

Transversal homotopy theory

Joint with Conor Smyth, inspired by Baez and Dolan
Details in [arXiv:0910.3322](https://arxiv.org/abs/0910.3322)

March, 2010

Homotopy groups

A **homotopy** of continuous maps $f, g : X \rightarrow Y$ is a continuous map

$$h : X \times [0, 1] \rightarrow Y$$

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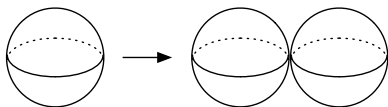
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For $n = 0$ it is a set, for $n = 1$ a group, and for $n \geq 2$ an abelian group where group operation arises from



Homotopy groups of spheres

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S^0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
S^1	\mathbb{Z}	0	0	0	0	0	0	0	0	0	0	0	0	0	0
S^2	0	\mathbb{Z}	\mathbb{Z}	\mathbb{Z}_2	\mathbb{Z}_2	\mathbb{Z}_{12}	\mathbb{Z}_2	\mathbb{Z}_2	\mathbb{Z}_3	\mathbb{Z}_{15}	\mathbb{Z}_2	\mathbb{Z}_2^2	$\mathbb{Z}_{12} \times \mathbb{Z}_2$	$\mathbb{Z}_{84} \times \mathbb{Z}_2^2$	\mathbb{Z}_2^2
S^3	0	0	\mathbb{Z}	\mathbb{Z}_2	\mathbb{Z}_2	\mathbb{Z}_{12}	\mathbb{Z}_2	\mathbb{Z}_2	\mathbb{Z}_3	\mathbb{Z}_{15}	\mathbb{Z}_2	\mathbb{Z}_2^2	$\mathbb{Z}_{12} \times \mathbb{Z}_2$	$\mathbb{Z}_{84} \times \mathbb{Z}_2^2$	\mathbb{Z}_2^2
S^4	0	0	0	\mathbb{Z}	\mathbb{Z}_2	\mathbb{Z}_2	$\mathbb{Z} \times \mathbb{Z}_{12}$	\mathbb{Z}_2^2	\mathbb{Z}_2^2	$\mathbb{Z}_{24} \times \mathbb{Z}_3$	\mathbb{Z}_{15}	\mathbb{Z}_2	\mathbb{Z}_2^3	$\mathbb{Z}_{120} \times \mathbb{Z}_{12} \times \mathbb{Z}_2$	$\mathbb{Z}_{84} \times \mathbb{Z}_2^5$
S^5	0	0	0	0	\mathbb{Z}	\mathbb{Z}_2	\mathbb{Z}_2	\mathbb{Z}_{24}	\mathbb{Z}_2	\mathbb{Z}_2	\mathbb{Z}_2	\mathbb{Z}_{30}	\mathbb{Z}_2	\mathbb{Z}_2^3	$\mathbb{Z}_{72} \times \mathbb{Z}_2$
S^6	0	0	0	0	0	\mathbb{Z}	\mathbb{Z}_2	\mathbb{Z}_2	\mathbb{Z}_{24}	0	\mathbb{Z}	\mathbb{Z}_2	\mathbb{Z}_{60}	$\mathbb{Z}_{24} \times \mathbb{Z}_2$	\mathbb{Z}_2^3
S^7	0	0	0	0	0	0	\mathbb{Z}	\mathbb{Z}_2	\mathbb{Z}_2	\mathbb{Z}_{24}	0	0	\mathbb{Z}_2	\mathbb{Z}_{120}	\mathbb{Z}_2^3
S^8	0	0	0	0	0	0	0	\mathbb{Z}	\mathbb{Z}_2	\mathbb{Z}_2	\mathbb{Z}_{24}	0	0	\mathbb{Z}_2	$\mathbb{Z} \times \mathbb{Z}_{120}$

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¹Table from en.wikipedia.org/wiki/Homotopy_groups_of_spheres.

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S^0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
S^1	\mathbb{Z}	0	0	0	0	0	0	0	0	0	0	0	0	0	0
S^2	0	\mathbb{Z}	\mathbb{Z}	\mathbb{Z}_2	\mathbb{Z}_2	\mathbb{Z}_{12}	\mathbb{Z}_2	\mathbb{Z}_2	\mathbb{Z}_3	\mathbb{Z}_{15}	\mathbb{Z}_2	\mathbb{Z}_2^2	$\mathbb{Z}_{12} \times \mathbb{Z}_2$	$\mathbb{Z}_{84} \times \mathbb{Z}_2^2$	\mathbb{Z}_2^2
S^3	0	0	\mathbb{Z}	\mathbb{Z}_2	\mathbb{Z}_2	\mathbb{Z}_{12}	\mathbb{Z}_2	\mathbb{Z}_2	\mathbb{Z}_3	\mathbb{Z}_{15}	\mathbb{Z}_2	\mathbb{Z}_2^2	$\mathbb{Z}_{12} \times \mathbb{Z}_2$	$\mathbb{Z}_{84} \times \mathbb{Z}_2^2$	\mathbb{Z}_2^2
S^4	0	0	0	\mathbb{Z}	\mathbb{Z}_2	\mathbb{Z}_2	$\mathbb{Z} \times \mathbb{Z}_{12}$	\mathbb{Z}_2^2	\mathbb{Z}_2^2	$\mathbb{Z}_{24} \times \mathbb{Z}_3$	\mathbb{Z}_{15}	\mathbb{Z}_2	\mathbb{Z}_2^3	$\mathbb{Z}_{120} \times \mathbb{Z}_{12} \times \mathbb{Z}_2$	$\mathbb{Z}_{84} \times \mathbb{Z}_2^5$
S^5	0	0	0	0	\mathbb{Z}	\mathbb{Z}_2	\mathbb{Z}_2	\mathbb{Z}_{24}	\mathbb{Z}_2	\mathbb{Z}_2	\mathbb{Z}_2	\mathbb{Z}_{30}	\mathbb{Z}_2	\mathbb{Z}_2^3	$\mathbb{Z}_{72} \times \mathbb{Z}_2$
S^6	0	0	0	0	0	\mathbb{Z}	\mathbb{Z}_2	\mathbb{Z}_2	\mathbb{Z}_{24}	0	\mathbb{Z}	\mathbb{Z}_2	\mathbb{Z}_{60}	$\mathbb{Z}_{24} \times \mathbb{Z}_2$	\mathbb{Z}_2^3
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► $\pi_n(S^k) = 0$ for $n < k$

