NOTES ON EULER CALCULUS IN AN O-MINIMAL STRUCTURE

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Abstract. The aim of this note is to give an elementary exposition of Euler calculus for functions constructible with respect to a fixed o-minimal structure. Our approach is to use natural identifications $CF(X) \cong K(X) \cong \mathbb{K}(X)$ of the bounded constructible functions $CF(X)$ on X, the 'small' Grothendieck group $K(X)$ of (definable) subspaces of X and the 'big' Grothendieck group $\mathbb{K}(X)$ of (definable) spaces over X. The operations of the Euler calculus have natural geometric interpretations in $\mathbb{K}(X)$ from which the usual properties follow. This allows us to develop the Euler calculus without recourse to any (co)homological or sheaf-theoretic interpretation.

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1. Introduction

Euler calculus was developed in the late 1980s by Viro [Vir88] and by Schapira [Sch91, Sch95]. It provides an integration theory for constructible functions which allows one to study the topology of constructible sets and functions. Viro started from the observation that compactly supported Euler characteristic χ_c, is additive and so is almost a measure, the only difference being that it is not necessarily positive. From this perspective he developed the Euler integral by analogy with integration with respect to a measure. His main applications were to complex geometry and singularity theory — [GZ10] is a survey of this circle of ideas and its more recent relations with motivic measure and other topics in algebraic geometry. Schapira started from the fact that (under suitable conditions) constructible functions are the Grothendieck group of the derived category of constructible sheaves. The operations of the Euler calculus then arise as 'de-categorifications' of the well-known operations on constructible sheaves. His applications were mainly in real analytic geometry, particularly to tomography and questions initiated from robotics. The recent survey paper [CGR12] focusses on yet other applications to sensing which have been developed by Baryshnikv, Ghrist and others; it also contains an extensive bibliography which we do not attempt to replicate here.

We take a slightly different approach to developing the Euler calculus (for spaces definable in some o-minimal structure), based on natural identifications

$$CF(X) \cong K(X) \cong \mathbb{K}(X) \tag{1}$$

of the bounded constructible functions $CF(X)$ on X, the 'small' Grothendieck group $K(X)$ of (definable) subspaces of X and the 'big' Grothendieck group $\mathbb{K}(X)$ of (definable) spaces over X. The operations of the Euler calculus have geometric interpretations in $\mathbb{K}(X)$ from which the usual properties follow. This approach seems natural, and we hope it will be attractive to students and others wishing to learn the subject without first digesting the more sophisticated machinery of sheaf-theory and homological algebra (which we do not require). This is close in spirit to Viro's approach, however we develop the full calculus of operations for functions constructible with respect to an arbitrary o-minimal structure. A limitation of working within the context of o-minimal structures is that we consider only spaces embedded in Euclidean space. It should be possible to globalise to an analytic-geometric category, in the sense of [vdDM96], but we do not do so here.

We develop the Euler calculus with integral coefficients, but one could work with coefficients in any ring. Indeed, in §7 we use the Euler calculus of $\mathbb{Z}/2$-valued constructible functions to define the 'constructible homology groups' $CH_*(X)$ of a definable space. These are isomorphic to the Borel–Moore homology groups $H_*^{\mathrm{BM}}(X; \mathbb{Z}/2)$, and the associated Euler characteristic is the compactly supported Euler characteristic χ_c. In particular this shows that χ_c is purely topological, independent of the o-minimal structure we use to define it. It also shows that χ_c is a proper homotopy invariant.

The only tools we require are two basic, and easily digestible, theorems on the structure of definable sets and maps in an o-minimal structure. These are the existence of cell decompositions, and the existence of trivialisations for definable maps. We use these as 'black boxes'.

A calculus is a collection of rules for computation. We summarise the rules of the Euler calculus here. For simplicity we restrict to the case of *locally compact* definable spaces, although many of these properties hold for general definable spaces. The bounded constructible functions on such a space from a ring $CF(X)$. It is generated by indicator functions of definable subsets and is equipped with an abelian group endomorphism $\mathbb{D}_X$ (the dual). A continuous definable map $\varphi\colon X \to Y$ induces functorial homomorphisms of abelian groups

$$\varphi_!, \varphi_* \colon CF(X) \to CF(Y) \quad \text{and} \quad \varphi^*, \varphi^! \colon CF(Y) \to CF(X),$$

moreover φ^* is a ring homomorphism. Here, by 'functorial' we mean that $\varphi_!\psi_! = (\varphi\psi)_!$ and $\psi^*\varphi^* = (\varphi\psi)^*$ and so on. These satisfy, and are determined by,

(a) $\varphi_!(1_A) = \chi_c(A)$ where χ_c is the compactly supported Euler characteristic, 1_A is the indicator function of definable $A \subset X$ and $\varphi\colon X \to \mathrm{pt}$;
(b) $\varphi^*(f) = f \circ \varphi$;

(c) (base change) $\Phi^*\Psi_! = \psi_!\varphi^*$ whenever

$$\begin{array}{ccc} W & \xrightarrow{\psi} & X \\ \downarrow{\scriptstyle\varphi} & & \downarrow{\scriptstyle\Phi} \\ Y & \xrightarrow{\Psi} & Z \end{array}$$

is a cartesian diagram;

(d) (projection formula) $\varphi_!(f \cdot \varphi^* g) = \varphi_! f \cdot g$;
(e) $\mathbb{D}_{\mathbb{R}^n}(1_C) = (-1)^{\dim C} 1_{\overline{C}}$ where $C \subset \mathbb{R}^n$ is a definable cell;
(f) $\mathbb{D}_X^2 = \mathrm{id}$;
(g) $\varphi_* = \mathbb{D}_Y \varphi_! \mathbb{D}_X$;
(h) $\varphi^! = \mathbb{D}_Y \varphi^* \mathbb{D}_X$;
(i) $\varphi_! \mathbb{D}_X = \mathbb{D}_Y \varphi_!$ when φ is proper;
(j) $\mathbb{D}_X \varphi^* = \varphi^* \mathbb{D}_Y$ when φ is a local homeomorphism.

To some extent this list is arbitrary, for instance (a) and (e) can be replaced by other (more parsimonious) characterisations, and there are many useful properties which we have not listed. Nevertheless, it captures the key properties of the Euler calculus.

The structure of the paper is as follows. §2 provides the necessary background in o-minimal structures on which our approach rests. In §3 we show that definable sets in an o-minimal structure have well-defined compactly supported Euler characteristic. In §4 we construct the isomorphisms (1). The Euler integral, more generally the proper pushforward operation for constructible functions, is defined in §5, and its basic properties – functoriality, projection and base change formulæ — are developed. This section also contains a brief discussion of integral transforms, i.e. more general operations on constructible functions which can be described as 'integration against a kernel'. As an application of the calculus developed we compute the kernel of the composite of two such transforms, and deduce Schapira's inversion formula [Sch95]. In §6, we introduce the duality operation on constructible functions and show that duality is an involution for functions on a locally compact space. We also briefly discuss the operations obtained from proper pushforward and pullback by conjugating by the dual. In §7 we use the $\mathbb{Z}/2$-valued Euler calculus to construct Borel–Moore homology for locally-compact definable spaces and proper definable maps between them. The associated Euler characteristic is shown to be the compactly supported Euler characteristic.

2. O-minimal structures

We give a brief summary of the theory of o-minimal structures, introducing the properties we will use in later sections. Roughly, an o-minimal structure is a collection of 'tame' subsets of Euclidean space with which one can perform standard geometric and topological constructions. A good reference for further details and proofs is [vdD98].

2.1. **Definition and elementary properties.** A *structure* on $\mathbb{R}$ consists of a collection $\mathcal{S}_n$ of subsets of $\mathbb{R}^n$ for each $n \in \mathbb{N}$ such that

(1) $\mathcal{S}_n$ is a Boolean algebra containing $\mathbb{R}^n$, i.e. $\mathcal{S}_n$ is closed under intersection, union and complement;
(2) $\mathcal{S}_n$ contains the diagonals $\{(x_1, \ldots, x_n) \mid x_i = x_j\}$ for any $1 \le i < j \le n$;
(3) if $X \in \mathcal{S}_n$ then $X \times \mathbb{R} \in \mathcal{S}_{n+1}$;
(4) if $X \in \mathcal{S}_{n+1}$ then $\pi(X) \in \mathcal{S}_n$ where $\pi\colon \mathbb{R}^{n+1} \to \mathbb{R}^n$ is projection onto the first n coordinates;
(5) the graphs of addition and multiplication are in $\mathcal{S}_3$.

A structure is *o-minimal* (short for 'order minimal') if the collection $\mathcal{S}_1$ consists of finite unions of open intervals (possibly infinite) and points. We say $X \subset \mathbb{R}^n$ is *definable* if it is in $\mathcal{S}_n$. We say a map $\varphi \colon X \to Y$, where $X \in \mathcal{S}_m$ and $Y \in \mathcal{S}_n$, is *definable* if its graph $\{(x, y) \mid y = \varphi(x)\}$ is definable, i.e. is in $\mathcal{S}_{m+n}$.

The definition of o-minimal structure is asymmetric with respect to the coordinates. However, if $X \in \mathcal{S}_n$ and $f \colon \{1, \ldots, n\} \to \{1, \ldots, n\}$ is any map then it turns out that

$$\{(x_1, \ldots, x_n) \mid \left(x_{f(1)}, \ldots, x_{f(n)}\right) \in X\}$$

is in $\mathcal{S}_n$ too, i.e. we can permute and identify coordinates at will. Other nice properties are that

(1) the order relation $\{(x_1, x_2) \mid x_1 < x_2\}$ is definable;
(2) if X is definable then the interior X^0 and the closure $\overline{X}$ are definable, hence so too are the boundary $\partial X = \overline{X} - X^0$ and the frontier $\operatorname{fr} X = \overline{X} - X$;
(3) products of definable spaces and of definable maps are definable;
(4) if φ is a definable map, and $A \subset X$ and $B \subset Y$ are definable, then $\varphi(A)$ and $\varphi^{-1}(B)$ are definable;
(5) if $f, g \colon X \to \mathbb{R}$ are definable functions then so are kf for any $k \in \mathbb{R}$, $f + g$, $f \times g$, $\max\{f, g\}$, and $\min\{f, g\}$.

