An introduction to Wave diffraction & the Wiener–Hopf technique

Ian Thompson

Department of Mathematical Sciences

Loughborough University
What is the Wiener–Hopf technique?

- A method for solving problems involving discontinuous boundary conditions.
- Uses classical complex analysis:
  - poles & residues,
  - branch points,
  - Cauchy & Fourier integrals,
  - Liouville’s theorem,
  - Jordan’s lemma.
- Surrounded by an aura of fear, thanks largely to some terrible literature.
There is a fundamental physics problem here, as well as a difficult maths problem.
The Sommerfeld problem (continued)

- Arnold Sommerfeld solved his famous problem in 1896\(^1\).
- His method doesn’t seem to work for any other problems!
- Around 1931, rigorous work on integral equations by N. Wiener, E. Hopf and others\(^2\) was adapted to solve the Sommerfeld problem\(^3\).
- This came to be known as the Wiener–Hopf technique.
- It enables us to solve many otherwise intractable problems.
- Later, D. S. Jones found an easier way to apply the method using Fourier integrals.
- This strips away a lot of unnecessary mathematical apparatus.

\(^1\)Sommerfeld: ‘Optics’ (1964) [In English]
\(^2\)Paley & Wiener: ‘Fourier transforms in the complex domain’ (1934)
\(^3\)Wiener: ‘Collected Works’ 1981 [In English]
How it works

• Let $\Gamma$ be a contour that divides $\mathbb{C}$ into upper and lower regions.

• Represent the wavefield $\phi$ in the form

$$\phi(x, y) = \frac{1}{2\pi i} \int_{\Gamma} \hat{\phi}(\alpha; y)e^{-i\alpha x} \, d\alpha.$$ 

• Suppose that $\hat{\phi}(\alpha; 0) \to 0$ as $|\alpha| \to \infty$.

• Jordan's lemma says that for $x > 0$, we can close $\Gamma$ in the lower half plane, whereas for $x < 0$ we can close in the upper half plane.

• This is absolutely crucial:

Different singularity structures above and below $\Gamma$ give different results for $x > 0$ and $x < 0$. 

The wave equation

• We have
\[
\left( \nabla^2 - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \right) \Phi(x, y, t) = 0.
\]

• Assume that fields are time-harmonic, i.e.
\[
\Phi(x, y, t) = \Re \left[ \phi(x, y)e^{-i\omega t} \right].
\]

• Now we have to solve the Helmholtz equation
\[
(\nabla^2 + k^2)\phi(x, y) = 0, \quad k = \frac{\omega}{c},
\]

• Note propagation directions, e.g. \(e^{iky}\) propagates upwards, because
\[
\phi = e^{iky} \quad \Rightarrow \quad \Phi = \Re \left[ e^{i(ky-\omega t)} \right].
\]
Setting up the Sommerfeld problem

- Write $\phi^t = \phi^i + \phi^s$.
- The incident wave is known:
  $$\phi^i = e^{ik(x \cos \Theta + y \sin \Theta)}.$$
- We need to solve the Helmholtz equation subject to the b.c.
  $$\phi^s(x, 0) = -e^{ikx \cos \Theta} \quad \text{on} \quad x > 0.$$
- Also, $\nabla \phi^s \sim O(r^{-1/2})$ as $r = \sqrt{x^2 + y^2} \to 0$ (tip condition).
- $\phi^s$ must not include any contributions that are incoming toward the barrier, or that grow as $|y|$ is increased.
The first golden rule

• **Always** look to exploit symmetries.

• Decompose the incident wave into odd and even (in $y$) functions:

$$
\varphi^i_S(x, y) = \frac{1}{2} [\varphi^i(x, y) + \varphi^i(x, -y)] ,
\varphi^i_A(x, y) = \frac{1}{2} [\varphi^i(x, y) - \varphi^i(x, -y)] .
$$

• $\varphi^i_A(x, 0) = 0$, so this satisfies the b.c. on its own.

