Bloch wave excitation at the edge of a lattice

The last word (?)

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Introduction

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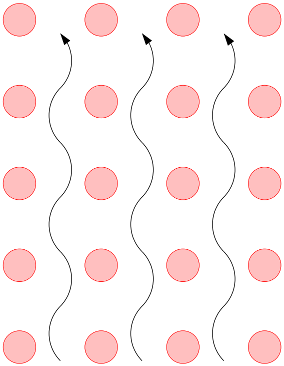

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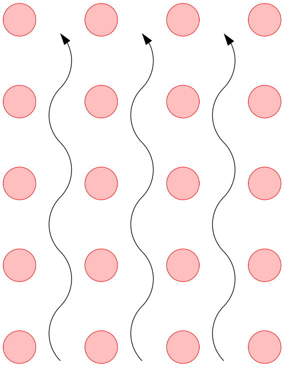
  - Periodic columns cylindrical columns standing in water
  - Composite elastic materials with periodic inner structures



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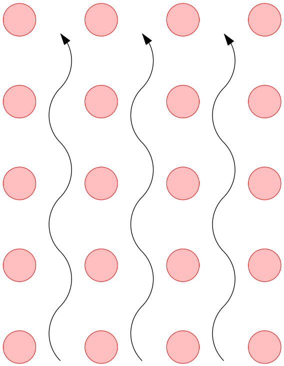

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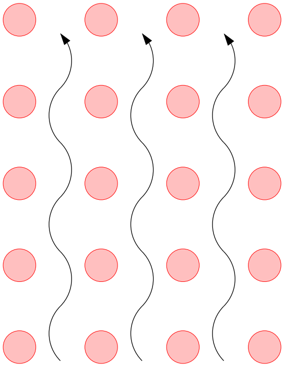

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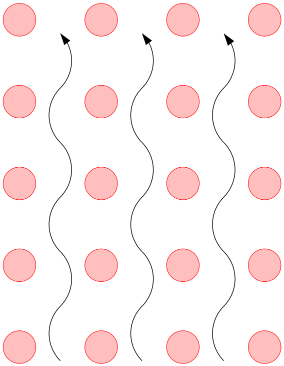

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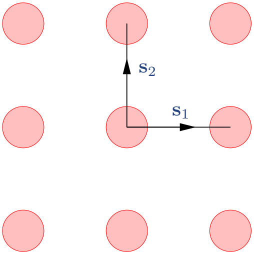
- $k = \omega/c$, $c = \sqrt{\mu/\rho}$, μ: shear modulus, ρ: density. $c = O(10^3 \mathrm{ms}^{-1})$ for metal & rock.

Bloch vectors

- If u represents a Bloch wave, then

$$u(\mathbf{r} + j\mathbf{s}_1 + p\mathbf{s}_2) = \mathrm{e}^{\mathrm{i}(j\mathbf{s}_1 + p\mathbf{s}_2)\cdot\boldsymbol{\beta}} u(\mathbf{r}),$$

for $j, p \in \mathbb{Z}$.



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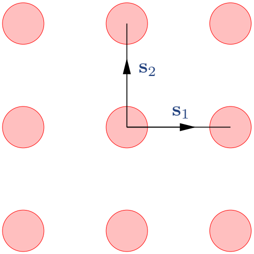
- The Bloch vector is not unique; e.g. if

$$\mathbf{s}^* \cdot \mathbf{s}_1 = 2q_1\pi \quad \text{and} \quad \mathbf{s}^* \cdot \mathbf{s}_2 = 2q_2\pi,$$

  for $q_1, q_2 \in \mathbb{Z}$ then $\boldsymbol{\beta}$ and $\boldsymbol{\beta} + \mathbf{s}^*$ are equivalent. Bloch waves do not have phase velocity!