A definable map need not be continuous, however as our focus is on topology we will always work with continuous definable maps. Let Sp be the category of definable spaces and continuous definable maps. It is a small category. It also has has fibre products: given $\alpha \colon A \to X$ and $\beta \colon B \to X$ the fibre product is

$$A \times_X B = \{(a, b) \in A \times B \mid \alpha(a) = \beta(b)\} \to X.$$

It is definable as it is the inverse image of the diagonal in $X \times X$ under the definable map $\alpha \times \beta \colon A \times B \to X \times X$. From this it follows, for example, that the fibre of a definable map is definable.

We describe two important examples; further examples can be found in [vdD98, vdDM96]. The smallest o-minimal structure is the *semi-algebraic sets*. These are the subsets of $\mathbb{R}^n$ which are finite Boolean combinations of subsets cut-out by (real) polynomial equations or inequalities, i.e. subsets of the form

$$\bigcup_{i \in I} \bigcap_{j \in J_i} X_{ij}$$

where I and each J_i are finite sets, and each X_{ij} is either of the form $\{p(x_1, \ldots, x_n) = 0\}$ or $\{p(x_1, \ldots, x_n) > 0\}$ for some polynomial $p \in \mathbb{R}[x_1, \ldots, x_n]$.

A larger example is provided by the *globally sub-analytic sets*. To define these we begin with *semi-analytic sets*. These are *local* analytic analogues of semi-algebraic sets — a subset $X \subset \mathbb{R}^n$ is semi-analytic if each $x \in X$ has a neighbourhood U such that $U \cap X$ is a finite Boolean combination of subsets cut-out by (real) analytic equations or inequalities. This class is not closed under projections, i.e. property (4) of an o-minimal structure is not satisfied. We refine it in two steps to fix this defect. A subset $X \subset \mathbb{R}^n$ is *sub-analytic* if each $x \in X$ has a neighbourhood U such that $U \cap X = \pi(Y)$ for some semi-analytic $Y \subset \mathbb{R}^{m+n}$ with compact closure, where $\pi \colon \mathbb{R}^{m+n} \to \mathbb{R}^n$ is projection onto the first n coordinates. The image of a sub-analytic set under a *proper* projection is sub-analytic, however property (4) still fails in general. A subanalytic $X \subset \mathbb{R}^n$ is *globally subanalytic* if it is also subanalytic at infinity, more precisely if $\imath(X) \subset \mathbb{R}^N$ is subanalytic, where $\imath \colon \mathbb{R}^n \to \mathbb{R}^N$ is the composite of the standard embedding $\mathbb{R}^n \hookrightarrow \mathbb{RP}^n$ and a (choice of) algebraic embedding $\mathbb{RP}^n \hookrightarrow \mathbb{R}^N$.

2.2. **Basic structure theorems.** In this section we state the key theorems about the structure of definable sets and maps which we will need to develop the theory of Euler calculus.

The first result will be the existence of cell decompositions of definable sets. Before stating it we recall the notions of cell and cell decomposition in an o-minimal structure. The cells in $\mathbb{R}^n$ are defined inductively on n by

(1) $\{0\}$ is a cell in $\mathbb{R}^0$;
(2) if $C \subset \mathbb{R}^n$ is a cell then
   (a) $C \times \mathbb{R}$ is a cell;
   (b) if $f: C \to \mathbb{R}$ is continuous and definable then the sets of those $(x,t) \in C \times \mathbb{R}$ such that $f(x) = t$, such that $f(x) < t$, and such that $f(x) > t$ are cells;
   (c) if $f, g: C \to \mathbb{R}$ are continuous definable functions with $f(x) < g(x)$ for all $x \in C$ then the set of $(x,t) \in C \times \mathbb{R}$ such that $f(x) < t < g(x)$ is a cell.

By construction each cell is definable. In fact, each cell is definably homeomorphic to a product of open intervals [vdDM96, §4]. As examples, the cells in $\mathbb{R}$ are either points or (possibly infinite) open intervals; the open unit disk $\{x \in \mathbb{R}^n \mid 1 > \|x\|\}$ is a cell; the interior of the standard simplex — i.e. the convex hull of the basis vectors $e_i = (0, \ldots, 0, 1, 0, \ldots, 0)$ in $\mathbb{R}^n$ — is a cell, and so on.

The notion of cell decomposition of $\mathbb{R}^n$ is also defined inductively. A *cell decomposition* of $\mathbb{R}$ is a finite partition of $\mathbb{R}$ into cells. A cell decomposition of $\mathbb{R}^{n+1}$ is a finite partition $\{C_i \mid i \in I\}$ of $\mathbb{R}^{n+1}$ into cells such that $\{\pi(C_i) \mid i \in I\}$ is a cell decomposition of $\mathbb{R}^n$, where $\pi: \mathbb{R}^{n+1} \to \mathbb{R}^n$ is projection onto the first n coordinates. A cell decomposition is *compatible with* A if A is a union of cells.

Theorem 2.1 (Cell decompositions [vdDM96, §4.8])**.** *Given definable $A_1, \ldots, A_n \subset \mathbb{R}^n$ there is a cell decomposition of $\mathbb{R}^n$ compatible with each of the A_i. Moreover, the cell decomposition can be chosen to satisfy the frontier condition, i.e. so that the closure of any cell is a union of cells.*

We will use the shorthand 'a cell decomposition of A' for 'a cell decomposition of $\mathbb{R}^n$ compatible with A'. It follows from the above theorem that any two cell decompositions of A have a common subdivision: simply take a cell decomposition compatible with the intersections of cells in the two original decompositions.

Any subdivision of a cell decomposition can be broken down into a sequence of 'elementary' subdivisions.

Definition 2.2. Suppose $\{C_i \mid i \in I\}$ is a cell decomposition of $\mathbb{R}^n$. Let $\pi_k: \mathbb{R}^n \to \mathbb{R}^{n-k}$ denote projection onto the first $n-k$ coordinates. An *elementary subdivision of depth* 0 of $\{C_i \mid i \in I\}$ is a cell decomposition obtained by replacing a cell C_i by two cells $C_i^{\pm}$ of dimension $\dim C_i$, and one cell C_i^0 of dimension $\dim C_i - 1$, such that $C_i = C_i^- \cup C_i^0 \cup C_i^+$ and $\pi_1 C_i^{\pm} = \pi_1 C_i^0 = \pi_1 C_i$.

An *elementary subdivision of depth* k of $\{C_i \mid i \in I\}$ is induced from an elementary subdivision of depth 0 of a cell $\pi_k C_i$ in the decomposition $\{\pi_k C_i \mid i \in I\}$ of $\mathbb{R}^{n-k}$ by replacing each cell C_j with $\pi_k C_j = \pi_k C_i$ by two cells

$$C_j^{\pm} = C_j \cap \pi_k^{-1}(\pi_k C_i)^{\pm}$$

of dimension $\dim C_j$, and one cell $C_j^0 = C_j \cap \pi_k^{-1}(\pi_k C_i)^0$ of dimension $\dim C_j - 1$.

Example 2.3. The decomposition $\mathbb{R}^n = \{x_{n-k} < 0\} \cup \{x_{n-k} = 0\} \cup \{x_{n-k} > 0\}$ is a depth k elementary subdivision of the trivial cell decomposition of $\mathbb{R}^n$, i.e. the cell decomposition with only one cell.

Lemma 2.4. *Suppose $\{C_i\}$ is a cell decomposition of $\mathbb{R}^n$ satisfying the frontier condition. Then the cell decomposition $\{\pi_k C_i\}$ of $\mathbb{R}^{n-k}$ satisfies the frontier condition too. Suppose further that $\{D_j\}$ is a depth k elementary subdivision of $\{C_i\}$ such that $\{\pi_k D_j\}$ satisfies the frontier condition. Then $\{D_j\}$ also satisfies the frontier condition.*

Proof. Suppose $\pi_k C_i \cap \overline{\pi_k C_j} \neq \emptyset$. Then there exists C with $\pi_k C = \pi_k C_j$ and $C_i \cap \overline{C} \neq \emptyset$. Thus $C_i \subset \overline{C}$ by the frontier condition and $\pi_k C_i \subset \pi_k\left(\overline{C}\right) \subset \overline{\pi_k C} = \overline{\pi_k C_j}$ which establishes the first claim.

For the second claim we need only check the condition for the 'new' cells in $\{D_j\}$. We do so those of the form C_i^-, the other cases being similar. Let $C = C_i$ and $D = C_i^-$. Suppose $D_j \cap \overline{D} \neq \emptyset$. Then, by the frontier condition for $\{C_i\}$, we have $D_j \subset \overline{C}$, and by the frontier condition for $\{\pi_k D_j\}$ we have $\pi_k D_j \subset \overline{\pi_k D}$. Therefore $D_j \subset \overline{C} \cap \pi_k^{-1}\left(\overline{\pi_k D}\right) = \overline{D}$ as required. □

Proposition 2.5. *Suppose $\{D_j \mid j \in J\}$ is a subdivision of the cell decomposition $\{C_i \mid i \in I\}$ of $\mathbb{R}^n$. Then there is a finite sequence of elementary subdivisions (of various depths) starting from $\{C_i \mid i \in I\}$ and ending at $\{D_j \mid j \in J\}$. Moreover, if both $\{C_i\}$ and $\{D_j\}$ satisfy the frontier condition then we may choose the sequence of elementary subdivisions so that each intermediate cell decomposition also satisfies the frontier condition.*

Proof. We prove this by induction on n. It is clear for $n = 1$. Suppose it is true for $n-1$, and let $\pi\colon \mathbb{R}^n \to \mathbb{R}^{n-1}$ be projection onto the first $n-1$ coordinates. By the inductive hypothesis it suffices to prove it for the special case when $\{\pi D_j \mid j \in J\}$ is an elementary subdivision of $\{\pi C_i \mid i \in I\}$. If this subdivision has depth k then by performing the corresponding elementary subdivision of $\{C_i \mid i \in I\}$ of depth $k+1$ we further reduce to the case in which $\{\pi C_i \mid i \in I\} = \{\pi D_j \mid j \in J\}$. Here it is clear that $\{D_j \mid j \in J\}$ is obtained from $\{C_i \mid i \in I\}$ by finitely many depth 0 elementary subdivisions. This establishes the first statement.

The more refined second statement is also proved by induction. Again it is clear for $n = 1$ since any cell decomposition of $\mathbb{R}$ satisfies the frontier condition. Suppose it is true for $n-1$. Then Lemma 2.4 shows that it suffices to consider the case in which $\{\pi C_i\} = \{\pi D_j\}$. In this situation the subdivision is achieved by a finite sequence of depth 0 elementary subdivisions. If these are carried out by subdividing cells C in order of increasing dimension of πC then so will all the intermediate cell decompositions. This is because the final subdivision $\{D_j\}$ satisfies the frontier condition and we always subdivide the boundary of any cell before we subdivide the cell itself. This completes the proof. □

The final fact we need about o-minimal structures is the following result about continuous definable maps.