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S^1	\mathbb{Z}	0	0	0	0	0	0	0	0	0	0	0	0	0	0
S^2	0	\mathbb{Z}	\mathbb{Z}	\mathbb{Z}_2	\mathbb{Z}_2	\mathbb{Z}_{12}	\mathbb{Z}_2	\mathbb{Z}_2	\mathbb{Z}_3	\mathbb{Z}_{15}	\mathbb{Z}_2	\mathbb{Z}_2^2	$\mathbb{Z}_{12} \times \mathbb{Z}_2$	$\mathbb{Z}_{84} \times \mathbb{Z}_2^2$	\mathbb{Z}_2^2
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- ▶ finite unless $k = n$ or $k = 2m, n = 4m - 1$ (Serre, 1951)

Geometrical interpretation



Lev Pontrjagin gave a geometric interpretation of the homotopy groups of spheres in terms of bordism theory of smooth manifolds (1938). Perhaps curiously for a topologist he was blind.

The Pontrjagin construction — preliminaries

Theorem (Smooth approximation)

A continuous map $f : M \rightarrow N$ of smooth manifolds, smooth on closed $A \subset M$, is homotopic rel A to a smooth map.

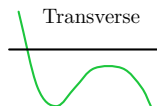
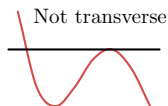
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Recall f is **transverse** to B if $df(T_x M) + T_{f(x)} B = T_{f(x)} N$ for all $x \in f^{-1} B$. This implies $f^{-1} B$ is a submanifold and df induces

$$Nf^{-1} B \cong f^* NB.$$



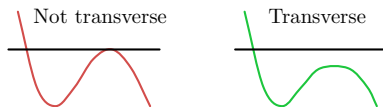
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Theorem (Transversal approximation)

A smooth map $f : M \rightarrow N$ is homotopic to a map transverse to a compact submanifold $B \subset N$ by a homotopy local to $f^{-1}B$.

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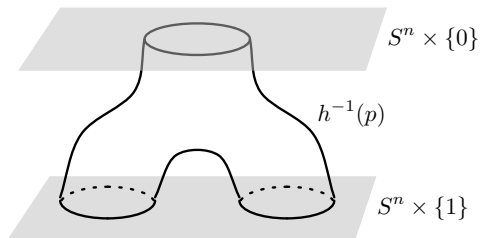
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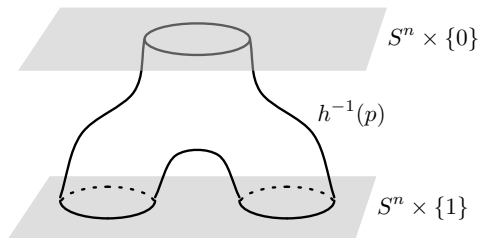
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Furthermore, $Nf^{-1}p \cong f^*Np \cong f^{-1}p \times \mathbb{R}^k$ is trivial, with given trivialisation, i.e. $f^{-1}p$ is **framed**, and similarly for h .

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$$U \cong NA \cong A \times \mathbb{R}^k \xrightarrow{\pi_2} \mathbb{R}^k \cong S^k - *$$

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Theorem (Pontrjagin, 1938)

$$\pi_n(S^k) \cong \Omega_{n-k}^{fr}(S^n)$$

Homotopy groups of spheres — reprise

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S^1	\mathbb{Z}	0	0	0	0	0	0	0	0	0	0	0	0	0	0
S^2	0	\mathbb{Z}	\mathbb{Z}	\mathbb{Z}_2	\mathbb{Z}_2	\mathbb{Z}_{12}	\mathbb{Z}_2	\mathbb{Z}_2	\mathbb{Z}_3	\mathbb{Z}_{15}	\mathbb{Z}_2	\mathbb{Z}_2^2	$\mathbb{Z}_{12} \times \mathbb{Z}_2$	$\mathbb{Z}_{84} \times \mathbb{Z}_2^2$	\mathbb{Z}_2^2
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S^4	0	0	0	\mathbb{Z}	\mathbb{Z}_2	\mathbb{Z}_2	$\mathbb{Z} \times \mathbb{Z}_{12}$	\mathbb{Z}_2^2	\mathbb{Z}_2^2	$\mathbb{Z}_{24} \times \mathbb{Z}_3$	\mathbb{Z}_{15}	\mathbb{Z}_2	\mathbb{Z}_2^3	$\mathbb{Z}_{120} \times \mathbb{Z}_{12} \times \mathbb{Z}_2$	$\mathbb{Z}_{84} \times \mathbb{Z}_2^5$
S^5	0	0	0	0	\mathbb{Z}	\mathbb{Z}_2	\mathbb{Z}_2	\mathbb{Z}_{24}	\mathbb{Z}_2	\mathbb{Z}_2	\mathbb{Z}_2	\mathbb{Z}_{30}	\mathbb{Z}_2	\mathbb{Z}_2^3	$\mathbb{Z}_{72} \times \mathbb{Z}_2$
S^6	0	0	0	0	0	\mathbb{Z}	\mathbb{Z}_2	\mathbb{Z}_2	\mathbb{Z}_{24}	0	\mathbb{Z}	\mathbb{Z}_2	\mathbb{Z}_{60}	$\mathbb{Z}_{24} \times \mathbb{Z}_2$	\mathbb{Z}_2^3
S^7	0	0	0	0	0	0	\mathbb{Z}	\mathbb{Z}_2	\mathbb{Z}_2	\mathbb{Z}_{24}	0	0	\mathbb{Z}_2	\mathbb{Z}_{120}	\mathbb{Z}_2^3
S^8	0	0	0	0	0	0	0	\mathbb{Z}	\mathbb{Z}_2	\mathbb{Z}_2	\mathbb{Z}_{24}	0	0	\mathbb{Z}_2	$\mathbb{Z} \times \mathbb{Z}_{120}$