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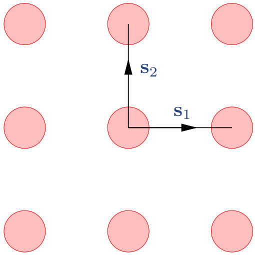
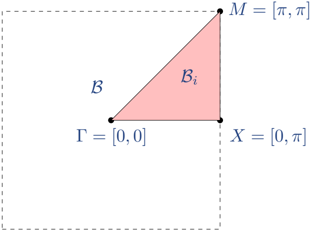
- The irreducible Brillouin zone (IBZ) contains the shortest possible representation for each Bloch vector (X, Γ, M coords for $\mathbf{s}_1 = [1, 0]$ $\mathbf{s}_2 = [0, 1]$).





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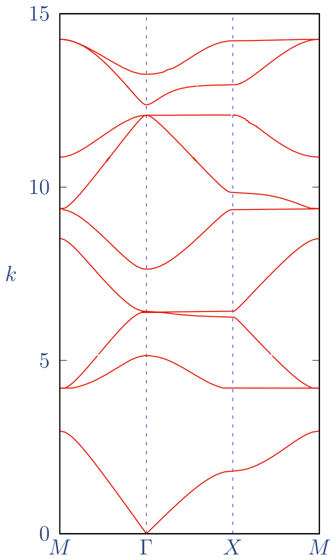

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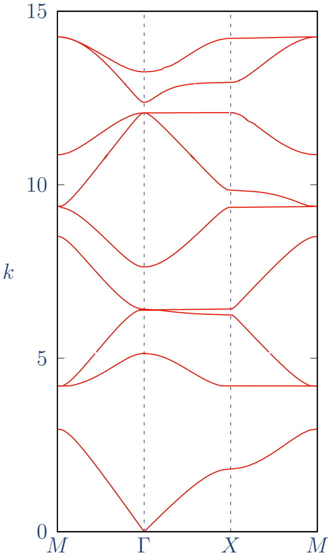
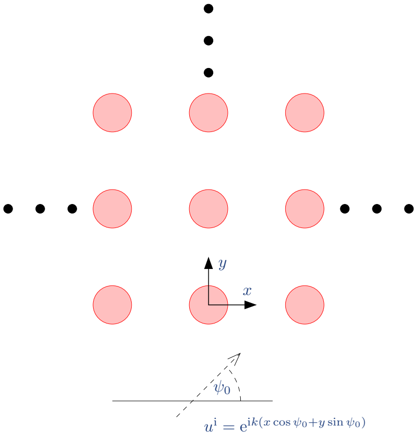
- This medium could be used to block signals for which $3 \lesssim \omega/c \lesssim 4$, where there is a gap.



The excitation problem

- To gain more information about pass bands we consider a wave in free space striking the edge of a periodic medium.



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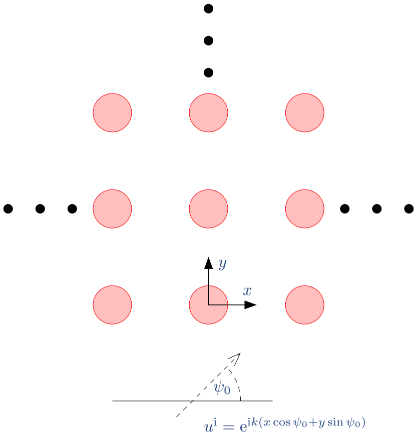

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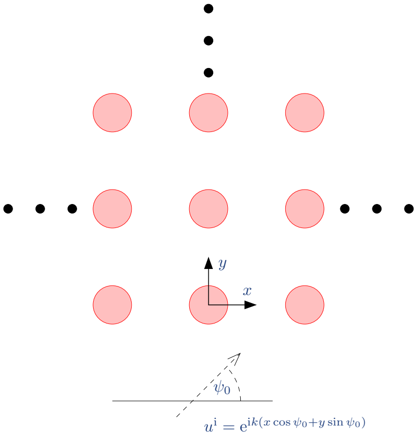

