Preface to the first edition

These notes are based on a course for graduate students entitled 'A beginner's guide to intersection homology theory' given in Oxford in 1987. The course was intended to be accessible to first year graduate students and to mathematicians from different areas of mathematics. The aim was to give some of the idea of the power, usefulness and beauty of intersection homology theory while only assuming fairly basic mathematical knowledge. To succeed at all in this it was necessary to give at most briefly sketched proofs of the important theorems and to concentrate on explaining the main ideas and definitions. The result is that these notes do not constitute in any sense an introductory textbook on intersection homology. Rather they are intended to be a piece of propaganda on its behalf. The hope is that mathematicians of very varied backgrounds with interests in singular spaces should find the notes readable and should be stimulated to learn in greater depth about intersection homology and use it in their work.

Over the last century ordinary homology theory for manifolds has been applied with enormous success to all sorts of different parts of mathematics. Often however ordinary homology is not as successful in dealing with problems involving singular spaces as with problems involving manifolds. In such situations it is possible that intersection homology (which coincides with ordinary homology for manifolds) may be more successful. Many examples of this phenomenon have been found since intersection homology was introduced a decade ago. It was because exactly this phenomenon has occurred in my own work in the last few years that I became an enthusiast for intersection homology, and, although by no means an expert on the subject, decided to give this course.

The goal I had in mind was to explain enough of the theory of intersection homology to be able to give a sketch (following Bernstein [15]) of the proof of Kazhdan–Lusztig conjecture (Kazhdan–Lusztig [104, 105]). This relates the representation theory of complex Lie algebras to the theory of Hecke algebras via D-modules and intersection homology, and was in fact important motivation in the development of intersection homology theory (cf. Brylinski [36]). It seemed a suitable target at which to aim, though much of the material covered on the way is just as interesting (or more so, depending on one's point of view) in its own right.

This goal influenced the structure of the second half of the course and thus the lecture notes. This first half consists of an elementary introduction to intersection homology theory. The introductory chapter, which is intended as motivation for the reader, describes three situations in which intersection homology is more successful than ordinary homology in dealing with singular spaces. The second chapter describes briefly some standard homology theory and sheaf theory; it would be helpful but not essential for the reader to be already familiar with this material. There are several different ways of defining intersection homology which vary in difficulty and elegance: Chapter 4 gives the most elementary of these and describes some of its basic properties.

The singular spaces given most attention throughout the notes are complex varieties, but intersection homology is defined for more general spaces as well (the most general being topological pseudomanifolds). The fourth chapter discusses the relationship between the intersection homology of singular complex projective varieties and an analytically defined cohomology theory, L^2-cohomology, which is a generalisation of de Rham cohomology for compact manifolds. Chapter 7 describes the important sheaf-theoretic construction and characterisations of intersection homology, due to Deligne and developed in Goresky and MacPherson [70], which imply that intersection homology is a topological invariant.

The final three chapters lead towards the proof of the Kazhdan–Lusztig conjecture which is described in Chapter 12. The tenth chapter discusses the relationship of the intersection homology with the Weil conjectures and the arithmetic of algebraic varieties defined over finite fields, while Chapter 11 describes briefly the theory of D-modules and the Riemann–Hilbert correspondence relating D-modules to intersection homology.

Nothing in these lecture notes is original work. The papers I have used most heavily are those listed in the references by Goresky and MacPherson, Borel, Bernstein, and Beilinson, Bernstein and Deligne. I would like to thank Joseph Bernstein for first suggesting several years ago that I should look at intersection homology, and all those who attended the 'beginner's guide' last year for pointing out many slips and errors. I am also grateful to Valerie Siviter for typing the original manuscript and to Terri Moss for typing the final version.

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April 1988

Preface to the second edition

As a beginning graduate student trying to learn about intersection homology, I found the first edition of this book invaluable, giving, as it did, an accessible treatment with clear and simple sketches of the main ideas. Having digested it I had the confidence to go on to grapple with more specialist, technical texts and a basic framework within which to place them. Since I found it so useful, I am very pleased to be given the opportunity to co-author a second, updated edition.

This edition differs from the first in two respects. Firstly, a number of new topics have been included; some, such as Witt spaces and their bordism groups, signatures for singular spaces, perverse sheaves, and Zucker's conjecture, represent strands of thought which were omitted from the first edition, and others, such as the combinatorial construction of intersection cohomology for fans, represent subsequent developments. Secondly, some of the basic material has been revised and supplemented. The treatment of sheaf cohomology has been expanded, and given its own chapter, and more emphasis has been placed on intersection homology as a topological theory and on the rôle of generalised Poincaré duality. These changes reflect the structure and approach of a graduate course, rather unimaginatively entitled Intersection Cohomology, which I gave in Cambridge in Spring 2004.