Homotopy groups of spheres — reprise

	π_1	π_2	π_3	π_4	π_5	π_6	π_7	π_8	π_9	π_{10}	π_{11}	π_{12}	π_{13}	π_{14}	π_{15}
S^0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
S^1	\mathbb{Z}	0	0	0	0	0	0	0	0	0	0	0	0	0	0
S^2	0	\mathbb{Z}	\mathbb{Z}	\mathbb{Z}_2	\mathbb{Z}_2	\mathbb{Z}_{12}	\mathbb{Z}_2	\mathbb{Z}_2	\mathbb{Z}_3	\mathbb{Z}_{15}	\mathbb{Z}_2	\mathbb{Z}_2^2	$\mathbb{Z}_{12} \times \mathbb{Z}_2$	$\mathbb{Z}_{84} \times \mathbb{Z}_2^2$	\mathbb{Z}_2^2
S^3	0	0	\mathbb{Z}	\mathbb{Z}_2	\mathbb{Z}_2	\mathbb{Z}_{12}	\mathbb{Z}_2	\mathbb{Z}_2	\mathbb{Z}_3	\mathbb{Z}_{15}	\mathbb{Z}_2	\mathbb{Z}_2^2	$\mathbb{Z}_{12} \times \mathbb{Z}_2$	$\mathbb{Z}_{84} \times \mathbb{Z}_2^2$	\mathbb{Z}_2^2
S^4	0	0	0	\mathbb{Z}	\mathbb{Z}_2	\mathbb{Z}_2	$\mathbb{Z} \times \mathbb{Z}_{12}$	\mathbb{Z}_2^2	\mathbb{Z}_2^2	$\mathbb{Z}_{24} \times \mathbb{Z}_3$	\mathbb{Z}_{15}	\mathbb{Z}_2	\mathbb{Z}_2^3	$\mathbb{Z}_{120} \times \mathbb{Z}_{12} \times \mathbb{Z}_2$	$\mathbb{Z}_{84} \times \mathbb{Z}_2^5$
S^5	0	0	0	0	\mathbb{Z}	\mathbb{Z}_2	\mathbb{Z}_2	\mathbb{Z}_{24}	\mathbb{Z}_2	\mathbb{Z}_2	\mathbb{Z}_2	\mathbb{Z}_{30}	\mathbb{Z}_2	\mathbb{Z}_2^3	$\mathbb{Z}_{72} \times \mathbb{Z}_2$
S^6	0	0	0	0	0	\mathbb{Z}	\mathbb{Z}_2	\mathbb{Z}_2	\mathbb{Z}_{24}	0	\mathbb{Z}	\mathbb{Z}_2	\mathbb{Z}_{60}	$\mathbb{Z}_{24} \times \mathbb{Z}_2$	\mathbb{Z}_2^3
S^7	0	0	0	0	0	0	\mathbb{Z}	\mathbb{Z}_2	\mathbb{Z}_2	\mathbb{Z}_{24}	0	0	\mathbb{Z}_2	\mathbb{Z}_{120}	\mathbb{Z}_2^3
S^8	0	0	0	0	0	0	\mathbb{Z}	\mathbb{Z}_2	\mathbb{Z}_2	\mathbb{Z}_2	\mathbb{Z}_{24}	0	0	\mathbb{Z}_2	$\mathbb{Z} \times \mathbb{Z}_{120}$