• In other words, $\varphi^s_A(x, y) \equiv 0$, and so $\varphi^s(x, y) = \varphi^s_S(x, y)$.

• $\varphi^s(x, 0)$ must be continuous on $x < 0$ (where there is no barrier), and it’s an even function, so

$$
\frac{\partial \varphi^s}{\partial y} = 0 \quad y = 0, x < 0.
$$
The second golden rule

- Exhaust all elementary arguments before resorting to W–H.
- We need to solve for $\phi^s(x, y)$ in $y \geq 0$.
- Start by writing
  $$\phi^s(x, y) = \frac{1}{2\pi i} \int_\Gamma \hat{\phi}(\alpha; y)e^{-i\alpha x} \, d\alpha.$$ 
- Apply operator $(\nabla^2 + k^2)$ (ok to commute with integral):
  $$\int_\Gamma \left[ \frac{\partial^2 \hat{\phi}}{\partial y^2} - (\alpha^2 - k^2) \right] e^{-i\alpha x} \, d\alpha = 0.$$ 
- This can only be true $\forall x$ if $[\cdot] \equiv 0$, and hence
  $$\hat{\phi}(\alpha; y) = A(\alpha)e^{-\gamma(\alpha)y} + B(\alpha)e^{\gamma(\alpha)y},$$
  where $\gamma(\alpha) = (\alpha^2 - k^2)^{1/2}$. 
The functions $A$ and $B$

$$\phi^s(x, y) = \frac{1}{2\pi i} \int_{\Gamma} \left[ A(\alpha)e^{-\gamma(\alpha)y} + B(\alpha)e^{\gamma(\alpha)y} \right] e^{-i\alpha x} \, d\alpha.$$ 

- At points on $\Gamma$ where $\Re[\gamma(\alpha)] = \Re[(\alpha^2 - k^2)^{1/2}] \neq 0$:
  
  If $e^{-\gamma(\alpha)y} \to 0$ as $y \to \infty$, then $|e^{\gamma(\alpha)y}| \to \infty$ & vice-versa.

- At points on $\Gamma$ where $\Re[\gamma(\alpha)] = \Re[(\alpha^2 - k^2)^{1/2}] = 0$:
  
  If $e^{-\gamma(\alpha)y}$ represents an outgoing contribution then $e^{\gamma(\alpha)y}$ represents an incoming contribution & vice-versa.

- One solution is always unphysical—doesn’t matter which because we have yet to choose the branch of $\gamma(\alpha)$.

- It is conventional to take $B(\alpha) \equiv 0$.

- Also, $A(\alpha) \sim O(\alpha^{-\eta})$ with $\eta > 1$ as $\alpha \to \infty \in \Gamma$. 

The contour $\Gamma$

$$\phi^s(x, y) = \frac{1}{2\pi i} \int_{\Gamma} A(\alpha) e^{-\gamma(\alpha)y} e^{-i\alpha x} \, d\alpha.$$ 

- Recall: $\Gamma$ divides $\mathbb{C}$ into upper and lower regions.
- $\phi^s$ contains only decaying ($\Re[\gamma] > 0$) and outgoing ($\Im[\gamma] < 0$) contributions. So, . . .
The contour $\Gamma$

\[ \phi^s(x, y) = \frac{1}{2\pi i} \int_{\Gamma} A(\alpha) e^{-\gamma(\alpha) y} e^{-i\alpha x} \, d\alpha. \]

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- $\gamma(\alpha) = \sqrt{\alpha^2 - k^2}$ for real $\alpha$ with $|\alpha| > k$. 

\[ \begin{array}{c}
\Re[\alpha] \\
\Im[\alpha]
\end{array} \]

\[ \begin{array}{c}
-k \\
-\times \\
k
\end{array} \]

\[ \begin{array}{c}
\times \\
\times \\
\times
\end{array} \]
The contour \( \Gamma \)