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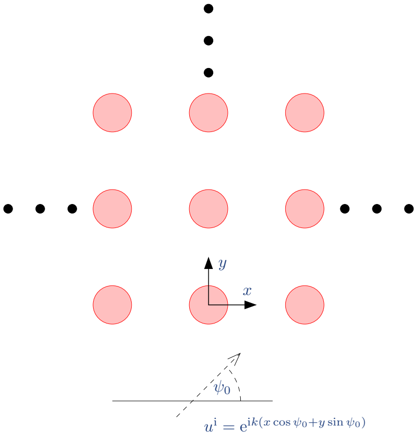

A remark from a presentation in 2014

There is more to do: the method used here is not easy to implement at high frequencies.

Multipoles

- The scattered field can be represented as a sum of singular wavefunctions, centred at each scatterer:

$$u^{\mathrm{s}}(\mathbf{r}) = \sum_{n=-\infty}^{\infty} \sum_{p=0}^{\infty} \sum_{j=-\infty}^{\infty} A_n^{j,p} \mathcal{H}_n(\mathbf{r} - j\mathbf{s}_1 - p\mathbf{s}_2)$$

where $\mathcal{H}_n(\mathbf{r}) = \mathrm{H}_n^{(1)}(kr)\mathrm{e}^{\mathrm{i}n\theta}$, $\mathbf{r} = r[\cos\theta, \sin\theta]$ and $\mathrm{H}_n^{(1)}(\cdot)$ is a Hankel function of the first kind.

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  - $\sum_n$ Different modes radiating from each scatterer. Exponentially convergent.
  - $\sum_j$ Position in x. Can be evaluated exactly.
  - $\sum_p$ Position in y. Very slowly convergent if Bloch waves are excited.

- Applying the boundary condition leads to a linear system of equations for the coefficients A_n^p:

$$A_n^q + Z_n \sum_{m=-\infty}^{\infty} \sum_{p=0}^{\infty} A_m^p S_{m-n}^{q-p} = T_n^q, \quad n \in \mathbb{Z},\ q = 0, 1, \ldots$$

where the RHS is given by $-Z_n \mathrm{i}^n \mathrm{e}^{\mathrm{i}qk\mathbf{s}_2\cdot[\cos\psi_0, \sin\psi_0]} \mathrm{e}^{-\mathrm{i}n\psi_0}$.

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- S_n^q is obtained by summing contributions along one row:

$$S_n^q = {\sum_{j=-\infty}^{\infty}}' \mathrm{e}^{\mathrm{i}jks_1 \cos\psi_0} \mathcal{H}_n(q\mathbf{s}_2 - j\mathbf{s}_1) \quad (' \text{ means omit } j = 0 \text{ if } q = 0).$$

Sums of this type were evaluated by Twersky in the 1960s.

The radiation condition

- We now have two problems:
  1. The linear system for A_n^p cannot be solved by truncation if Bloch waves are excited, because $A_n^p \not\to 0$ as $p \to \infty$.
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- To deal with these, start by finding all possible Bloch modes.
- Then calculate the mean energy flux across $\mathbf{s}_1$ for each mode, using

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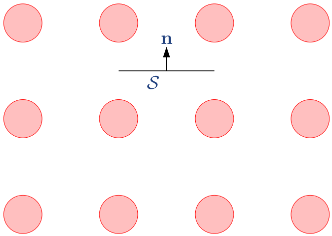

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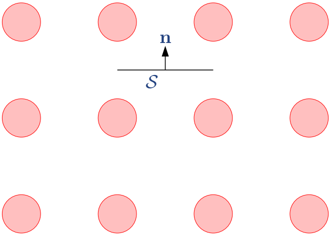

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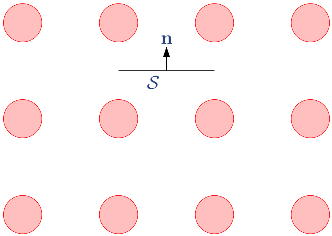

- Modes with $\langle E \rangle < 0$ cannot be excited by scattering at $y = 0$ (y component of group velocity is < 0).
- This is the radiation condition for Bloch wave excitation (Sommerfeld does not apply here, because Bloch waves have no phase velocity).