► $\pi_n(S^k) \cong \Omega_{n-k}^{\text{fr}}(S^n) = 0$ for $n < k$

Homotopy groups of spheres — reprise

	π_1	π_2	π_3	π_4	π_5	π_6	π_7	π_8	π_9	π_{10}	π_{11}	π_{12}	π_{13}	π_{14}	π_{15}
S^0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
S^1	\mathbb{Z}	0	0	0	0	0	0	0	0	0	0	0	0	0	0
S^2	0	\mathbb{Z}	\mathbb{Z}	\mathbb{Z}_2	\mathbb{Z}_2	\mathbb{Z}_{12}	\mathbb{Z}_2	\mathbb{Z}_2	\mathbb{Z}_3	\mathbb{Z}_{15}	\mathbb{Z}_2	\mathbb{Z}_2^2	$\mathbb{Z}_{12} \times \mathbb{Z}_2$	$\mathbb{Z}_{84} \times \mathbb{Z}_2^2$	\mathbb{Z}_2^2
S^3	0	0	\mathbb{Z}	\mathbb{Z}_2	\mathbb{Z}_2	\mathbb{Z}_{12}	\mathbb{Z}_2	\mathbb{Z}_2	\mathbb{Z}_3	\mathbb{Z}_{15}	\mathbb{Z}_2	\mathbb{Z}_2^2	$\mathbb{Z}_{12} \times \mathbb{Z}_2$	$\mathbb{Z}_{84} \times \mathbb{Z}_2^2$	\mathbb{Z}_2^2
S^4	0	0	0	\mathbb{Z}	\mathbb{Z}_2	\mathbb{Z}_2	$\mathbb{Z} \times \mathbb{Z}_{12}$	\mathbb{Z}_2^2	\mathbb{Z}_2^2	$\mathbb{Z}_{24} \times \mathbb{Z}_3$	\mathbb{Z}_{15}	\mathbb{Z}_2	\mathbb{Z}_2^3	$\mathbb{Z}_{120} \times \mathbb{Z}_{12} \times \mathbb{Z}_2$	$\mathbb{Z}_{84} \times \mathbb{Z}_2^5$
S^5	0	0	0	0	\mathbb{Z}	\mathbb{Z}_2	\mathbb{Z}_2	\mathbb{Z}_{24}	\mathbb{Z}_2	\mathbb{Z}_2	\mathbb{Z}_2	\mathbb{Z}_{30}	\mathbb{Z}_2	\mathbb{Z}_2^3	$\mathbb{Z}_{72} \times \mathbb{Z}_2$
S^6	0	0	0	0	0	\mathbb{Z}	\mathbb{Z}_2	\mathbb{Z}_2	\mathbb{Z}_{24}	0	\mathbb{Z}	\mathbb{Z}_2	\mathbb{Z}_{60}	$\mathbb{Z}_{24} \times \mathbb{Z}_2$	\mathbb{Z}_2^3
S^7	0	0	0	0	0	0	\mathbb{Z}	\mathbb{Z}_2	\mathbb{Z}_2	\mathbb{Z}_{24}	0	0	\mathbb{Z}_2	\mathbb{Z}_{120}	\mathbb{Z}_2^3
S^8	0	0	0	0	0	0	0	\mathbb{Z}	\mathbb{Z}_2	\mathbb{Z}_2	\mathbb{Z}_{24}	0	0	\mathbb{Z}_2	$\mathbb{Z} \times \mathbb{Z}_{120}$

- ▶ $\pi_n(S^k) \cong \Omega_{n-k}^{\text{fr}}(S^n) = 0$ for $n < k$
- ▶ $\pi_n(S^n) \cong \Omega_0^{\text{fr}}(S^n) \cong \mathbb{Z}$

Homotopy groups of spheres — reprise

	π_1	π_2	π_3	π_4	π_5	π_6	π_7	π_8	π_9	π_{10}	π_{11}	π_{12}	π_{13}	π_{14}	π_{15}
S^0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
S^1	\mathbb{Z}	0	0	0	0	0	0	0	0	0	0	0	0	0	0
S^2	0	\mathbb{Z}	\mathbb{Z}	\mathbb{Z}_2	\mathbb{Z}_2	\mathbb{Z}_{12}	\mathbb{Z}_2	\mathbb{Z}_2	\mathbb{Z}_3	\mathbb{Z}_{15}	\mathbb{Z}_2	\mathbb{Z}_2^2	$\mathbb{Z}_{12} \times \mathbb{Z}_2$	$\mathbb{Z}_{84} \times \mathbb{Z}_2^2$	\mathbb{Z}_2^2
S^3	0	0	\mathbb{Z}	\mathbb{Z}_2	\mathbb{Z}_2	\mathbb{Z}_{12}	\mathbb{Z}_2	\mathbb{Z}_2	\mathbb{Z}_3	\mathbb{Z}_{15}	\mathbb{Z}_2	\mathbb{Z}_2^2	$\mathbb{Z}_{12} \times \mathbb{Z}_2$	$\mathbb{Z}_{84} \times \mathbb{Z}_2^2$	\mathbb{Z}_2^2
S^4	0	0	0	\mathbb{Z}	\mathbb{Z}_2	\mathbb{Z}_2	$\mathbb{Z} \times \mathbb{Z}_{12}$	\mathbb{Z}_2^2	\mathbb{Z}_2^2	$\mathbb{Z}_{24} \times \mathbb{Z}_3$	\mathbb{Z}_{15}	\mathbb{Z}_2	\mathbb{Z}_2^3	$\mathbb{Z}_{120} \times \mathbb{Z}_{12} \times \mathbb{Z}_2$	$\mathbb{Z}_{84} \times \mathbb{Z}_2^5$
S^5	0	0	0	0	\mathbb{Z}	\mathbb{Z}_2	\mathbb{Z}_2	\mathbb{Z}_{24}	\mathbb{Z}_2	\mathbb{Z}_2	\mathbb{Z}_2	\mathbb{Z}_{30}	\mathbb{Z}_2	\mathbb{Z}_2^3	$\mathbb{Z}_{72} \times \mathbb{Z}_2$
S^6	0	0	0	0	0	\mathbb{Z}	\mathbb{Z}_2	\mathbb{Z}_2	\mathbb{Z}_{24}	0	\mathbb{Z}	\mathbb{Z}_2	\mathbb{Z}_{60}	$\mathbb{Z}_{24} \times \mathbb{Z}_2$	\mathbb{Z}_2^3
S^7	0	0	0	0	0	0	\mathbb{Z}	\mathbb{Z}_2	\mathbb{Z}_2	\mathbb{Z}_{24}	0	0	\mathbb{Z}_2	\mathbb{Z}_{120}	\mathbb{Z}_2^3
S^8	0	0	0	0	0	0	0	\mathbb{Z}	\mathbb{Z}_2	\mathbb{Z}_2	\mathbb{Z}_{24}	0	0	\mathbb{Z}_2	$\mathbb{Z} \times \mathbb{Z}_{120}$

- ▶ $\pi_n(S^k) \cong \Omega_{n-k}^{\text{fr}}(S^n) = 0$ for $n < k$
- ▶ $\pi_n(S^n) \cong \Omega_0^{\text{fr}}(S^n) \cong \mathbb{Z}$
- ▶ $\pi_{n+1}(S^n) \cong \Omega_1^{\text{fr}}(S^{n+1})$