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- Recall: \( \Gamma \) divides \( \mathbb{C} \) into upper and lower regions.
- \( \phi^s \) contains only decaying (\( \Re[\gamma] > 0 \)) and outgoing (\( \Im[\gamma] < 0 \)) contributions. So, . . .
- \( \gamma(\alpha) = \sqrt{\alpha^2 - k^2} \) for real \( \alpha \) with \( |\alpha| > k \).
- Go clockwise around \( \alpha = -k \) to reduce the argument from 0 to \( -\pi/2 \).
The contour $\Gamma$

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- $\gamma(\alpha) = \sqrt{\alpha^2 - k^2}$ for real $\alpha$ with $|\alpha| > k$.
- Go clockwise around $\alpha = -k$ to reduce the argument from 0 to $-\pi/2$.
- Go anticlockwise around $\alpha = k$ to change it back to 0.
The contour $\Gamma$

\[ \phi^s(x, y) = \frac{1}{2\pi i} \int_{\Gamma} A(\alpha) e^{-\gamma(\alpha)y} e^{-i\alpha x} \, d\alpha. \]

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- Go clockwise around $\alpha = -k$ to reduce the argument from $0$ to $-\pi/2$.

- Go anticlockwise around $\alpha = k$ to change it back to $0$.

- $D^\pm = \{\alpha \pm iu : \, \alpha \in \Gamma, \, u \geq 0\}.$
Boundary Condition on $x > 0$

$$\phi^s(x, y) = \frac{1}{2\pi i} \int_{\Gamma} A(\alpha) e^{-i\alpha x} \, d\alpha.$$  

- We need to satisfy $\phi^s(x, 0) = -e^{ikx \cos \Theta}$ on $x > 0$.

- So, if $x > 0$, the only contribution must come from a simple pole at $\alpha = -k \cos \Theta$, otherwise we get the wrong $x$ dependence.

- $\Gamma$ passes above $\alpha = -k \cos \Theta$, and we can write

  $$A(\alpha) = \frac{1}{\alpha + k \cos \Theta} + A^-(\alpha),$$

- $A^-(\alpha)$ is unknown, but it is analytic in $\mathcal{D}^-$, and $A^-(\alpha) \to 0$ as $\alpha \to \infty \in \mathcal{D}^-$.  

- Jordan’s lemma applies (close $\Gamma$ in $\mathcal{D}^-$); the BC on $x > 0$ is satisfied.
Symmetry Condition on $x < 0$

$$\frac{\partial \phi^s}{\partial y}(x, 0) = \frac{-1}{2\pi i} \int_{\Gamma} G(\alpha) e^{-i\alpha x} \, d\alpha \quad \text{where} \quad G(\alpha) = \gamma(\alpha) A(\alpha).$$

- We need to satisfy $\frac{\partial \phi^s}{\partial y}(x, 0) = 0$ on $x < 0$.
- If $x < 0$, there must be no contributions to the integral.
- Hence, $G(\alpha) = G^+(\alpha)$, i.e. it’s is analytic in $\mathcal{D}^+$, and $G^+(\alpha) \to 0$ as $\alpha \to \infty \in \mathcal{D}^+$.
- Jordan’s lemma applies again (close $\Gamma$ in $\mathcal{D}^+$); the symmetry condition on $x > 0$ is now satisfied.
Wiener–Hopf equation

• We have $G(\alpha) = \gamma(\alpha)A(\alpha)$, and hence
  \[ G^+(\alpha) = \gamma(\alpha) \left[ \frac{1}{\alpha + k \cos \Theta} + A^-(\alpha) \right]. \]  
  (\*)

• We can product split $\gamma(\alpha) = (\alpha^2 - k^2)^{1/2}$:
  \[ \gamma^+(\alpha) = (\alpha + k)^{1/2}; \quad \gamma^-(\alpha) = (\alpha - k)^{1/2}. \]

• Use in (\*):
  \[ \frac{G^+(\alpha)}{\gamma^+(\alpha)} = \frac{\gamma^-(\alpha)}{\alpha + k \cos \Theta} + \gamma^-(\alpha)A^-(\alpha). \]

• LHS is analytic in $\mathcal{D}^+$. RHS is analytic in $\mathcal{D}^- \setminus \{-k \cos \Theta\}$.