Theorem 2.6 (Trivialisation theorem [vdDM96, §4.11])**.** *Suppose $f\colon A \to B$ is a continuous definable map between definable spaces. Then B has a cell decomposition $B = B_1 \sqcup \cdots \sqcup B_k$ such that there exist definable homeomorphisms $h_i\colon f^{-1}B_i \to B_i \times F_i$ making the diagram*

$$\begin{array}{ccc} f^{-1}B_i & \xrightarrow{\quad h_i \quad} & B_i \times F_i \\ & {\scriptstyle f}\searrow \quad \swarrow{\scriptstyle p_1} & \\ & B_i & \end{array}$$

(where p_1 is projection onto the first coordinate) commute.

3. Compactly supported Euler characteristic

A definable space has a well-defined compactly supported Euler characteristic; it is given by the alternating sum of numbers of cells of each dimension in (any) cell decomposition.

Proposition 3.1. *Let X be a definable space. Then the* compactly supported Euler characteristic

$$\chi_c(X) = \sum_{i \in I} (-1)^{\dim C_i}$$

is well-defined, independent of the cell decomposition $X = \bigsqcup_{i \in I} C_i$ used to compute it.

Proof. Recall that any cell decomposition is finite so that the above sum makes sense. Moreover, any two cell decompositions have a common subdivision and by Proposition 2.5 any subdivision can be achieved by a finite sequence of elementary subdivisions. Hence it suffices to show that $\chi_c(X)$ is the same when computed via a cell decomposition and any elementary subdivision thereof. This is clear since in any elementary subdivision a number of cells are each replaced by two cells of the same dimension and one of one dimension less, and this does not alter the alternating sum of the numbers of cells of each dimension. □

Example 3.2. Considering the trivial cell decompositions, with only one cell, we see that $\chi_c(\mathbb{R}^n) = (-1)^n$. In particular, unlike the usual Euler characteristic, *compactly supported* Euler characteristic is not a homotopy invariant.

Example 3.3. The sphere $S^d = \{x_0^2 + \cdots + x_d^2 = 1\}$ is a definable subset of $\mathbb{R}^{d+1}$ in any o-minimal structure. The image under the projection onto the last factor is the union of an open d-cell and S^{d-1}. From this observation we can inductively construct a cell decomposition of S^d with two i-cells for each $0 \leq i \leq d$. Hence

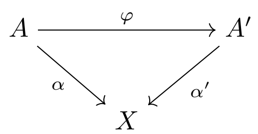
$$\chi_c(S^d) = \begin{cases} 2 & d \text{ even} \\ 0 & d \text{ odd.} \end{cases}$$

4. Constructible functions and Grothendieck groups

In this section we show for a definable space X that the ring of bounded constructible functions $X \to \mathbb{Z}$, the 'small' Grothendieck group of definable subspaces of X, and the 'big' Grothendieck group of definable spaces over X are naturally isomorphic.

4.1. **The 'big' Grothendieck group.** The 'big' Grothendieck group $\mathbb{K}(X)$ is the abelian group generated by elements of the form $[\alpha\colon A \to X]$, where α is a continuous definable map, subject to the relations:

(1) $[\alpha\colon A \to X] = [\alpha'\colon A' \to X]$ whenever there is a commutative diagram



in Sp with φ a definable homeomorphism;

(2) $[\alpha\colon A \to X] = [\alpha|_B\colon B \to X] + [\alpha|_{A-B}\colon A - B \to X]$ for any definable subspace $B \subset A$.

We make $\mathbb{K}(X)$ into a ring with multiplication given by the fibre product

$$[\alpha\colon A \to X] \cdot [\beta\colon B \to X] = [\alpha \times_X \beta\colon A \times_X B \to X].$$

When no confusion is likely to arise we will denote the class $[\alpha\colon A \to X]$ more succinctly either by $[A \to X]$ or by $[\alpha]$.

Let $\varphi\colon X \to Y$ be a continuous definable map. We define a ring homomorphism $\varphi^*\colon \mathbb{K}(Y) \to \mathbb{K}(X)$ by

$$\varphi^*[A \to Y] = [X \times_Y A \to X].$$

This makes $\mathbb{K}(-) : \mathrm{Sp}^{\mathrm{op}} \to \mathrm{Ring}$ into a functor.

4.2. **The 'small' Grothendieck group.** The 'small' Grothendieck group $K(X)$ is generated by the *injective* continuous definable maps $[\alpha\colon A \hookrightarrow X]$ with the same relations as the 'big' Grothendieck group. The first relation above means that injections with the same image give the same class in $K(X)$. Hence we may equivalently consider the generators of $K(X)$ to be of the form $[A]$ where $A \subset X$ is a definable subspace, with relations $[A] = [B] + [B - A]$ for any definable $B \subset A$ and product $[A] \cdot [B] = [A \cap B]$.

The 'small' Grothendieck group is a functor $K(-) : \mathrm{Sp}^{\mathrm{op}} \to \mathrm{Ring}$: for continuous definable $\varphi\colon X \to Y$ we define $\varphi^*\colon K(Y) \to K(X)$ by $[A] \mapsto [\varphi^{-1}(A)]$.

4.3. **Constructible functions.** The bounded constructible functions $CF(X)$ are the subset of bounded maps $f\colon X \to \mathbb{Z}$ such that $f^{-1}(n)$ is definable for each $n \in \mathbb{Z}$. An elementary check shows that constructible functions form a ring under point-wise addition and multiplication of functions. If $\varphi\colon X \to Y$ is a continuous definable map then composition defines a ring homomorphism

$$\varphi^*\colon CF(Y) \to CF(X) : f \mapsto f \circ \varphi,$$

and this makes $CF(-) : \mathrm{Sp}^{\mathrm{op}} \to \mathrm{Ring}$ into a functor.

The most basic, and most important, example of a constructible function is the indicator function $1_A\colon X \to \mathbb{Z}$ of a definable subset $A \subset X$:

$$1_A(x) = \begin{cases} 1 & x \in A \\ 0 & x \notin A. \end{cases}$$

Note that any *bounded* constructible function is a linear combination of indicator functions; explicitly

$$f = \sum_{n \in \mathbb{Z}} n 1_{f^{-1}(n)}.$$

As f is bounded there exists $N \in \mathbb{N}$ such that $f^{-1}(n) = \emptyset$ for $|n| > N$. Hence we can even find a cell decomposition of X such that each $f^{-1}(n)$ is a union of cells, and any $f \in CF(X)$ can be expressed as a linear combination of indicator functions of cells.

4.4. **Natural isomorphisms.** The 'big' and 'small' Grothendieck groups, and the constructible functions are three views of the same object. More precisely, we construct natural ring homomorphisms

$$\begin{array}{ccc} K(X) & \xrightarrow{\quad \mathcal{I}_X \quad} & \mathbb{K}(X) \\ & {}^{\mathcal{C}_X}\nwarrow \qquad \swarrow{}^{\mathcal{E}_X} & \\ & CF(X) & \end{array}$$

and show that the composite of any two is inverse to the third. In each case we define the homomorphism on generators and extend $\mathbb{Z}$-linearly. (We verify that these actually are homomorphisms with the stated target in the sequel.) The definitions are $\mathcal{C}_X(1_A) = [A]$, and $\mathcal{I}_X([A]) = [A \hookrightarrow X]$, and

$$\mathcal{E}_X([A \xrightarrow{\alpha} X]) = x \mapsto \chi_c\left(\alpha^{-1}x\right).$$

When X is a point the claim follows quite simply. It is immediate that $CF(\mathrm{pt}) \cong K(\mathrm{pt}) \cong \mathbb{Z}$. Given a class $[A] \in \mathbb{K}(\mathrm{pt})$ we may choose a cell decomposition $A = C_1 \sqcup \cdots \sqcup C_k$. Then

$$[A] = \sum_i [C_i] = \sum_i (-1)^{\dim C_i}[\mathrm{pt}] = \chi_c(A)\,[\mathrm{pt}].$$

It follows that $\mathbb{K}(\mathrm{pt}) \cong \mathbb{Z}$ too. One can easily verify that the given maps provide the identifications. Henceforth, for simplicity, we will suppress these maps.

Proposition 4.1. *The map $\mathcal{C}_X\colon CF(X) \to K(X)$ is a natural isomorphism of rings.*

Proof. Firstly, it is a ring homomorphism because

$$\mathcal{C}_X(1_A 1_B) = \mathcal{C}_X(1_{A\cap B}) = [A \cap B] = [A]\cdot[B].$$

Secondly, it is an isomorphism because the inverse assignment $[A] \mapsto 1_A$ is well-defined. Finally, it is natural because

$$\mathcal{C}_X \circ \varphi^*(1_A) = \mathcal{C}_X(1_{\varphi^{-1}A}) = [\varphi^{-1}A] = \varphi^*[A] = \varphi^* \circ \mathcal{C}_Y([A])$$

for any definable $A \subset Y$. $\square$

Proposition 4.2. *The map $\mathcal{E}_X\colon \mathbb{K}(X) \to CF(X)$ is a natural ring homomorphism.*

Proof. To see that $\mathcal{E}_X$ is a ring homomorphism from $\mathbb{K}(X)$ to integer-valued functions on X, recall that the ring structure on functions is defined pointwise and note that

$$\mathcal{E}_X([A \xrightarrow{\alpha} X])(x) = \chi_c\left(\alpha^{-1}x\right) = [\alpha^{-1}x] = \imath_x^*[A \xrightarrow{\alpha} X] \in \mathbb{K}(\mathrm{pt}),$$

where $\imath_x$ is the inclusion of the point x. It is a natural transformation because

$$\begin{aligned}
\mathcal{E}_X \circ \varphi^*([B \xrightarrow{\beta} Y])(x) &= \imath_x^*\varphi^*([B \xrightarrow{\beta} Y]) \\
&= \imath_{\varphi x}^*([B \xrightarrow{\beta} Y]) \\
&= \chi_c\left(\beta^{-1}(\varphi x)\right) \\
&= \mathcal{E}_Y([B \xrightarrow{\beta} Y])(\varphi x) \\
&= \varphi^* \circ \mathcal{E}_Y([B \xrightarrow{\beta} Y])(x).
\end{aligned}$$