Homotopy groups of spheres — reprise

	π_1	π_2	π_3	π_4	π_5	π_6	π_7	π_8	π_9	π_{10}	π_{11}	π_{12}	π_{13}	π_{14}	π_{15}
S^0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
S^1	\mathbb{Z}	0	0	0	0	0	0	0	0	0	0	0	0	0	0
S^2	0	\mathbb{Z}	\mathbb{Z}	\mathbb{Z}_2	\mathbb{Z}_2	\mathbb{Z}_{12}	\mathbb{Z}_2	\mathbb{Z}_2	\mathbb{Z}_3	\mathbb{Z}_{15}	\mathbb{Z}_2	\mathbb{Z}_2^2	$\mathbb{Z}_{12} \times \mathbb{Z}_2$	$\mathbb{Z}_{84} \times \mathbb{Z}_2^2$	\mathbb{Z}_2^2
S^3	0	0	\mathbb{Z}	\mathbb{Z}_2	\mathbb{Z}_2	\mathbb{Z}_{12}	\mathbb{Z}_2	\mathbb{Z}_2	\mathbb{Z}_3	\mathbb{Z}_{15}	\mathbb{Z}_2	\mathbb{Z}_2^2	$\mathbb{Z}_{12} \times \mathbb{Z}_2$	$\mathbb{Z}_{84} \times \mathbb{Z}_2^2$	\mathbb{Z}_2^2
S^4	0	0	0	\mathbb{Z}	\mathbb{Z}_2	\mathbb{Z}_2	$\mathbb{Z} \times \mathbb{Z}_{12}$	\mathbb{Z}_2^2	\mathbb{Z}_2^2	$\mathbb{Z}_{24} \times \mathbb{Z}_3$	\mathbb{Z}_{15}	\mathbb{Z}_2	\mathbb{Z}_2^3	$\mathbb{Z}_{120} \times \mathbb{Z}_{12} \times \mathbb{Z}_2$	$\mathbb{Z}_{84} \times \mathbb{Z}_2^5$
S^5	0	0	0	0	\mathbb{Z}	\mathbb{Z}_2	\mathbb{Z}_2	\mathbb{Z}_{24}	\mathbb{Z}_2	\mathbb{Z}_2	\mathbb{Z}_2	\mathbb{Z}_{30}	\mathbb{Z}_2	\mathbb{Z}_2^3	$\mathbb{Z}_{72} \times \mathbb{Z}_2$
S^6	0	0	0	0	0	\mathbb{Z}	\mathbb{Z}_2	\mathbb{Z}_2	\mathbb{Z}_{24}	0	\mathbb{Z}	\mathbb{Z}_2	\mathbb{Z}_{60}	$\mathbb{Z}_{24} \times \mathbb{Z}_2$	\mathbb{Z}_2^3
S^7	0	0	0	0	0	0	\mathbb{Z}	\mathbb{Z}_2	\mathbb{Z}_2	\mathbb{Z}_{24}	0	0	\mathbb{Z}_2	\mathbb{Z}_{120}	\mathbb{Z}_2^3
S^8	0	0	0	0	0	0	0	\mathbb{Z}	\mathbb{Z}_2	\mathbb{Z}_2	\mathbb{Z}_{24}	0	0	\mathbb{Z}_2	$\mathbb{Z} \times \mathbb{Z}_{120}$

- ▶ $\pi_n(S^k) \cong \Omega_{n-k}^{\text{fr}}(S^n) = 0$ for $n < k$
- ▶ $\pi_n(S^n) \cong \Omega_0^{\text{fr}}(S^n) \cong \mathbb{Z}$
- ▶ $\pi_{n+1}(S^n) \cong \Omega_1^{\text{fr}}(S^{n+1}) \cong \pi_1(SO(n))$

Homotopy groups of spheres — reprise

	π_1	π_2	π_3	π_4	π_5	π_6	π_7	π_8	π_9	π_{10}	π_{11}	π_{12}	π_{13}	π_{14}	π_{15}
S^0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
S^1	\mathbb{Z}	0	0	0	0	0	0	0	0	0	0	0	0	0	0
S^2	0	\mathbb{Z}	\mathbb{Z}	\mathbb{Z}_2	\mathbb{Z}_2	\mathbb{Z}_{12}	\mathbb{Z}_2	\mathbb{Z}_2	\mathbb{Z}_3	\mathbb{Z}_{15}	\mathbb{Z}_2	\mathbb{Z}_2^2	$\mathbb{Z}_{12} \times \mathbb{Z}_2$	$\mathbb{Z}_{84} \times \mathbb{Z}_2^2$	\mathbb{Z}_2^2
S^3	0	0	\mathbb{Z}	\mathbb{Z}_2	\mathbb{Z}_2	\mathbb{Z}_{12}	\mathbb{Z}_2	\mathbb{Z}_2	\mathbb{Z}_3	\mathbb{Z}_{15}	\mathbb{Z}_2	\mathbb{Z}_2^2	$\mathbb{Z}_{12} \times \mathbb{Z}_2$	$\mathbb{Z}_{84} \times \mathbb{Z}_2^2$	\mathbb{Z}_2^2
S^4	0	0	0	\mathbb{Z}	\mathbb{Z}_2	\mathbb{Z}_2	$\mathbb{Z} \times \mathbb{Z}_{12}$	\mathbb{Z}_2^2	\mathbb{Z}_2^2	$\mathbb{Z}_{24} \times \mathbb{Z}_3$	\mathbb{Z}_{15}	\mathbb{Z}_2	\mathbb{Z}_2^3	$\mathbb{Z}_{120} \times \mathbb{Z}_{12} \times \mathbb{Z}_2$	$\mathbb{Z}_{84} \times \mathbb{Z}_2^5$
S^5	0	0	0	0	\mathbb{Z}	\mathbb{Z}_2	\mathbb{Z}_2	\mathbb{Z}_{24}	\mathbb{Z}_2	\mathbb{Z}_2	\mathbb{Z}_2	\mathbb{Z}_{30}	\mathbb{Z}_2	\mathbb{Z}_2^3	$\mathbb{Z}_{72} \times \mathbb{Z}_2$
S^6	0	0	0	0	0	\mathbb{Z}	\mathbb{Z}_2	\mathbb{Z}_2	\mathbb{Z}_{24}	0	\mathbb{Z}	\mathbb{Z}_2	\mathbb{Z}_{60}	$\mathbb{Z}_{24} \times \mathbb{Z}_2$	\mathbb{Z}_2^3
S^7	0	0	0	0	0	0	\mathbb{Z}	\mathbb{Z}_2	\mathbb{Z}_2	\mathbb{Z}_{24}	0	0	\mathbb{Z}_2	\mathbb{Z}_{120}	\mathbb{Z}_2^3
S^8	0	0	0	0	0	0	0	\mathbb{Z}	\mathbb{Z}_2	\mathbb{Z}_2	\mathbb{Z}_{24}	0	0	\mathbb{Z}_2	$\mathbb{Z} \times \mathbb{Z}_{120}$