Filtering

- Consider the case where one Bloch wave is excited. Then

$$A_n^p = b\mathrm{e}^{\mathrm{i}p\mathbf{s}_2\cdot\boldsymbol{\beta}} B_n + \hat{A}_n^p.$$

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$$C_n^p = \begin{cases} A_n^p & \text{if } p = 0 \\ A_n^p - \mathrm{e}^{\mathrm{i}\mathbf{s}_2\cdot\boldsymbol{\beta}} A_n^{p-1} & \text{otherwise.} \end{cases} \qquad (*)$$

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- The Bloch wave contributes to C_0 only. Also, $C_p \to 0$ as $p \to \infty$.

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$$C_n^p = \begin{cases} A_n^p & \text{if } p = 0 \\ A_n^p - \mathrm{e}^{\mathrm{i}\mathbf{s}_2\cdot\boldsymbol{\beta}}A_n^{p-1} & \text{otherwise.} \end{cases} \qquad (*)$$
- For $p > 0$, we have $C_n^p = \hat{A}_n^p - \mathrm{e}^{\mathrm{i}\mathbf{s}_2\cdot\boldsymbol{\beta}}\hat{A}_n^{p-1}$.
- The Bloch wave contributes to C_0 only. Also, $C_p \to 0$ as $p \to \infty$.
- We can solve $(*)$ to obtain
$$A_n^p = \sum_{j=0}^{p} C_n^j \mathrm{e}^{\mathrm{i}(p-j)\mathbf{s}_2\cdot\boldsymbol{\beta}}.$$

- Substitute into the system for A_n^p ...

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- We can filter any number of Bloch waves by repeatedly applying the same transformation (proof: Thompson & Brougham, 2017).

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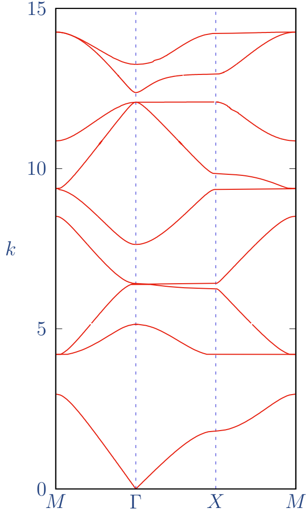

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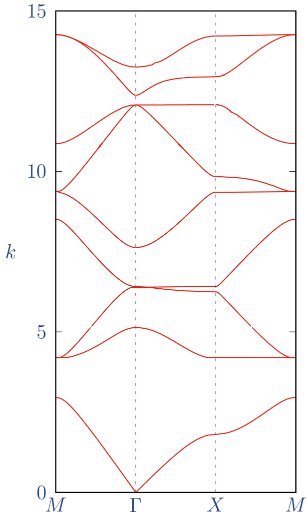

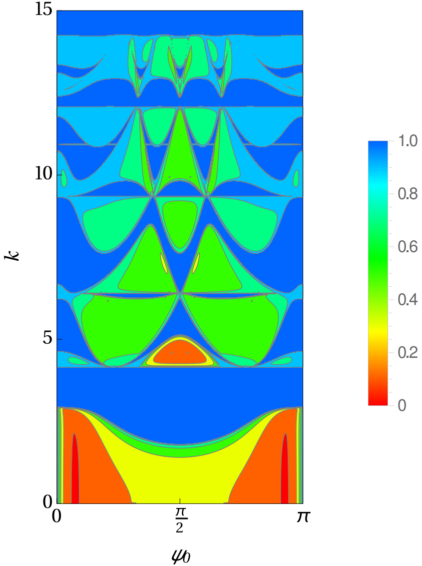

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- Filtering will work in the same way for more complicated systems:
  - In-plane elastic wave problem, 3D lattices composed using spheres, ...