in W(\mathbb{Z}_p) \subset W(\mathbb{Q})$:

1. choose an integral matrix B with even diagonal entries representing β;

2. plumb according to B to obtain a $4k$-manifold with boundary $(M, \partial M)$;

3. collapse the boundary to obtain $M/\partial M$.

Then $M/\partial M$ is Witt but not IP because

$$IH_{2k-1}(\partial M) \cong \mathbb{Z}_p.$$

The linking form represents the class in $W(\mathbb{Z}_p)$.

2. Witt bordism and Witt groups

Bordism of Witt spaces is a homology theory.

Theorem 4. *Witt space bordism is the connective version of Ranicki's free rational L-theory.*

Another description: to each X we can assign its *PL-constructible bounded derived category of sheaves* $D_b^c(X;\mathbb{Q})$.

There is a contravariant triangulated functor

$$D_{\mathsf{PV}} : D_b^c(X;\mathbb{Q}) \to D_b^c(X;\mathbb{Q})$$

with $D_{\mathsf{PV}}^2 = 1$ (Poincaré–Verdier duality) given by

$$D_{\mathsf{PV}}(-) = RHom(-, \mathcal{C}_X)$$

where $\mathcal{C}_X$ is the sheaf complex of chains with closed support.

In this situation there are 4-periodic Balmer–Witt groups

$$W_i\left(D_b^c(X;\mathbb{Q})\right)$$

generated by isomorphism classes of self-dual objects up to a Witt equivalence relation.

Theorem 5. *These Balmer–Witt groups form a homology theory and*

$$\begin{array}{rcl} \Omega_i^{\textit{Witt}}(X) & \longrightarrow & W_i\left(D_b^c(X;\mathbb{Q})\right) \\ [f: W \to X] & \longmapsto & [Rf_*\mathcal{IC}_W] \end{array}$$

is an isomorphism for $i > \dim X$ *where* $\mathcal{IC}_W$ *is the sheaf complex of intersection chains with closed support.*

Slogan: "bordism invariants of Witt spaces = Witt equivalence invariants of self-dual sheaves".

3. Cappell and Shaneson's formula

A map $f : X \to Y$ between stratified spaces is **stratified** if

1. $f^{-1}S$ is a (possibly empty) union of strata for each stratum S of Y;

2. $f^{-1}S \to S$ is a locally trivial fibre bundle (with fibre a stratified space).

Examples: Morse functions, algebraic and analytic maps of complex varieties.

General set up:

1. W is a $4k$-dim stratified Witt space;

2. X is a stratified space with only even dim singular strata S;

3. $f : W \to X$ is a stratified map.

Theorem 6. [Cappell–Shaneson]
If the strata of X are simply-connected then

$$\sigma(W) = \sum_{S \subset X} \sigma(\overline{S})\sigma(F_S) \qquad (1)$$

where $\overline{S}$ is the closure of the stratum S and the F_S are certain Witt spaces depending only on $f : W \to X$.

Theorem 7. *There is a Witt bordism (over X)*

$$W \sim \bigsqcup_{S \subset X} E_S \qquad (2)$$

where E_S is a Witt space over $\overline{S}$ with fibre F_S over points in S and point fibres over $\overline{S} - S$.

To obtain (1) from (2) apply signature and use

$$\pi_1 S = 1 \quad \Rightarrow \quad \sigma(E_S) = \sigma(\overline{S})\sigma(F_S)$$

(follows from Cappell–Shaneson's methods).

Key idea: Novikov additivity (after Siegel).

Pinch bordism of Witt spaces:

$$
\begin{aligned}
M \cup_\partial M' &\sim M/\partial M + M'/\partial M' \\
\Rightarrow \sigma(M \cup_\partial M') &= \sigma(M/\partial M) + \sigma(M'/\partial M') \\
&= \sigma(M, \partial M) + \sigma(M', \partial M')
\end{aligned}
$$

4. Witt groups of perverse sheaves

By varying the constructibility condition we obtain other triangulated categories with duality eg. if X is a complex algebraic variety

$$D_b^{\text{alg-c}}(X;\mathbb{Q}) \subset D_b^c(X;\mathbb{Q})$$

Riemann–Hilbert correspondence $\Rightarrow$

$$D_b^{\text{alg-c}}(X;\mathbb{Q}) \cong D_b(\text{Perv}(X))$$

Theorem of Balmer $\Rightarrow$

$$W_0\left(D_b(\text{Perv}(X))\right) \cong W\left(\text{Perv}(X)\right)$$

$W(\text{Perv}(X))$ is generated by inner products on local systems on the nonsingular parts of irreducible subvarieties of X. It is functorial under proper maps. If $f : Y \to X$ is a proper algebraic map then Y determines a class

$$f_*[\mathcal{IC}_Y] \in W(\text{Perv}(X)).$$

Theorem 8. *If* $V = \{f = 0\} \subset X$ *is a hypersurface then there is a split surjection*

$$W(\mathrm{Perv}(X)) \to W(\mathrm{Perv}(V))$$

induced by the perverse vanishing cycles functor ${}^p\varphi_f$*.*

Conjecture 9 (c.f. Youssin)**.** $W(\mathrm{Perv}(X))$ *decomposes as a direct sum indexed by the simple objects of* $\mathrm{Perv}(X)$*.*

Example: if Y is smooth and $f : Y \to \mathbb{C}$ has a single isolated singularity at 0 then the class

$$f_*[\mathcal{IC}_Y] \in W(\mathrm{Perv}(\mathbb{C}))$$

maps to

$$\begin{cases} 0 & z \neq 0 \\ [I_F] & z = 0 \end{cases}$$

in $W(\mathrm{Perv}(z)) \cong W(\mathbb{Q})$ where I_F is the intersection form on the middle homology of the Milnor fibre F of the singularity.