It remains to show that $\mathcal{E}_X([A \xrightarrow{\alpha} X])$ is constructible. To do so we appeal to Theorem 2.6. We may choose a cell decomposition $X = C_1 \sqcup \cdots \sqcup C_k$ so that $\alpha^{-1}C_i \cong C_i \times F_i$, as spaces over X, for some definable spaces F_i. Therefore

$$\begin{aligned}
[A \to X] &= \sum_i [\alpha^{-1}C_i \to X] \\
&= \sum_i [C_i \times F_i] \\
&= \sum_i [C_i \hookrightarrow X]\cdot\pi^*[F_i \to \mathrm{pt}] \\
&= \sum_i [C_i \hookrightarrow X]\cdot\pi^*\left(\chi_c(F_i)\,[\mathrm{pt}]\right) \\
&= \sum_i \chi_c(F_i)\,[C_i \hookrightarrow X]
\end{aligned}$$

where $\pi\colon X \to \mathrm{pt}$. It follows that $\mathcal{E}_X([A \xrightarrow{\alpha} X]))$ is constructible on X. □

Proposition 4.3. *The map $\mathcal{I}_X : K(X) \to \mathbb{K}(X)$ is a natural ring homomorphism, and moreover is surjective.*

Proof. The constructions of the 'big' and 'small' Grothendieck groups are the same, except that the former has more generators. The map $\mathcal{I}_X$ is induced by the inclusion of this subset of generators. It follows that it is a ring homomorphism, and is natural. It remains only to see that it is surjective. This follows from the final computation of the previous proof,

$$[A \to X] = \sum_i \chi_c(F_i)\,[C_i \hookrightarrow X],$$

since the right hand side is in the image of $\mathcal{I}_X$. □

Corollary 4.4. *The composite of any two of the homomorphisms $\mathcal{C}_X$, $\mathcal{I}_X$ and $\mathcal{E}_X$ is inverse to the third. In particular all three are isomorphisms.*

Proof. We compute

$$\mathcal{C}_X\mathcal{E}_X\mathcal{I}_X([A]) = \mathcal{C}_X\mathcal{E}_X([A \hookrightarrow X]) = \mathcal{C}_X(1_A) = [A].$$

It follows that the composite $\mathcal{C}_X \circ \mathcal{E}_X \circ \mathcal{I}_X$ is the identity. In particular $\mathcal{I}_X$ must be injective. Since it is surjective by Proposition 4.3, it is an isomorphism. We also know by Proposition 4.1 that $\mathcal{C}_X$ is an isomorphism, so we deduce that $\mathcal{E}_X$ is an isomorphism too. The result follows. □

5. Euler calculus

5.1. **Proper Pushforward.** Recall that the 'big' Grothendieck group is a contravariant functor under pullback

$$\varphi^*([B \to Y]) = [B \times_Y X \to X]$$

along continuous definable $\varphi\colon X \to Y$. We can also define the *proper pushforward*

$$\varphi_!\colon \mathbb{K}(X) \to \mathbb{K}(Y) : [A \xrightarrow{\alpha} X] \mapsto [A \xrightarrow{\varphi\alpha} Y]$$

by composing. It is easy to check that this is a well-defined homomorphism of abelian groups (not of rings), and that this makes the 'big' Grothendieck group into a covariant functor from definable spaces to abelian groups.

Using the isomorphisms of the previous section we can define proper pushforwards for the 'small' Grothendieck group, and for constructible functions too. Explicitly these are given on generators by

$$\varphi_!([A]) = \sum_n n[Y_n]$$

where $Y_n = \{y \in Y \mid \chi_c(\varphi^{-1}y \cap A) = n\}$, and by $\varphi_!(1_A)(y) = \chi_c(\varphi^{-1}y \cap A)$, respectively. Their value on more general elements is computed by linearly extending. When $Y = \mathrm{pt}$ and $f \in CF(X)$ we have

$$\varphi_!(f) = \varphi_!\left(\sum_n n1_{f^{-1}(n)}\right) = \sum_n n\chi_c\left(f^{-1}(n)\right), \tag{2}$$

and in particular $\varphi_!(1_X) = \chi_c(X)$.

Remark 5.1. Following [Vir88, Sch91] we use the notation $\varphi_!(f) = \int_X f\,d\chi$ when $\varphi\colon X \to \mathrm{pt}$ is the map to a point, and refer to proper pushforward to a point as taking the *Euler integral* or *integral with respect to the Euler characteristic* of the constructible function f.

Lemma 5.2. *Let $X = \bigsqcup_{i\in I} X_i$ be a decomposition into subsets X_i each of which is definably homeomorphic to a cell. Then $\chi_c(X) = \sum_{i\in I}(-1)^{\dim X_i}$.*

Proof. This is immediate:

$$\chi_c(X) = \varphi_!(1_X) = \varphi_!\left(\sum_{i\in I} 1_{X_i}\right) = \sum_{i\in I}\chi_c(X_i) = \sum_{i\in I}(-1)^{\dim X_i}.$$

□

This absolves us from working with the rather restrictive, and asymmetric, definition of cell decomposition in an o-minimal structure; for instance we can compute the compactly supported Euler characteristic from any definable triangulation.

Remark 5.3. By construction the transformations $\mathcal{C}_X$, $\mathcal{I}_X$, and $\mathcal{E}_X$ are natural with respect to proper pushforward, i.e. commute with proper pushforward.

Standard properties of the interplay between proper pushforward and pullback for the 'big' Grothendieck group reduce to standard properties of fibred products.

Lemma 5.4 (Projection formula)**.** *Suppose $\varphi\colon X \to Y$ is a continuous definable map. Then $\varphi_!(a\cdot\varphi^*b) = \varphi_!a\cdot b$ in $\mathbb{K}(Y)$ for any classes $a \in \mathbb{K}(X)$ and $b \in \mathbb{K}(Y)$.*

Proof. Since both sides are linear, it suffices to prove this for generators, i.e. for $a = [A \to X]$ and $b = [B \to Y]$. In that case we have

$$\begin{aligned}\varphi_!(a\cdot\varphi^*b) &= \varphi_!([A\times_X (X\times_Y B)\to X])\\ &= [A\times_Y B \to Y]\\ &= [A\to Y]\cdot[B\to Y]\\ &= \varphi_!a\cdot b\end{aligned}$$

as required. □

Lemma 5.5 (Base change)**.** *Suppose that*

$$\begin{array}{ccc} W & \xrightarrow{\psi} & X\\ \downarrow{\scriptstyle\varphi} & & \downarrow{\scriptstyle\Phi}\\ Y & \xrightarrow{\Psi} & Z\end{array}$$

is a Cartesian diagram, i.e. that $W \cong X\times_Z Y = \{(x,y) \mid \Phi(x) = \Psi(y)\}$. Then $\Phi^*\Psi_!(a) = \psi_!\varphi^*(a)$ for any $a\in\mathbb{K}(Y)$.

Proof. Again, it suffices to prove this equality for a generator $a = [A\to Y]$. In this case

$$\begin{aligned}\Phi^*\Psi_!([A\to Y]) &= \Phi^*([A\to Z])\\ &= [X\times_Z A\to X]\\ &= [(X\times_Z Y)\times_Y A\to X]\\ &= [W\times_Y A\to X]\\ &= \psi_![W\times_Y A\to W]\\ &= \psi_!\varphi^*[A\to Y]\end{aligned}$$

as required. □

Corollary 5.6. *The projection and base change formulae hold for classes in the 'small' Grothendieck group, and for constructible functions.*

Proof. This follows immediately from the above two lemmas, Remark 5.3 and Corollary 4.4. ☐

Corollary 5.7. *The proper pushforward of a constructible function along a continuous definable map $\varphi \colon X \to Y$ is given by taking the Euler integral along the fibre:*

$$\varphi_!(f)(y) = \imath_y^* \varphi_!(f) = \varphi_! \imath_{\varphi^{-1}y}^*(f) = \varphi_!(f|_{\varphi^{-1}y}) = \int_{\varphi^{-1}y} f|_{\varphi^{-1}y}\, d\chi$$

by base change and (2), where $\imath_y$ and $\imath_{\varphi^{-1}y}$ denote the respective inclusions.

Example 5.8. Suppose that $C \subset \mathbb{R}^n$ is a cell with compact closure $\overline{C}$. Let $\pi \colon \mathbb{R}^n \to \mathbb{R}^{n-1}$ be the projection onto the first $n-1$ coordinates. Using the functoriality of proper pushforward, Corollary 5.7, and the easily-checked fact that $\chi_c\left(\overline{D}\right) = 1$ for any cell $D \subset \mathbb{R}$ with compact closure, we have

$$\chi_c\left(\overline{C}\right) = \int_{\overline{C}} 1_{\overline{C}}\, d\chi = \int_{\pi\overline{C}} \pi_! 1_{\overline{C}}\, d\chi = \int_{\pi\overline{C}} 1_{\pi\overline{C}}\, d\chi = \chi_c\left(\pi\overline{C}\right).$$

Hence $\chi_c\left(\overline{C}\right) = 1$ by induction on n.

Example 5.9 (Riemann–Hurwitz formula)**.** Suppose $\varphi \colon X \to Y$ is a definable d-fold cover, ramified over a finite set of points in Y. Then

$$\begin{aligned}
\chi_c(X) &= \int_Y \varphi_! 1_X\, d\chi \\
&= \int_Y \chi_c\left(\varphi^{-1}y\right) d\chi \\
&= \int_Y \left(d \cdot 1_Y + \sum_{y \in Y} (\#\{\varphi^{-1}y\} - d) 1_y \right) d\chi \\
&= \chi_c(Y) - \sum_{y \in Y} (\#\{\varphi^{-1}y\} - d).
\end{aligned}$$

Example 5.10 (Morse theory)**.** Suppose $\varphi \colon X \to \mathbb{R}$ is a definable Morse function on a compact manifold. Let $x_1, \ldots, x_k$ be the critical points, with respective indices $\alpha_1, \ldots, \alpha_k$ and ordered so that $\varphi(x_1) < \cdots < \varphi(x_k)$. Let $\pi \colon X \to \mathbb{R}$ be the map to a point. Then, for $\epsilon < \min\{|f(x_i) - f(x_{i-1})|\|$,

$$\begin{aligned}
\chi_c(X) = \pi_! \varphi_!(1_X) &= \sum_{i=1}^k \left(\chi_c(\varphi^{-1}(\varphi x_i)) - \chi_c(\varphi^{-1}(\varphi x_i - \epsilon)) \right) \\
&= \sum_{i=1}^k \left(1 - \chi_c(S^{\alpha_i - 1})\right) = \sum_{i=1}^k (-1)^{\alpha_i}
\end{aligned}$$

because the fibre $\varphi^{-1}(\varphi x_i))$ is obtained from the fibre $\varphi^{-1}(\varphi x_i - \epsilon)$ by collapsing an $(\alpha_i - 1)$-sphere to a point, and $\chi_c(S^d) = 1 + (-1)^d$.