• Subtract $\frac{\gamma^-(\alpha)}{\alpha + k \cos \Theta}$ from both sides. . .
Wiener–Hopf equation (continued)

\[
\frac{G^+(\alpha)}{\gamma^+(\alpha)} - \frac{\gamma^-(\alpha)}{\alpha + k \cos \Theta} = \frac{\gamma^-(\alpha)}{\alpha + k \cos \Theta} + \gamma^-(\alpha) A^-(\alpha).
\]

Analytic in \( \mathcal{D}^+ \) \hspace{1cm} Analytic in \( \mathcal{D}^- \)

- Both sides represent the same function! Call it \( J(\alpha) \).
- This is analytic in \( \mathcal{D}^+ \cup \mathcal{D}^- = \mathbb{C} \), i.e. it’s entire.
- We need to determine its behaviour as \(|\alpha| \to \infty\).
- From the LHS, \( J(\alpha) \to 0 \) as \( \alpha \to \infty \in \mathcal{D}^+ \).
- It’s not clear what happens to \( \gamma^{-}(\alpha) A^{-}(\alpha) \) as \( \alpha \to \infty \in \mathcal{D}^- \).
Tauberian Theorems

• Suppose that

\[ f(x) = \frac{1}{2\pi i} \int_{\Gamma} F^{-}(\alpha)e^{-i\alpha x} \, d\alpha, \]

where \( F^{-}(\alpha) \to 0 \) as \( \alpha \to \infty \in \mathcal{D}^{-} \), so that \( f(x) = 0 \) for \( x > 0 \).

• If \( f(x) \sim O(x^{\eta}) \) as \( x \to 0^{-} \) (with \( \eta > -1 \)) then \( F^{-}(\alpha) \sim O(\alpha^{-(\eta+1)}) \) as \( \alpha \to \infty \in \mathcal{D}^{-} \).

• We have to apply this to \( \phi^{t} \), because \( \phi^{t}(x, 0) = 0 \) for \( x > 0 \).

\[
\phi^{t}(x, 0) = \frac{1}{2\pi i} \int_{\Gamma} \left[ \frac{1}{\alpha + k \cos \Theta} + A^{-}(\alpha) \right] e^{-i\alpha x} \, d\alpha
\]

(the residue at \( \alpha = -k \cos \Theta \) is \( e^{ikx \cos \Theta} = \phi^{i}(x, 0) \)).

• \( \nabla \phi^{t} \sim O(r^{-1/2}) \) as \( r \to 0 \), so \( \phi^{t}(x, 0) \sim O(1) \) as \( x \to 0^{-} \) (at most). \( \implies A^{-}(\alpha) \sim O(\alpha^{-1}) \) as \( \alpha \to \infty \in \mathcal{D}^{-} \) (at most).
Solving the Wiener–Hopf equation

- At last, we have everything we need. . .

\[
\frac{G^+(\alpha)}{\gamma^+(\alpha)} - \frac{\gamma^-(\alpha - k \cos \Theta)}{\alpha + k \cos \Theta} = \frac{\gamma^-(\alpha) - \gamma^-(\alpha - k \cos \Theta)}{\alpha + k \cos \Theta} + \gamma^-(\alpha)A^-(\alpha) = J(\alpha).
\]

\( \rightarrow 0 \) as \( \alpha \rightarrow \infty \) in \( D^+ \)  \( \rightarrow 0 \) as \( \alpha \rightarrow \infty \) in \( D^- \)

- \( J(\alpha) \rightarrow 0 \) as \( \alpha \rightarrow \infty \) in any direction, and it’s an entire function.

- Liouville’s theorem says that \( J(\alpha) \equiv 0 \).