▶ $\pi_n(S^k) \cong \Omega_{n-k}^{\text{fr}}(S^n) = 0$ for $n < k$

▶ $\pi_n(S^n) \cong \Omega_0^{\text{fr}}(S^n) \cong \mathbb{Z}$

▶ $\pi_{n+1}(S^n) \cong \Omega_1^{\text{fr}}(S^{n+1}) \cong \pi_1(SO(n)) \cong \begin{cases} 0 & n < 2 \\ \mathbb{Z} & n = 2 \\ \mathbb{Z}/2\mathbb{Z} & n > 2 \end{cases}$

Homotopy groups of spheres — reprise

	π_1	π_2	π_3	π_4	π_5	π_6	π_7	π_8	π_9	π_{10}	π_{11}	π_{12}	π_{13}	π_{14}	π_{15}
S^0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
S^1	\mathbb{Z}	0	0	0	0	0	0	0	0	0	0	0	0	0	0
S^2	0	\mathbb{Z}	\mathbb{Z}	\mathbb{Z}_2	\mathbb{Z}_2	\mathbb{Z}_{12}	\mathbb{Z}_2	\mathbb{Z}_2	\mathbb{Z}_3	\mathbb{Z}_{15}	\mathbb{Z}_2	\mathbb{Z}_2^2	$\mathbb{Z}_{12} \times \mathbb{Z}_2$	$\mathbb{Z}_{84} \times \mathbb{Z}_2^2$	\mathbb{Z}_2^2
S^3	0	0	\mathbb{Z}	\mathbb{Z}_2	\mathbb{Z}_2	\mathbb{Z}_{12}	\mathbb{Z}_2	\mathbb{Z}_2	\mathbb{Z}_3	\mathbb{Z}_{15}	\mathbb{Z}_2	\mathbb{Z}_2^2	$\mathbb{Z}_{12} \times \mathbb{Z}_2$	$\mathbb{Z}_{84} \times \mathbb{Z}_2^2$	\mathbb{Z}_2^2
S^4	0	0	0	\mathbb{Z}	\mathbb{Z}_2	\mathbb{Z}_2	$\mathbb{Z} \times \mathbb{Z}_{12}$	\mathbb{Z}_2^2	\mathbb{Z}_2^2	$\mathbb{Z}_{24} \times \mathbb{Z}_3$	\mathbb{Z}_{15}	\mathbb{Z}_2	\mathbb{Z}_2^3	$\mathbb{Z}_{120} \times \mathbb{Z}_{12} \times \mathbb{Z}_2$	$\mathbb{Z}_{84} \times \mathbb{Z}_2^5$
S^5	0	0	0	0	\mathbb{Z}	\mathbb{Z}_2	\mathbb{Z}_2	\mathbb{Z}_{24}	\mathbb{Z}_2	\mathbb{Z}_2	\mathbb{Z}_2	\mathbb{Z}_{30}	\mathbb{Z}_2	\mathbb{Z}_2^3	$\mathbb{Z}_{72} \times \mathbb{Z}_2$
S^6	0	0	0	0	0	\mathbb{Z}	\mathbb{Z}_2	\mathbb{Z}_2	\mathbb{Z}_{24}	0	\mathbb{Z}	\mathbb{Z}_2	\mathbb{Z}_{60}	$\mathbb{Z}_{24} \times \mathbb{Z}_2$	\mathbb{Z}_2^3
S^7	0	0	0	0	0	0	\mathbb{Z}	\mathbb{Z}_2	\mathbb{Z}_2	\mathbb{Z}_{24}	0	0	\mathbb{Z}_2	\mathbb{Z}_{120}	\mathbb{Z}_2^3
S^8	0	0	0	0	0	0	0	\mathbb{Z}	\mathbb{Z}_2	\mathbb{Z}_2	\mathbb{Z}_{24}	0	0	\mathbb{Z}_2	$\mathbb{Z} \times \mathbb{Z}_{120}$

▶ $\pi_n(S^k) \cong \Omega_{n-k}^{\text{fr}}(S^n) = 0$ for $n < k$

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▶ $\pi_{4n-1}(S^{2n}) \cong \Omega_{2n-1}^{\text{fr}}(S^{4n-1})$