Let me list the major revisions and supplements in more detail. The first four chapters constitute the elementary material. The introduction motivates the subject by giving examples of the utility of intersection homology. The old second chapter has been split into two, the first part reviewing simplicial and singular homology, and the second reviewing sheaf cohomology from both the Čech and derived functor viewpoints. The latter contains new material on derived categories of sheaves, which are a fundamental technical tool. The treatment of intersection homology in the fourth chapter has been revised and expanded to apply to pseudomanifolds rather than just to complex projective varieties. The latter now appear as a nice class of examples with especially good properties.

Rational intersection homology satisfies generalised Poincaré duality for a class of singular spaces called Witt spaces. These include all pseudomanifolds with only even dimensional strata, such as complex projective varieties, and also those pseudomanifolds satisfying a certain condition on the links of any odd codimensional strata. It is possible to define a bordism invariant signature for a Witt space. Chapter 5 discusses this material (which was not in the first edition). It culminates in a sketch of Siegel's beautiful computation [162] equating the bordism groups of $4n$-dimensional Witt spaces with the Witt group of symmetric rational bilinear forms.

Chapter 6 explains the relation of intersection cohomology to the analytically defined L^2-cohomology. It now contains a (very) brief introduction to the Hodge theory of L^2-cohomology and a new section, based on Zucker [184], on locally symmetric varieties and Zucker's conjecture.

Chapter 7 explains how the intersection homology groups can be obtained as the (hyper)cohomology of an intersection sheaf complex. This complex can be axiomatically characterised independently of the stratification, leading to a proof of intersection homology's topological invariance. It also has a new section on constructible sheaves and Verdier duality. This duality, a contravariant equivalence on the constructible derived category of sheaves, plays a fundamental rôle in intersection homology theory.

There is a beautiful Abelian subcategory, the perverse sheaves, which is preserved by Verdier duality and whose simple objects are intersection sheaf complexes, possibly with twisted coefficients, supported on the strata. Chapter 8 gives a simple introduction to this deep theory. The nearby and vanishing cycles of a fibre of a complex analytic map are introduced as important examples of perverse sheaves. An amplified section on Beilinson, Bernstein, Deligne and Gabber's decomposition theorem completes the chapter.

The new Chapter 9 provides an elementary treatment of the combinatorial intersection cohomology of a fan. When the fan is rational there is a corresponding toric variety whose intersection cohomology agrees with this combinatorial invariant of the fan. In this situation, deep results, such as the decomposition theorem, have relatively simple combinatorial proofs. The chapter ends with a discussion of Stanley's conjectures on the generalised h-vector of a fan.

The discussion of the Weil conjectures, $\mathcal{D}$-modules, the Riemann–Hilbert correspondence and the Kazhdan–Lusztig conjecture in Chapters 9, 10 and 11 is virtually unchanged, apart from some corrections, in particular to the definition of étale cohomology.

I have tried to write in the spirit of the first edition, maintaining the book as an introductory guide, or even a piece of propaganda on behalf of the subject, rather than a textbook. This means that many results are quoted, or presented with only a sketch proof. In order that the interested reader can delve further I have attempted to provide a comprehensive bibliography. Each chapter concludes with a brief section suggesting further reading. Nevertheless, intersection homology is a large and growing subject, touching on many aspects of topology, geometry and algebra and with a correspondingly large research literature. I will undoubtedly have made omissions and oversights, for which I can only apologise. One topic which is prominent by its absence is Saito's theory of mixed Hodge modules and the existence of a Hodge structure on the intersection cohomology of a complex projective variety.

None of the results in this book are original, and I owe many debts to the clear expositions in the references, whilst accepting full responsibility for any errors.

Most of this second edition was written during my time at Christ's College, Cambridge and I am very grateful for their financial, social and culinary support. I would like to thank the students who sat through my course and remained cheerful until the end. I am also very grateful to Aaron Lauda for LaTeX-ing the original manuscript of the first edition and to Ivan Smith for his indefatigable proof-reading and numerous helpful comments and suggestions (though again, the remaining errors are mine).

Special thanks go to Soumhya for her patience and encouragement, particularly during my more irascible moments. Finally, I wish to thank Frances Kirwan for her invaluable help during the writing of this second edition and for introducing me to intersection homology and infecting me with her enthusiasm for the subject.

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October 2005

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