5.2. **Integral transforms.** Let X and Y be definable spaces, and consider a class $b \in \mathbb{K}(X \times Y)$. The *integral transform with kernel c* is defined to be the map

$$\mathbb{K}(X) \to \mathbb{K}(Y) : a \mapsto \pi_{2!}\left(\pi_1^* a \cdot b\right),$$

where $X \xleftarrow{\pi_1} X \times Y \xrightarrow{\pi_2} Y$ are the projections. In general it is a homomorphism of abelian groups, but not of rings.

Examples 5.11. (1) Let $b = [X \times Y]$. Then, either by direct calculation or by the projection formula,

$$\begin{aligned}\pi_{2!}\left(\pi_1^*[A] \cdot c\right) &= \pi_{2!}[A \times_X (X \times Y) \times_{X \times Y} (X \times Y)] \\ &= \pi_{2!}[A \times Y] \\ &= \chi_c(A)\,[Y].\end{aligned}$$

More generally, $\pi_{2!}\left(\pi_1^* a \cdot c\right) = (\pi_! a)[Y]$ where $\pi \colon X \to \mathrm{pt}$.

(2) Let $b = [\Gamma_\varphi]$ be the class of the graph of a continuous definable map $\varphi \colon X \to Y$. Then the corresponding integral transform $\mathbb{K}(X) \to \mathbb{K}(Y)$ is simply the proper pushforward $\varphi_!$ since, for a generator $[A \hookrightarrow X] \in \mathbb{K}(X)$, we have

$$\begin{aligned}\pi_{2!}\left(\pi_1^*[A] \cdot [\Gamma_\varphi]\right) &= \pi_{2!}[A \times_X (X \times Y) \times_{X \times Y} \Gamma_\varphi] \\ &= \pi_{2!}[A \times Y \times_{X \times Y} \Gamma_\varphi] \\ &= \pi_{2!}[\Gamma_{\varphi|_A}] \\ &= \varphi_![A].\end{aligned}$$

Transposing we may equally consider the graph as a class in $\mathbb{K}(Y \times X)$ giving an integral transform $\mathbb{K}(Y) \to \mathbb{K}(X)$. In this case we obtain the pullback φ^* — given a generator $[B \hookrightarrow Y] \in \mathbb{K}(Y)$ we compute

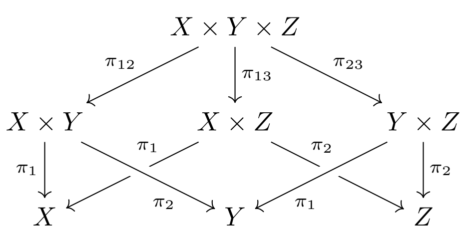
$$\begin{aligned}\pi_{1!}\left(\pi_2^*[B] \cdot [\Gamma_\varphi]\right) &= \pi_{1!}[B \times_Y (X \times Y) \times_{X \times Y} \Gamma_\varphi] \\ &= \pi_{1!}[X \times B \times_{X \times Y} \Gamma_\varphi] \\ &= \pi_{1!}[\Gamma_{\varphi|_{\varphi^{-1}B}}] \\ &= \varphi^*[B].\end{aligned}$$

Proposition 5.12. *The composite of the integral transforms with respective kernels $b \in \mathbb{K}(X \times Y)$ and $c \in \mathbb{K}(Y \times Z)$ is the integral transform with kernel*

$$\pi_{13!}\left(\pi_{12}^* b \cdot {\pi_{23}}^* c\right) \in \mathbb{K}(X \times Z)$$

where π_{ij} is projection onto the ith and jth factors of $X \times Y \times Z$.

Proof. Consider the commutative diagram



in which the central 'diamond' is Cartesian, and all maps are projections. (In what follows the ambiguity arising from denoting all projections onto the first factor by π_1, and so on, is easily resolved by context.) Then for $a \in \mathbb{K}(X)$, $b \in \mathbb{K}(X \times Y)$ and $c \in \mathbb{K}(Y \times Z)$ we have, using base change, the projection formula, functoriality of proper pushforward and pullback, and the fact that pullback is a ring homomorphism,

$$\begin{aligned}\pi_{2!}\left(\pi_1^*\pi_{2!}\left(\pi_1^*a\cdot b\right)\cdot c\right) &= \pi_{2!}\left(\pi_{23!}\pi_{12}^*\left(\pi_1^*a\cdot b\right)\cdot c\right)\\ &= \pi_{2!}\left(\pi_{23!}\left(\pi_{12}^*\pi_1^*a\cdot \pi_{12}^*b\right)\cdot c\right)\\ &= \pi_{2!}\left(\pi_{23!}\left(\pi_{13}^*\pi_1^*a\cdot \pi_{12}^*b\right)\cdot c\right)\\ &= \pi_{2!}\pi_{23!}\left(\pi_{13}^*\pi_1^*a\cdot \pi_{12}^*b\cdot \pi_{23}^*c\right)\\ &= \pi_{2!}\pi_{13!}\left(\pi_{13}^*\pi_1^*a\cdot \pi_{12}^*b\cdot \pi_{23}^*c\right)\\ &= \pi_{2!}\left(\pi_1^*a\cdot \pi_{13!}\left(\pi_{12}^*b\cdot \pi_{23}^*c\right)\right)\end{aligned}$$

as claimed. □

Example 5.13. Let $b = [\Gamma_\varphi]$ and $c = [\Gamma_\phi]$ be the classes of graphs of continuous definable maps $\varphi\colon X \to Y$ and $\psi\colon Y \to Z$. Then $\pi_{12}^*b \cdot \pi_{23}{}^*c$ is the class of

$$\{(x, \varphi(x), \psi\varphi(x)) \in X \times Y \times Z\}\,.$$

Hence the kernel of the composite transform is the class of the graph $\Gamma_{\psi\varphi}$ of the composite. This gives a common restatement of the fact that $(\psi\varphi)^* = \varphi^*\psi^*$ and $(\psi\varphi)_! = \psi_!\varphi_!$.

The inversion formula of [Sch95] is a corollary of the above proposition.

Corollary 5.14 (Schapira's inversion formula)**.** *Suppose that* $B \hookrightarrow X \times Y$ *and* $C \hookrightarrow Y \times X$ *and consider the projection*

$$\pi_{13}\colon \{(x, y, x') \mid (x, y) \in B \text{ and } (y, x') \in C\} \to X^2.$$

Suppose further that

$$\chi_c\left(\pi_{13}^{-1}(x, x')\right) = \begin{cases} m & x = x' \\ n & x \neq x'. \end{cases}$$

Then the composite of the transforms with respective kernels $b = [B \to X \times Y]$ *and* $c = [C \to Y \times X]$ *is*

$$\mathbb{K}(X) \to \mathbb{K}(X) : a \mapsto (m - n)a + (n\,\pi_!a)[X],$$

where $\pi\colon X \to \mathrm{pt}$ *is the map to a point.*

Proof. The given conditions imply that

$$\pi_{13!}\left(\pi_{12}^*b \cdot \pi_{23}^*c\right) = m[\Delta] + n[X^2 - \Delta] = (m - n)[\Delta] + n[X^2]$$

where $\Delta \subset X^2$ is the diagonal. The result then follows from the calculations in Examples 5.11. □

Some surprising applications of this inversion formula to tomography can be found in [Sch95].

6. Duality

The most important operation we have not yet discussed is duality. From the sheaf-theoretic perspective this is the de-categorification of Poincaré–Verdier duality on the constructible derived category. We give an elementary construction of the dual in terms of constructible functions.

Definition 6.1. Fix definable $X \subset \mathbb{R}^n$. For a cell $C \subset X$ we define the dual to be $\mathbb{D}_X(1_C) = (-1)^{\dim C} 1_{\overline{C}}$ where $\overline{C} \subset X$ is the closure of C. The dual $\mathbb{D}_X : CF(X) \to CF(X)$ is defined by extending linearly, using the fact that any constructible function can be expressed as a linear combination of indicator functions of cells.

Lemma 6.2. *The dual is a well-defined homomorphism, and*

$$\mathbb{D}_X(f)(x) = \int_{B_\epsilon(x)} f\, d\chi$$

for any sufficiently small $\epsilon > 0$*, where* $B_\epsilon(x) = \{y \in \mathbb{R}^n \mid \max\{\|x_i - y_i\| < \epsilon\}$ *is the open* ϵ*-ball about* x *in the* L_∞ *metric. In particular,* $\mathbb{D}_X(1_Y) = \chi_c(Y \cap B_\epsilon(x))$ *for any definable* $Y \subset X$*.*

Proof. To show that $\mathbb{D}_X$ is well-defined it suffices to show that $\mathbb{D}_X(1_C)$ is the same when computed using any cell decomposition $\{D_i \mid i \in I\}$ of a cell C. Fix $x \in \overline{C}$. Let $B = B_\epsilon(x)$. Cell decompositions are finite so that, for any sufficiently small $\epsilon > 0$,

$$x \in \overline{D_i} \iff B \cap D_i \neq \emptyset.$$

Moreover, $B \cap C$ is a cell of the same dimension as C and $\{B \cap D_i \neq \emptyset\}$ is a cell decomposition of $B \cap C$ with $\dim(B \cap D_i) = \dim(D_i)$ whenever the intersection is non-empty. Therefore

$$\sum_{x \in \overline{D_i}} (-1)^{\dim D_i} = \sum_{B \cap D_i \neq \emptyset} (-1)^{\dim(B \cap D_i)} = \chi_c(B \cap C) = (-1)^{\dim C}$$

and $\mathbb{D}_X(1_C)$ can indeed be computed using the cell decomposition $\{D_i\}$ of C. The integral formula for $\mathbb{D}_X(f)$ follows immediately from the above computation. The dual $\mathbb{D}_X$ is a homomorphism by construction. □

Example 6.3. On a point the dual is the identity. On the real line we have $\mathbb{D}_\mathbb{R}(1_t) = 1_t$, and $\mathbb{D}_\mathbb{R}(1_{(s,t)}) = -1_{[s,t]}$ with the obvious analogue for the indicator function of an infinite open interval. Note that $\mathbb{D}_R^2$ is the identity on $CF(\mathbb{R})$ since

$$\mathbb{D}_\mathbb{R}^2(1_{(s,t)}) = -\mathbb{D}_\mathbb{R}(1_{[s,t]}) = -\mathbb{D}_\mathbb{R}(1_s + 1_{(s,t)} + 1_t) = -1_s + 1_{[s,t]} - 1_t = 1_{(s,t)}$$

and so on. In the next section we will see that $\mathbb{D}_X^2$ is the identity for any locally-compact X.