Homotopy groups of spheres — reprise

	π_1	π_2	π_3	π_4	π_5	π_6	π_7	π_8	π_9	π_{10}	π_{11}	π_{12}	π_{13}	π_{14}	π_{15}
S^0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
S^1	\mathbb{Z}	0	0	0	0	0	0	0	0	0	0	0	0	0	0
S^2	0	\mathbb{Z}	\mathbb{Z}	\mathbb{Z}_2	\mathbb{Z}_2	\mathbb{Z}_{12}	\mathbb{Z}_2	\mathbb{Z}_2	\mathbb{Z}_3	\mathbb{Z}_{15}	\mathbb{Z}_2	\mathbb{Z}_2^2	$\mathbb{Z}_{12} \times \mathbb{Z}_2$	$\mathbb{Z}_{84} \times \mathbb{Z}_2^2$	\mathbb{Z}_2^2
S^3	0	0	\mathbb{Z}	\mathbb{Z}_2	\mathbb{Z}_2	\mathbb{Z}_{12}	\mathbb{Z}_2	\mathbb{Z}_2	\mathbb{Z}_3	\mathbb{Z}_{15}	\mathbb{Z}_2	\mathbb{Z}_2^2	$\mathbb{Z}_{12} \times \mathbb{Z}_2$	$\mathbb{Z}_{84} \times \mathbb{Z}_2^2$	\mathbb{Z}_2^2
S^4	0	0	0	\mathbb{Z}	\mathbb{Z}_2	\mathbb{Z}_2	$\mathbb{Z} \times \mathbb{Z}_{12}$	\mathbb{Z}_2^2	\mathbb{Z}_2^2	$\mathbb{Z}_{24} \times \mathbb{Z}_3$	\mathbb{Z}_{15}	\mathbb{Z}_2	\mathbb{Z}_2^3	$\mathbb{Z}_{120} \times \mathbb{Z}_{12} \times \mathbb{Z}_2$	$\mathbb{Z}_{84} \times \mathbb{Z}_2^5$
S^5	0	0	0	0	\mathbb{Z}	\mathbb{Z}_2	\mathbb{Z}_2	\mathbb{Z}_{24}	\mathbb{Z}_2	\mathbb{Z}_2	\mathbb{Z}_2	\mathbb{Z}_{30}	\mathbb{Z}_2	\mathbb{Z}_2^3	$\mathbb{Z}_{72} \times \mathbb{Z}_2$
S^6	0	0	0	0	0	\mathbb{Z}	\mathbb{Z}_2	\mathbb{Z}_2	\mathbb{Z}_{24}	0	\mathbb{Z}	\mathbb{Z}_2	\mathbb{Z}_{60}	$\mathbb{Z}_{24} \times \mathbb{Z}_2$	\mathbb{Z}_2^3
S^7	0	0	0	0	0	0	\mathbb{Z}	\mathbb{Z}_2	\mathbb{Z}_2	\mathbb{Z}_{24}	0	0	\mathbb{Z}_2	\mathbb{Z}_{120}	\mathbb{Z}_2^3
S^8	0	0	0	0	0	0	0	\mathbb{Z}	\mathbb{Z}_2	\mathbb{Z}_2	\mathbb{Z}_{24}	0	0	\mathbb{Z}_2	$\mathbb{Z} \times \mathbb{Z}_{120}$

▶ $\pi_n(S^k) \cong \Omega_{n-k}^{\text{fr}}(S^n) = 0$ for $n < k$

▶ $\pi_n(S^n) \cong \Omega_0^{\text{fr}}(S^n) \cong \mathbb{Z}$

▶ $\pi_{n+1}(S^n) \cong \Omega_1^{\text{fr}}(S^{n+1}) \cong \pi_1(SO(n)) \cong \begin{cases} 0 & n < 2 \\ \mathbb{Z} & n = 2 \\ \mathbb{Z}/2\mathbb{Z} & n > 2 \end{cases}$

▶ $\pi_{4n-1}(S^{2n}) \cong \Omega_{2n-1}^{\text{fr}}(S^{4n-1}) \cong \mathbb{Z} \oplus ?$

Stabilisation

There is a commutative diagram

$$\begin{array}{ccc} \pi_n(S^k) & \xlongequal{\quad} & \Omega_{n-k}^{\text{fr}}(S^n) \\ \text{Suspension} \downarrow & & \downarrow \text{Equatorial embedding} \\ \pi_{n+1}(S^{k+1}) & \xlongequal{\quad} & \Omega_{n-k}^{\text{fr}}(S^{n+1}) \end{array}$$

Stabilisation

There is a commutative diagram

$$\begin{array}{ccc} \pi_n(S^k) & \xlongequal{\quad} & \Omega_{n-k}^{\text{fr}}(S^n) \\ \text{Suspension} \downarrow & & \downarrow \text{Equatorial embedding} \\ \pi_{n+1}(S^{k+1}) & \xlongequal{\quad} & \Omega_{n-k}^{\text{fr}}(S^{n+1}) \end{array}$$

- ▶ Freudenthal suspension thm \Rightarrow LHS stabilises for $2k \geq n + 2$.

Stabilisation

There is a commutative diagram

$$\begin{array}{ccc} \pi_n(S^k) & \xlongequal{\quad} & \Omega_{n-k}^{\text{fr}}(S^n) \\ \text{Suspension} \downarrow & & \downarrow \text{Equatorial embedding} \\ \pi_{n+1}(S^{k+1}) & \xlongequal{\quad} & \Omega_{n-k}^{\text{fr}}(S^{n+1}) \end{array}$$

- ▶ Freudenthal suspension thm \Rightarrow LHS stabilises for $2k \geq n + 2$.
- ▶ Whitney embedding thm \Rightarrow RHS stabilises for $2k \geq n + 2$.