- Now we can complete the solution using either \( G^+ \) or \( A^- \):

\[
G^+(\alpha) = \gamma(\alpha)A(\alpha) \quad \Longrightarrow \quad A(\alpha) = \frac{\gamma^-(\alpha - k \cos \Theta)}{\gamma^-(\alpha)(\alpha + k \cos \Theta)}.
\]
What does it all mean?

\[ \phi^s(x, y) = \frac{1}{2\pi i} \int_\Gamma \frac{\gamma^-(\alpha) \gamma^+(-k \cos \Theta)}{\gamma^-(\alpha)(\alpha + k \cos \Theta)} e^{-\gamma(\alpha)|y|} e^{-i\alpha x} d\alpha. \]

- This satisfies all of the required conditions.
- In this (very special) case, the integral can be evaluated exactly(!)

\[ \phi^s(r, \theta) = \frac{e^{ikr}}{2} \left\{ w \left[ e^{i\pi/4} \sqrt{2kr} \sin \left( \frac{\theta + \Theta}{2} \right) \right] + w \left[ e^{i\pi/4} \sqrt{2kr} \sin \left( \frac{\theta - \Theta}{2} \right) \right] \right\}. \]

- \( w(z) \) is the scaled complex error function; \( w(z) = e^{-z^2} [1 - \text{erf}(z)] \).
- This is Sommerfeld’s original solution.
- The error function rapidly but continuously (de)activates the plane wave terms as the optical boundaries are crossed.
- There is also a circular wave that is \( O[(kr)^{-1/2}] \) in the far field, away from the boundaries.
$k = 1.0, \Theta = \pi/4$

(Plane wave terms included)
\( k = 1.0, \ \Theta = \pi/4 \)

(Plane wave terms excluded—diffraction only)
$k = 1.0, \Theta = \pi/4$

(Plane wave terms excluded—diffraction only)
One more thing. . .

• To solve the Sommerfeld problem, we had to write

$$\gamma(\alpha) = \gamma^+(\alpha)\gamma^-(\alpha).$$

• This is called a kernel factorisation.

• It was easy to find $\gamma^\pm(\alpha)$; just ‘share out’ the singularities:

$$\gamma(\alpha) = (\alpha^2 - k^2)^{1/2}, \quad \gamma^\pm(\alpha) = (\alpha \pm k)^{1/2}.$$

• It doesn’t take much to make this hard, eg.

$$f(\alpha) = 1 + \gamma(\alpha), \quad f^\pm(\alpha) = ?$$
Cauchy integrals

• Suppose that \( K(\alpha) \to 1 \) as \( \alpha \to \infty \in \Gamma \). Define

\[
I^+(\alpha) = \frac{1}{2\pi i} \int_{\Gamma} \frac{\log[K(z)]}{z - \alpha} \, dz, \quad \alpha \in \mathcal{D}^+,
\]

\[
I^-(\alpha) = \frac{-1}{2\pi i} \int_{\Gamma} \frac{\log[K(z)]}{z - \alpha} \, dz, \quad \alpha \in \mathcal{D}^-.
\]

• Then \( K^\pm(\alpha) = \exp[I^\pm(\alpha)] \), because

\[
K^+(\alpha)K^-(\alpha) = \exp[I^+(\alpha) + I^-(\alpha)] = \exp[\log[K(\alpha)]].
\]

• We use functions with known factorisations to make \( K(\alpha) \to 1 \):

\[
f(\alpha) = 1 + \gamma(\alpha), \quad K(\alpha) = \frac{1 + \gamma(\alpha)}{\gamma(\alpha)}, \quad f^\pm(\alpha) = \gamma^\pm(\alpha)K^\pm(\alpha).
\]

• It is usually very hard to evaluate these integrals, even numerically.
References

• The Sommerfeld problem appears in lots of books, eg.
  • Jones (1964) ‘The Theory of Electromagnetism’,

• Linton & Mclver solve one more problem via the Wiener–Hopf method.

• Only the dreaded

seems to contain further examples.

• Don’t read this after dark when there is a full moon.

• Prof. David Abrahams is in the process of writing a new book...