Definition 6.4. A subset $M \subset \mathbb{R}^n$ is a *k-dimensional definable manifold* if it is locally definably homeomorphic to $\mathbb{R}^k$, i.e. if for each $x \in M$ there is a definable open neighbourhood $U_x \subset M$ which is definably homeomorphic to an open neighbourhood of 0 in $\mathbb{R}^k$.

Clearly $\mathbb{R}^k$ is a definable manifold, as is the sphere S^k (with coordinate projections providing the local homeomorphisms). The next lemma follows immediately from the fact that the dual is local, and that $\mathbb{D}_{\mathbb{R}^k} 1_{\mathbb{R}^k} = (-1)^k 1_{\mathbb{R}^k}$.

Lemma 6.5. *If* M *is a* k*-dimensional definable manifold then* $\mathbb{D}_M(1_M) = (-1)^k 1_M$.

Remark 6.6. The dual can also be interpreted using integral transforms — this is the approach taken, for example, in [Sch91]. Let $\Delta_\epsilon = \{(x, x') \in X \mid \epsilon > ||x - x'||\}$ where $||x - x'|| = \max_i\{|x_i - x'_i|\}$. For $f \in CF(X)$ we define

$$\mathbb{D}_X(f) = \lim_{\epsilon \to 0} \pi_{2!}\left(\pi_1^* f \cdot 1_{\Delta_\epsilon}\right).$$

The value at $x \in X$ is therefore

$$\mathbb{D}_X(f)(x) = \lim_{\epsilon \to 0} \int_{X \times x} f \pi_1|_{\Delta_\epsilon}\, d\chi = \lim_{\epsilon \to 0} \int_{B_\epsilon(x)} f\, d\chi,$$

in agreement with Lemma 6.2. One can give a direct argument, without reference to cell decompositions, that this limit is well-defined. It suffices to do so when $f = 1_A$ is an indicator function. In this case $\mathbb{D}_X(1_A)(x) = \lim_{\epsilon \to 0} \chi_c(A \cap B_\epsilon(x))$. Consider the class of the definable subspace $(A \times \mathbb{R}) \cap C \to \mathbb{R}$ in $\mathbb{K}(\mathbb{R})$, where

$$C = \{(x', \epsilon) \in X \times \mathbb{R} \mid \epsilon > ||x' - x||^2\}$$

and the map is projection onto the second factor. Applying $\mathcal{E}_{\mathbb{R}}$ to this class we obtain a constructible function on $\mathbb{R}$ whose value at ϵ is $\chi_c(A \cap B_\epsilon(x))$. It follows that this quantity is constant for ϵ in some interval $(0,t)$ for $t>0$.

6.1. **The dual on locally-compact spaces.** Observe that $(\mathbb{D}_X - 1)1_C = 1_{\partial C}$ for even-dimensional cells, and $(\mathbb{D}_X + 1)1_C = -1_{\partial C}$ for odd-dimensional cells. Hence, by considering the dimension of the support, we see that $\mathbb{D}_X^2 - 1$ is nilpotent. In this section we show that $\mathbb{D}_X^2 - 1 = 0$ when X is locally-compact; there are easy examples which show this is false for more general X.

We introduce an operation of 'fibrewise dual' on a product space $X \times Y$. The *fibrewise dual in the second factor* is the limit as $\epsilon \to 0$ of the integral transform with kernel

$$\{(x,y,x,y') \in (X\times Y)^2 \mid \epsilon > ||y-y'||\}.$$

Extending our previous notation (which corresponds to the case in which $X = \mathrm{pt}$) we denote the resulting operation by $\mathbb{D}_Y \colon CF(X\times Y) \to CF(X \times Y)$. Explicitly

$$(\mathbb{D}_Y f)(x,y) = \lim_{\epsilon\to 0} \int_{x\times B_\epsilon(y)} f\, d\chi = \mathbb{D}_Y(f|_{x\times Y})(y).$$

An analogous argument to that in Remark 6.6 shows that this is well-defined, and is a homomorphism.

Lemma 6.7. *The dual on the product $X \times Y$ is given by*

$$\mathbb{D}_Y(\mathbb{D}_X f|_{x\times Y})(y) = (\mathbb{D}_{X\times Y} f)(x,y) = \mathbb{D}_X(\mathbb{D}_Y f|_{X\times y})(x),$$

or, for short, $\mathbb{D}_X\mathbb{D}_Y = \mathbb{D}_{X\times Y} = \mathbb{D}_Y\mathbb{D}_X$.

Proof. For sufficiently small $\epsilon > 0$

$$(\mathbb{D}_{X\times Y} f)(x,y) = \int_{B_\epsilon(x,y)} f\, d\chi = \int_{B_\epsilon(x)} \left(\int_{x\times B_\epsilon(y)} f\, d\chi \right) d\chi = \mathbb{D}_X\left(\mathbb{D}_Y f|_{X\times y}\right)(x).$$

By symmetry the result follows. □

Corollary 6.8. *The dual on $\mathbb{R}^n$ is involutory, i.e. $\mathbb{D}_{\mathbb{R}^n}^2 = \mathrm{id}$.*

Proof. Write $\mathbb{R}^n = X_1 \times \cdots \times X_n$ where $X_i = \langle e_i \rangle$ with e_i the ith standard basis vector. By Lemma 6.7 and Example 6.3 $\mathbb{D}_{\mathbb{R}^n}^2 = \mathbb{D}_{X_1}\cdots\mathbb{D}_{X_n}\cdot\mathbb{D}_{X_n}\cdots\mathbb{D}_{X_1} = \mathrm{id}$. □

Lemma 6.9. *Suppose $\imath \colon X \hookrightarrow Y$ is a definable closed embedding of a subspace of $Y \subset \mathbb{R}^n$. Then $\imath_!\mathbb{D}_X = \mathbb{D}_Y\imath_!$.*

Proof. Since X is closed, and $\imath_! f$ is the extension by zero of $f \in CF(X)$ to a function in $CF(Y)$,

$$\mathbb{D}_Y \imath_! f(y) = \begin{cases} \int_{B_\epsilon(y)} f\, d\chi & y \in X \\ 0 & y \notin X. \end{cases}$$

This is the function $\imath_!\mathbb{D}_X(f)$. □

Proposition 6.10. *Suppose that X is a locally compact definable space. Then $\mathbb{D}_X$ is an involution on $CF(X)$.*

Proof. The dual is local, so we may assume that X is actually compact. In particular, we may assume that the embedding $\imath\colon X \hookrightarrow \mathbb{R}^n$ is closed. Then

$$\mathbb{D}_X^2 = \imath^*\imath_!\mathbb{D}_X^2 = \imath^*\mathbb{D}_{\mathbb{R}^n}^2\imath_! = \imath^*\imath_! = \mathrm{id}$$

by Corollary 6.8, Lemma 6.9, and the fact that $\imath^*\imath_! = \mathrm{id}$. □

6.2. **The dual, pushforward and pullback.** In this section we discuss the interaction between the dual and both proper pushforward and pullback. We have already seen in Lemma 6.9 that the dual commutes with the proper pushforward of a closed embedding. Here is another easy case:

Lemma 6.11. *Suppose X is compact and $\pi\colon X \to \mathrm{pt}$ is the map to a point. Then $\pi_! \mathbb{D}_X = \pi_! = \mathbb{D}_{\mathrm{pt}} \pi_!$.*

Proof. It suffices to compute for the indicator function 1_C of a cell $C \subset X$. In this case

$$\pi_! \mathbb{D}_X 1_C = (-1)^{\dim C} \pi_! 1_{\overline{C}} = (-1)^{\dim C} \chi_c \left(\overline{C}\right) = (-1)^{\dim C} = \pi_! 1_C$$

by Example 5.8 because the closure of C in X and in $\mathbb{R}^n$ is the same, and is compact. The second equality is obvious since $\mathbb{D}_{\mathrm{pt}}$ is the identity. □

Corollary 6.12. *If M is a compact odd-dimensional manifold then $\chi_c(M) = 0$.*

Proof. By Lemmas 6.11 and 6.5, $\chi_c(M) = \pi_! 1_M = \pi_! \mathbb{D}_M(1_M) = \pi_!(-1_M) = -\chi_c(M)$. Hence $\chi_c(M) = 0$. □

Proposition 6.13. *Suppose that $\varphi\colon X \to Y$ is a proper continuous definable map, and that Y is locally compact space. Then $\varphi_! \mathbb{D}_X = \mathbb{D}_Y \varphi_!$.*

Proof. This is a local question on Y. Since Y is locally compact and φ proper we may assume both X and Y are compact. Factorise φ as $X \xrightarrow{\imath} X \times Y \xrightarrow{\pi_2} Y$ where $\imath$ is the inclusion of the graph. Since $\imath$ is a closed embedding Lemma 6.9 allows us to reduce to proving that $\pi_{2!} \mathbb{D}_{X \times Y} = \mathbb{D}_Y \pi_{2!}$. The fibre $X \times y$ of π_2 is compact, so for sufficiently small $\epsilon > 0$,

$$\int_{\varphi^{-1} y} \mathbb{D}_Y \mathbb{D}_X f \, d\chi = \int_{\varphi^{-1} y} \int_{x \times B_\epsilon(y)} \mathbb{D}_X f \, d\chi \, d\chi = \int_{B_\epsilon(y)} \int_{\varphi^{-1} y'} f \, d\chi \, d\chi.$$

It follows that $\pi_{2!} \mathbb{D}_{X \times Y} = \mathbb{D}_Y \pi_{2!} \mathbb{D}_X$. The result then follows from Lemma 6.11 because X is compact. □

Remark 6.14. The dual is intrinsic, i.e. $\mathbb{D}_X$ does not depend on the embedding of X into $\mathbb{R}^n$, but only on the definable homeomorphism type of X. This follows immediately from the last result and the fact that homeomorphisms are proper. This gives us some freedom in how we compute. For instance, if $C \subset X$ is any subset definably homeomorphic to a cell then $\mathbb{D}_X 1_C = (-1)^{\dim C} 1_{\overline{C}}$. Similarly, we do not have to use cubical cells in the integral formula:

$$\mathbb{D}_X f(x) = \int_C f \, d\chi$$

for any sufficiently small cell $C \ni x$, indeed for any such C definably homeomorphic to a cell.

