Transversal Homotopy Theory and the Tangle Hypothesis

Work in progress, joint with Conor Smyth.

November, 2010

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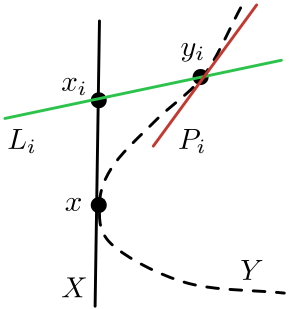
Basepoint given by stratified transversal map $* \to M$.

Whitney's condition B

Suppose X and Y are strata and $x \in X \cap \overline{Y}$ with sequences $x_i \to x$ and $y_i \to x$ in X and Y respectively.



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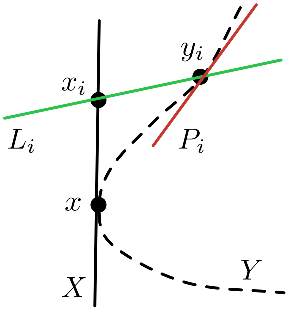

Whitney's condition B: If secant lines $L_i = \overline{x_i y_i} \to L$ and tangent planes $P_i = T_{y_i} Y \to P$ then $L \subset P$.

Transversal homotopy monoids

Definition

For Whitney stratified manifold M let

$$\psi_k(M) = \{f : I^k \to M \mid f \text{ transversal}, f(\partial I^k) = *\}/\sim$$

where $I = [0,1]$ and $f \sim g$ if there is a homotopy *through such transversal maps*.

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By Pontrjagin–Thom $\psi_k(\mathbb{S}^m)$ is ambient isotopy classes of framed codim-m submanifolds of $(0,1)^k$.

Transversal homotopy monoids

Functoriality

ψ_k is a functor on Whitney stratified manifolds and stratified transversal maps. There is a natural transformation $\psi_k \to \pi_k$.

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Example

The linking number of a framed link is given by

$$\begin{array}{ccc} \psi_3\left(\mathbb{S}^2\right) & \longrightarrow & \pi_3(S^2)\ . \\ \| & & \| \\ \{\text{framed links}\} & & \mathbb{Z} \end{array}$$

(Topologists' framing, not knot theorists'!)

Replacing spheres by other Thom spectra we can get plain-vanilla links, oriented links etc and higher-dimensional variants.

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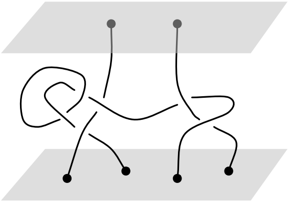
Example

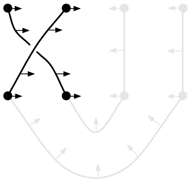
By Pontrjagin–Thom $\psi^1_2\left(\mathbb{S}^2\right) \simeq {}^{\mathrm{fr}}\mathrm{Tang}^1_2$ is category of framed tangles:



Monoidal categories with duals

Examples

The category of framed tangles is monoidal with duals; we can turn inputs into outputs, and vice versa (provided we dualise them):



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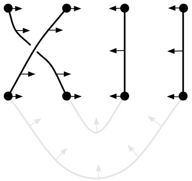

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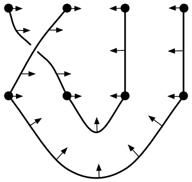

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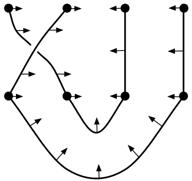

The category of finite dim vector spaces is another example, e.g.

$$\mathrm{Hom}(V, W) \cong \mathrm{Hom}(1, V^* \otimes W).$$

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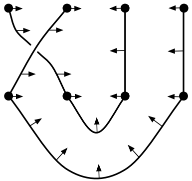

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Theorem (W '09)

$\psi_k^1(M)$ is a monoidal category with duals for $k > 0$, braided monoidal for $k > 1$ and symmetric monoidal for $k > 2$.

Transversal homotopy *n*-categories?

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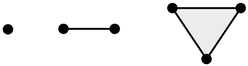

Simplicial?

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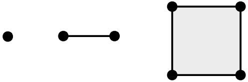

Cubical?