Stabilisation

There is a commutative diagram

$$\begin{array}{ccc} \pi_n(S^k) & \xlongequal{\quad} & \Omega_{n-k}^{\text{fr}}(S^n) \\ \text{Suspension} \downarrow & & \downarrow \text{Equatorial embedding} \\ \pi_{n+1}(S^{k+1}) & \xlongequal{\quad} & \Omega_{n-k}^{\text{fr}}(S^{n+1}) \end{array}$$

- ▶ Freudenthal suspension thm \Rightarrow LHS stabilises for $2k \geq n + 2$.
- ▶ Whitney embedding thm \Rightarrow RHS stabilises for $2k \geq n + 2$.

Hence whenever $2k \geq n + 2$ we have

$$\pi_{n-k}^S := \pi_n(S^k) \cong \Omega_{n-k}^{\text{fr}}(S^n) =: \Omega_{n-k}^{\text{fr}}.$$

Transversal homotopy theory — first example

What happens if we insist that each slice of a homotopy h is transverse to p ?

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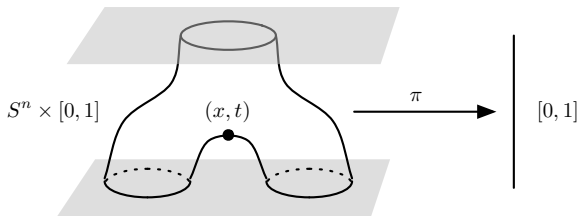
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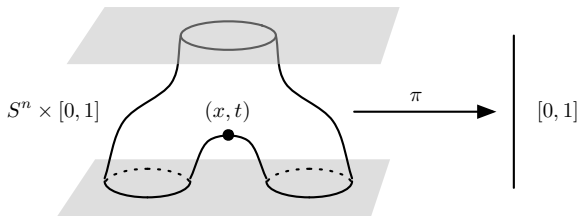


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Let $f = h(-, t)$. Then

$$\dim df(T_x S^n) < \dim S^n - \dim h^{-1}p = \operatorname{codim} p$$

so f is not transverse to p .

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Example: $\psi_3(S^2)$ is set of framed links in S^3 up to ambient isotopy.

Whitney stratified manifolds

A **Whitney stratified manifold** M is a manifold with a locally-finite partition into disjoint locally-closed submanifolds $\{S_i\}$ (the strata) satisfying Whitney's condition B .

Examples: (M, N) , $\mathbb{R}P^n$, $\mathbb{C}P^n$, Grassmannians, flag varieties . . .

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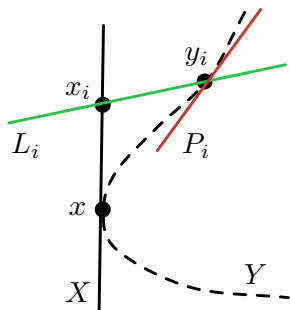
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Whitney stratified manifolds and stratified transversal maps form a category. Basepoints given by stratified transversal map $* \rightarrow M$.

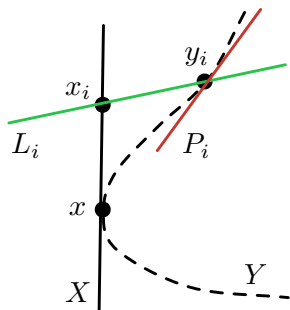
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Whitney's condition B: If secant lines $L_i = \overline{x_i y_i} \rightarrow L$ and tangent planes $P_i = T_{y_i} Y \rightarrow P$ then $L \subset P$.

Transversal homotopy theory

For Whitney stratified manifold M define

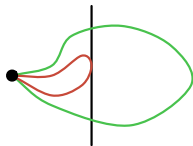
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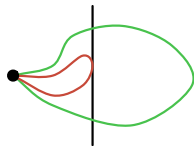


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For instance, $\psi_0(M) \cong \{\text{components of open strata}\}$,

$$\psi_1(\mathbb{S}^1) \cong \text{free monoid on } a \text{ and } a^\dagger$$

and $\psi_2(\mathbb{S}^2) \cong \text{free abelian monoid on } a \text{ and } a^\dagger \cong \mathbb{N}^2$.

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Each ψ_n is functorial under stratified transversal maps. There is a natural transformation $\psi_n \rightarrow \pi_n$.

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(Replacing spheres by other Thom spectra we can get plain-vanilla links, oriented links etc and higher-dimensional variants.)

Hoped-for higher versions

Expect there to be a ' k -tuply monoidal n -category' $\psi_{k,n+k}(M)$ with

0-morphisms transversal $S^k \rightarrow M$

1-morphisms transversal $S^k \times [0, 1] \rightarrow M$

...

n -morphisms transversal $S^k \times [0, 1]^n \rightarrow M$ up to htpy through such.

The case $n = 1$ can be made precise.

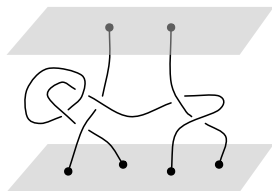
Theorem (W '09)

For any Whitney stratified manifold $\psi_{k,k+1}(M)$ is a

- ▶ rigid monoidal dagger category for $k > 0$
- ▶ rigid braided monoidal dagger category for $k > 1$
- ▶ rigid symmetric monoidal dagger category for $k > 2$.

The Tangle hypothesis

Example: $\psi_{2,3}(\mathbb{S}^2)$ is the category of framed tangles.



More generally, expect

$$\psi_{k,n+k}(\mathbb{S}^k) \simeq \text{frTang}_{k,n+k}^k.$$

Baez and Dolan's Tangle Hypothesis says the RHS is the 'free k -tuply monoidal n -category with duals' on one generator. From the LHS this can be viewed as a (massive) generalisation of

$$\pi_k(\mathbb{S}^k) \cong \mathbb{Z} \cong \text{free group on one generator.}$$