In general it is not true that $\varphi_! \mathbb{D}_X = \mathbb{D}_Y \varphi_!$. We define a new operation of *direct image* $\varphi_*\colon CF(X) \to CF(Y)$ by

$$\varphi_* = \mathbb{D}_Y \varphi_! \mathbb{D}_X. \tag{3}$$

It follows from the above Proposition that $\varphi_* = \varphi_!$ when φ is proper and Y locally compact. If both X and Y are locally compact then, without any assumptions on φ, we have $\varphi_! \mathbb{D}_X = \mathbb{D}_Y \varphi_*$ and $\varphi_* \mathbb{D}_X = \mathbb{D}_Y \varphi_!$.

We conclude this section with a parallel discussion of the interaction between the dual and pullback.

Proposition 6.15. *Suppose that $\varphi\colon X \to Y$ is a local homeomorphism. Then $\varphi^* \mathbb{D}_Y = \mathbb{D}_X \varphi^*$.*

Proof. Since the dual is local $\jmath^*\mathbb{D}_X = \mathbb{D}_U\jmath^*$ when $\jmath\colon U \hookrightarrow X$ is an open embedding. Hence we can reduce to the case in which φ is a homeomorphism. The result then follows from Proposition 6.13 since $\varphi^* = \varphi_!^{-1}$ and φ is proper. □

In general it is not true that $\varphi^*\mathbb{D}_Y = \mathbb{D}_X\varphi^*$. We define a new operation of *exceptional inverse image* $\varphi^!\colon CF(Y) \to CF(X)$ by

$$\varphi^! = \mathbb{D}_X\varphi^*\mathbb{D}_Y. \tag{4}$$

For example, if $\pi\colon X \to \mathrm{pt}$ is the map to a point then $\pi^! 1_{\mathrm{pt}} = \mathbb{D}_X\pi^*\mathbb{D}_{\mathrm{pt}}1_{\mathrm{pt}} = \mathbb{D}_X 1_X$. When φ is a local homeomorphism and X (hence also Y) is locally compact $\varphi^! = \varphi^*$. Without any assumptions on φ we have $\varphi^!\mathbb{D}_Y = \mathbb{D}_X\varphi^*$ and $\varphi^*\mathbb{D}_Y = \mathbb{D}_X\varphi^!$ when X and Y are both locally compact.

6.3. **Euler characteristic.** Using duality we can define the *Euler characteristic* $\chi(X) = \pi_!\mathbb{D}_X 1_X = \pi_!\pi^! 1_{\mathrm{pt}}$. This can be computed in terms of a decomposition $X = \bigsqcup_{i\in I} X_i$ in which each X_i is definably homeomorphic to a cell:

$$\chi(X) = \pi_!\mathbb{D}_X 1_X = \pi_!\left(\sum_{i\in I}(-1)^{\dim X_i} 1_{\overline{X_i}}\right) = \sum_{i\in I}(-1)^{\dim X_i}\chi_c\left(\overline{X_i}\right).$$

In general the Euler characteristic differs from its compactly supported cousin. For example, it is easy to check that $\chi(\mathbb{R}^n) = 1$ for all n.

Lemma 6.16. *If X is compact then $\chi(X) = \chi_c(X)$.*

Proof. Let $X = \bigsqcup_{i\in I} X_i$ be a decomposition in which each X_i is definably homeomorphic to a cell. Since X is compact each $\overline{X_i}$ is a closed cell and so $\chi_c\left(\overline{X_i}\right) = 1$ by Example 5.8. Hence

$$\chi(X) = \sum_{i\in I}(-1)^{\dim X_i}\chi_c\left(\overline{X_i}\right) = \sum_{i\in I}(-1)^{\dim X_i} = \chi_c(X).$$

□

7. Constructible homology

In this section we construct a homology theory for locally-compact definable spaces from $\mathbb{Z}/2$-valued constructible functions and the dual. The associated Euler characteristic is χ_c. We use $\mathbb{Z}/2$-coefficients to avoid introducing orientations, so that we can work purely in the context of constructible functions. This approach is different from the o-minimal homology theories developed in [EW08]; firstly, that paper treats homology with *compact* support, and secondly, it adapts the standard simplicial and singular homology approaches to the o-minimal setting, whereas we take a more geometric approach working directly with definable subspaces.

Let $\widetilde{CF}(X)$ denote the constructible functions on X with values in $\mathbb{Z}/2$. The whole theory developed in §3,4,5, and §6 goes through with coefficients in $\mathbb{Z}/2$. We use the same notation to denote the various functors for $\mathbb{Z}/2$-valued constructible functions, and the duality on $\widetilde{CF}(X)$. In particular $\widetilde{CF}(X) \cong \widetilde{K}(X)$ is isomorphic to the 'small' Grothendieck group of subspaces with $\mathbb{Z}/2$ coefficients via the map $f \mapsto f^{-1}(1)$. The latter is purely geometric — it is (isomorphic to) the set of definable subspaces of X equipped with the addition $[A] + [B] = [A \cup B - A \cap B]$ and the multiplication $[A] \cdot [B] = [A \cap B]$.

7.1. **Constructible homology.** The abelian group $\widetilde{CF}(X)$ is filtered by the dimension of supports: let $\widetilde{CF}_{\leq i}(X) = \{f \mid \dim \operatorname{supp} f \leq i\}$ be the subgroup of functions with support of dimension less than i. Let $\widetilde{CF}_*(X)$ be the associated graded group, with

$$\widetilde{CF}_i(X) = \widetilde{CF}_{\leq i}(X) \,/\, \widetilde{CF}_{\leq i-1}(X)\,.$$

Lemma 7.1. *Suppose X is locally-compact. Then the operation $\partial = \mathbb{D}_X + 1$ makes $\widetilde{CF}_*(X)$ into a chain complex.*

Proof. Recall from §6.1 that, with $\mathbb{Z}/2$ coefficients, $(\mathbb{D}_X + 1)1_C = 1_{\partial C}$ for any cell C in X. It follows that $\partial = \mathbb{D}_X + 1$ descends to a homomorphism

$$\widetilde{CF}_i(X) \to \widetilde{CF}_{i-1}(X)\,.$$

Since X is locally-compact $(\mathbb{D}_X + 1)^2 = \mathbb{D}_X^2 + 2\mathbb{D}_X + 1 = 1 + 1 = 0$, again because we use $\mathbb{Z}/2$ coefficients. □

Definition 7.2. For locally-compact X define the *constructible homology groups* $CH_*(X)$ to be the homology groups of $\widetilde{CF}_*(X)$. Clearly $CH_i(X)$ vanishes for $i < 0$ and for $i > \dim X$. In the next section we will see that $CH_*(X)$ is finite-dimensional.

Proposition 7.3. *Homology is functorial for proper continuous definable maps.*

Proof. Suppose $\varphi\colon X \to Y$ is a proper continuous definable map between locally-compact spaces. Since $\operatorname{supp}(\varphi_! f) \subset \varphi(\operatorname{supp}(f))$, it is clear that $\dim \operatorname{supp}(\varphi_! f) \leq \dim \operatorname{supp}(f)$. By Proposition 6.13 $(\mathbb{D}_Y + 1)\varphi_! f = \varphi_!(\mathbb{D}_X + 1)f$ so that $\varphi_!$ induces a chain map $\widetilde{CF}_*(X) \to \widetilde{CF}_*(Y)$, and thence a homomorphism $CH_*(X) \to CH_*(Y)$. □

The proof of the following corollary is standard; we omit it since we give an alternative proof in Corollary 7.13.

Corollary 7.4. *Constructible homology is a proper definable homotopy invariant.*

Remark 7.5. Recall that a semialgebraic set is definable in any o-minimal structure. Sullivan [Sul71] proved that the Euler characteristic of the link L_x of any point x in an algebraic set $V \subset \mathbb{R}^n$ is even. Thus, working in $\mathbb{Z}/2$,

$$\mathbb{D}_X 1_V(x) = \chi_c\left(B_\epsilon(x) \cap V; \mathbb{Z}/2\right) = 1 + \chi_c\left(L_x\right) = 1$$

and 1_V is a cycle for constructible homology. That is, any algebraic set V has a fundamental class in $[1_V] \in CH_{\dim V}(V)$.

7.2. **Cellular constructible homology.** In order to relate homology to Euler characteristic, and more generally to compute the homology groups, we now consider homology defined with respect to a given cell decomposition X. Fix a cell decomposition $\mathcal{D}$ satisfying the frontier condition, and let $\widetilde{CF}^{\mathcal{D}}(X)$ denote the functions constructible with respect to this decomposition, i.e. the subgroup of $\widetilde{CF}(X)$ generated by the indicator functions of cells in this decomposition. We require the frontier condition to hold in order that the dual $\mathbb{D}_X$ restricts to an endomorphism of $\widetilde{CF}^{\mathcal{D}}(X)$. The construction of the previous section can be carried out in the same way in this context, to define a chain complex $\widetilde{CF}^{\mathcal{D}}_*(X)$ and homology groups $CH^{\mathcal{D}}_*(X)$. The group $\widetilde{CF}^{\mathcal{D}}_i(X)$ is (isomorphic to) the $\mathbb{Z}/2$-vector space with basis the indicator functions of the i-cells in the decomposition $\mathcal{D}$. In particular it is finite-dimensional, so that $CH^{\mathcal{D}}_*(X)$ is finite dimensional.