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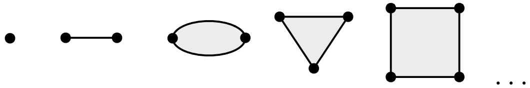

In Morrison and Walker's definition of *n*-category 'all' shapes are allowed. They work in the PL context; we give a smooth version of their definition.

Morrison–Walker n-categories

Terminology

Fix $n \in \mathbb{N}$. Henceforth,

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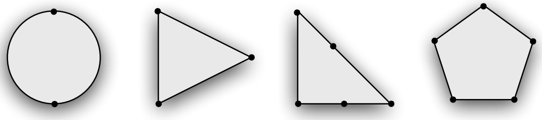
Examples

Examples of 2-cells for $n = 2$ with stratifications indicated (only the middle two are diffeomorphic):



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Axiom 1: Morphisms

For $0 \leq k \leq n$ there is a functor

$$\mathcal{C}^k : k\text{-cells and diffeomorphisms} \to \text{sets and bijections}$$

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Lemma

$\mathcal{C}^k$ extends to functor on k-dim spaces and diffeomorphisms.

Morrison–Walker n-categories

Axiom 2: Boundaries

For each k-cell $(B, \partial B)$ there is a natural transformation

$$\partial : \mathcal{C}^k(B) \to \mathcal{C}^{k-1}(\partial B).$$

The boundary is the domain and codomain rolled into one.

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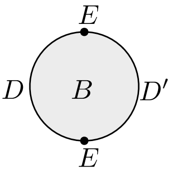

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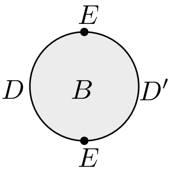

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Denote the image by $\mathcal{C}(\partial B; E)$, and preimage under ∂ by $\mathcal{C}(B; E)$.

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Axiom 3: Composition

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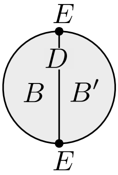

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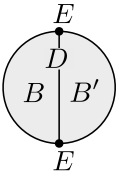

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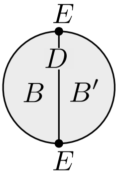

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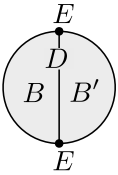

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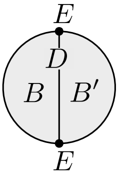

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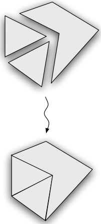

Morrison–Walker n-categories

Axiom 4: Existence of identities

If B is a k-cell and D a d-cell (with $d + k \leq n$) then there is a map

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(In fact require such maps for every 'pinched product'.)

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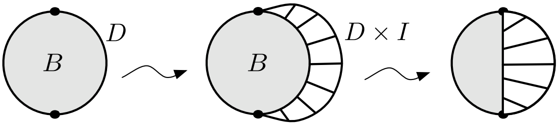

$$\mathcal{C}(B) \longrightarrow \mathcal{C}\left(B \cup (D \times I)\right) \longrightarrow \mathcal{C}(B)$$

Examples of Morrison–Walker $(n+k)$-categories

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- Framed tangles: ${}^{\mathrm{fr}}\mathrm{Tang}_k^n(B^i) =$

  $\{$*codim-k framed submanifolds of* B^i *which are transverse to strata of some cellular stratification*$\}$

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  (up to isotopy when $i = n + k$).
- Transversal homotopy: $\psi_k^n(M)(B^i) =$

  $\{f : B^i \to M \mid \exists$ *cellular stratification with* $f|_S$ *transverse* $\forall S$ *and* $f^{-1}(*) \supset \bigcup_{codim\, S<k} S\}$

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k-tuply monoidal n-categories

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${}^{\mathrm{fr}}\mathrm{Tang}_k^n$ and $\psi_k^n(M)$ are k-tuply monoidal n-category with duals.

Functors between Morrisson–Walker n-categories

Definition

The most general definition of functor is not completely clear, however any reasonable definition must include a system

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of natural transformations compatible with boundaries, compositions and identitites.

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of natural transformations compatible with boundaries, compositions and identitites.

Examples

- stratified transversal $f : M \to N$ induces $\psi_k^n(M) \to \psi_k^n(N)$;
- taking preimage stratification induces $\psi_k^n(\mathbb{S}^k) \to {}^{\mathrm{fr}}\mathrm{Tang}_k^n$.