Fix cell decompositions $\mathcal{D}$ of X and $\mathcal{D}'$ of Y. Suppose $\varphi\colon X \to Y$ is proper, and that for each cell $C \in \mathcal{D}$ the image $\varphi(C)$ is a union of cells of $\mathcal{D}'$ (equivalently, the preimage $\varphi^{-1}C'$ of each $C' \in \mathcal{D}'$ is contained in a single cell of $\mathcal{D}$). Then, as in Proposition 7.3, $\varphi_!$ induces a chain map $\widetilde{CF}_*^{\mathcal{D}}(X) \to \widetilde{CF}_*^{\mathcal{D}'}(Y)$ and thence a homomorphism $CH_*^{\mathcal{D}}(X) \to CH_*^{\mathcal{D}'}(Y)$. In particular, if φ is the identity map on $X = Y$ and $\mathcal{D}'$ is a subdivision of $\mathcal{D}$, the induced chain map is injective, given explicitly by

$$1_C \mapsto \sum_{C' \in I(C)} 1_{C'}, \tag{5}$$

where $C \in \mathcal{D}$ and $I(C) = \{C' \in \mathcal{D}' \mid C' \subset C, \dim C' = \dim C\}$. Since any constructible function can be written as a finite sum of indicator functions of cells in some decomposition, which may be chosen to satisfy the frontier condition,

$$\widetilde{CF}_*(X) \cong \operatorname{colim}_{\mathcal{D}} \widetilde{CF}_*^{\mathcal{D}}(X)\,.$$

Here the colimit is taken over cell decompositions of X satisfying the frontier condition and the injective chain maps between the associated complexes defined in (5). It follows that

$$CH_*(X) \cong \operatorname{colim}_{\mathcal{D}} CH_*^{\mathcal{D}}(X)\,. \tag{6}$$

Proposition 7.6. *For any cell decomposition of a locally-compact definable space X which satisfies the frontier condition $CH_*^{\mathcal{D}}(X) \cong CH_*(X)$.*

Proof. Using the colimit description of $CH_*(X)$ in (6) it suffices to prove that $CH_*^{\mathcal{D}}(X) \cong CH_*^{\mathcal{D}'}(X)$ for any subdivision $\mathcal{D}'$ of $\mathcal{D}$. Since (5) is injective, this is equivalent to showing that

$$\widetilde{CF}_*^{\mathcal{D}',\mathcal{D}}(X) = \widetilde{CF}_*^{\mathcal{D}'}(X) \big/ \widetilde{CF}_*^{\mathcal{D}}(X)$$

is acyclic whenever $\mathcal{D}'$ is a subdivision of $\mathcal{D}$. By Proposition 2.5 we can reduce to the case of an *elementary* subdivision. Suppose this has depth k with $\{C \mid \pi_k C = D\}$ being the subdivided cells. Then

$$\widetilde{CF}_*^{\mathcal{D}',\mathcal{D}}(X) = \langle 1_{C^\pm}, 1_{C^0} \mid \pi_k C = D \rangle \big/ \langle 1_{C^-} + 1_{C^+} \rangle$$

and $\partial 1_{C^\pm} = 1_{C^0} + f_{C^\pm}$ where $f_{C^\pm}$ is a sum of indicator functions of the form $1_{E^\pm}$. Suppose α is a cycle in $\widetilde{CF}_*^{\mathcal{D}',\mathcal{D}}(X)$. Let $\beta = \sum \alpha_{C^0} 1_{C^+}$ where α_{C^0} is the coefficient of 1_{C^0} in α. Then

$$\partial\beta - \alpha = \sum \gamma_{C^-} 1_{C^-} + \sum \gamma_{C^+} 1_{C^+}$$

for some coefficients $\gamma_{C^\pm}$. It follows that the coefficient of 1_{C^0} in $\partial(\partial\beta - \alpha)$ is $\gamma_{C^-} + \gamma_{C^+}$. Since $\partial(\partial\beta - \alpha) = 0$ we deduce that $\gamma_{C^-} = \gamma_{C^+}$ for all C. Therefore $\partial\beta - \alpha = 0$ because $1_{C^-} + 1_{C^+} = 0$ in $\widetilde{CF}_*^{\mathcal{D}',\mathcal{D}}(X)$. So α is a boundary, and $\widetilde{CF}_*^{\mathcal{D}',\mathcal{D}}(X)$ is acyclic as claimed. □

An immediate consequence is that constructible homology is finite-dimensional. This result also gives a method for computing constructible homology from a finite-dimensional chain complex.

Examples 7.7. By choosing the trivial decomposition with a single cell we see that $CH_i(\mathbb{R}^n) \cong \mathbb{Z}/2$ for $i = n$ and vanishes otherwise.

For the sphere S^d we can use the cell decomposition of Example 3.3. For this $\widetilde{CF}_i^{\mathcal{D}}(S^d) \cong (\mathbb{Z}/2)^2$ for $0 \le i \le d$, and vanishes otherwise, with each possibly non-zero boundary map given by the 2×2 matrix with all entries 1. Hence $CH_i(S^d) \cong \mathbb{Z}/2$ for $i = 0, d$, and vanishes otherwise.

Remark 7.8. We have three different viewpoints on constructible homology. The definition is roughly analogous to that of simplicial homology, in which chains are formal sums of contractible subspaces. Using the isomorphism $\widetilde{CF}(X) \cong \mathbb{K}(X;\mathbb{Z}/2)$ we get a description more akin to that of singular homology, in which chains are formal sums of maps into the space. Finally, fixing a cell decomposition we obtain an analogue $CH_*^{\mathcal{D}}(X)$ of cellular homology.

Corollary 7.9. *Suppose X is a locally-compact definable space. Then there is an isomorphism $CH_*(X) \cong H_*^{BM}(X;\mathbb{Z}/2)$ with the Borel–Moore homology of X with coefficients in $\mathbb{Z}/2$. In particular, $CH_*(X) \cong H_*(X;\mathbb{Z}/2)$ for compact X.*

Proof. Let $\widetilde{X}$ be the one point compactification of X, with the added point denoted by ∞. A cell decomposition $\mathcal{D}$ of X satisfying the frontier condition endows $\widetilde{X}$ with a CW-complex structure. There is an isomorphism of chain complexes

$$\widetilde{CF}_*^{\mathcal{D}}(X) \cong C_*(\widetilde{X};\mathbb{Z}/2)/C_*(\infty;\mathbb{Z}/2) = C_*(\widetilde{X},\infty;\mathbb{Z}/2)$$

where $C_*(Y;\mathbb{Z}/2)$ is the cellular chain complex, with coefficients in $\mathbb{Z}/2$, of a CW-complex Y. The Borel–Moore homology of X is, by definition, the relative group $H_*(\widetilde{X},\infty)$ so that the above chain isomorphism induces the required isomorphism $CH_*(X) \cong H_*^{\mathrm{BM}}(X;\mathbb{Z}/2)$. □

Remark 7.10. The constructible homology $CH_*(X)$ can be computed using any finite decomposition of X, satisfying the frontier condition, into subsets definably homeomorphic to cells. This is because any such decomposition gives a CW-structure on the one-point compactification $\widetilde{X}$ of X. For instance, using the decomposition of S^d into a point and its complement we see immediately that $CH_i(X) \cong \mathbb{Z}/2$ for $i = 0, d$ and vanishes otherwise.

Corollary 7.11. *The compactly supported Euler characteristic of a locally-compact space X is given by*

$$\chi_c(X) = \sum_i (-1)^i \dim CH_i(X) = \sum_i (-1)^i \dim H_*^{BM}(X;\mathbb{Z}/2).$$

Proof. For any cell decomposition $\mathcal{D}$ satisfying the frontier condition

$$\begin{aligned}\chi_c(X) &= \sum_{C\in\mathcal{D}} (-1)^{\dim C} \\ &= \sum_i (-1)^i \dim \widetilde{CF}_i^{\mathcal{D}}(X) \\ &= \sum_i (-1)^i \dim CH_i^{\mathcal{D}}(X) \\ &= \sum_i (-1)^i \dim CH_i(X) \\ &= \sum_i (-1)^i \dim H_*^{\mathrm{BM}}(X;\mathbb{Z}/2)\end{aligned}$$

by Proposition 7.6 and Corollary 7.9. □

Remark 7.12. It is a standard fact of homological algebra that the alternating sum of ranks of homology groups does not depend on the coefficients used; so equally $\chi_c(X) = \sum_i (-1)^i \dim H_*^{\mathrm{BM}}(X;\mathbb{Z})$.

Corollary 7.13. *The constructible homology $CH_*(X)$ and compactly supported Euler characteristic $\chi_c(X)$ are proper homotopy invariants of X. In particular they do not depend on which o-minimal structure one uses to define them, provided only that X is definable in that structure of course.*

Proof. This follows immediately since $H_*^{\mathrm{BM}}(X;\mathbb{Z}) \cong H^*(X;\mathscr{D}_X)$ is the cohomology of the dualising complex $\mathscr{D}_X$ for any locally-compact X, and this is a proper homotopy invariant [Ive84, §IX.1]. □

References

[CGR12] Justin Curry, Robert Ghrist, and Michael Robinson. Euler calculus with applications to signals and sensing. In *Advances in applied and computational topology*, volume 70 of *Proc. Sympos. Appl. Math.*, pages 75–145. Amer. Math. Soc., Providence, RI, 2012.

[EW08] Mário J. Edmundo and Arthur Woerheide. Comparison theorems for o-minimal singular (co)homology. *Trans. Amer. Math. Soc.*, 360(9):4889–4912, 2008.

[GZ10] S. M. Guseĭn-Zade. Integration with respect to the Euler characteristic and its applications. *Uspekhi Mat. Nauk*, 65(3(393)):5–42, 2010.

[Ive84] Birger Iversen. *Cohomology of sheaves*, volume 55 of *Lecture Notes Series*. Aarhus Universitet, Matematisk Institut, Aarhus, 1984.

[Sch91] Pierre Schapira. Operations on constructible functions. *J. Pure Appl. Algebra*, 72(1):83–93, 1991.

[Sch95] P. Schapira. Tomography of constructible functions. In *Applied algebra, algebraic algorithms and error-correcting codes (Paris, 1995)*, volume 948 of *Lecture Notes in Comput. Sci.*, pages 427–435. Springer, Berlin, 1995.

[Sul71] D. Sullivan. Combinatorial invariants of analytic spaces. In *Proceedings of Liverpool Singularities—Symposium, I (1969/70)*, pages 165–168. Springer, Berlin, 1971.

[vdD98] Lou van den Dries. *Tame topology and o-minimal structures*, volume 248 of *London Mathematical Society Lecture Note Series*. Cambridge University Press, Cambridge, 1998.

[vdDM96] Lou van den Dries and Chris Miller. Geometric categories and o-minimal structures. *Duke Math. J.*, 84(2):497–540, 1996.

[Vir88] O. Ya. Viro. Some integral calculus based on Euler characteristic. In *Topology and geometry—Rohlin Seminar*, volume 1346 of *Lecture Notes in Math.*, pages 127–138. Springer, Berlin, 1988.