D-framed tangles and collapse maps

Fix k-cell D and point $q \in D$.

D-framed tangles

Define ${}^{D\text{-fr}}\mathrm{Tang}_k^n$ as before but with compatible choices (for each tangle t) of tubular neighbourhood N and diffeomorphism

$$N \cong t \times D$$

identifying t and $t \times q$.

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Collapse map functor

For each D-framed tangle t define a map $B \to \mathbb{S}^k : t \mapsto p$ by choosing a transversal map

$$(D, \partial D) \to (\mathbb{S}^k, *) : q \mapsto p.$$

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This determines a functor ${}^{D\text{-fr}}\mathrm{Tang}_k^n \to \psi_k^n\left(\mathbb{S}^k\right)$.

Patchwork functors

Given k-tuply monoidal n-category with duals $\mathcal{C}$ and $c \in \mathcal{C}^k(D)$ define a patchwork functor

$$P_c : {}^{D\text{-fr}}\mathrm{Tang}_k^n \longrightarrow \mathcal{C}$$

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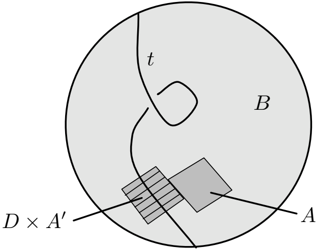

For $t \in {}^{D\text{-fr}}\mathrm{Tang}_k^n(B)$

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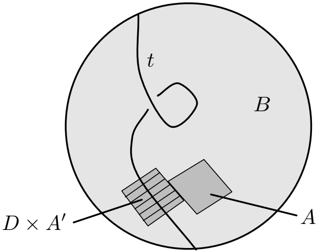

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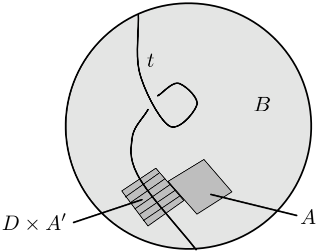

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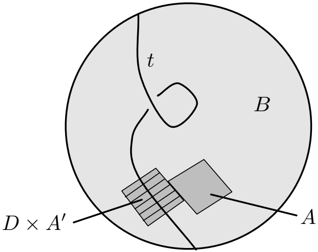
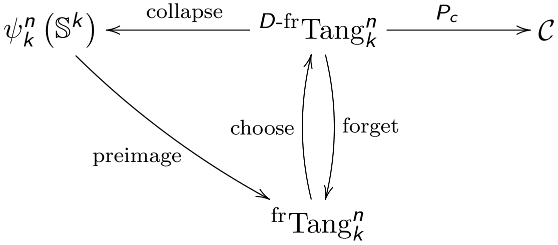


For $t \in {}^{D\text{-fr}}\mathrm{Tang}^n_k(B)$

- choose cellular stratification compatible with D-framing;
- assign $c \times A' \in \mathcal{C}(D \times A')$;
- assign $1 \times A \in \mathcal{C}(A)$ etc;
- composite is $P_c(B)(t)$.

Transversal Homotopy and the Tangle Hypothesis

We have sketched the construction of



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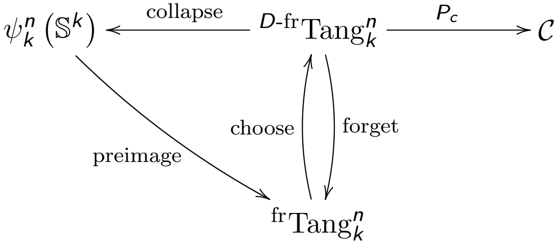

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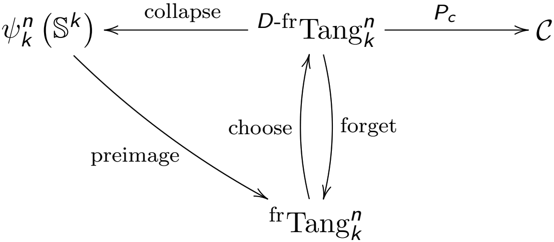

- LHS is Pontrjagin–Thom construction.
- RHS is Tangle Hypothesis: $^{\mathrm{fr}}\mathrm{Tang}_k^n$ is free k-tuply monoidal n-category with duals on one generator.