Diffraction in Mindlin plates

Ian Thompson

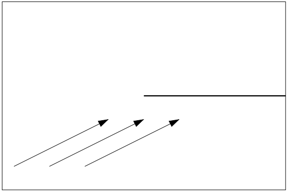
Department of Mathematical Sciences
University of Liverpool

Introduction

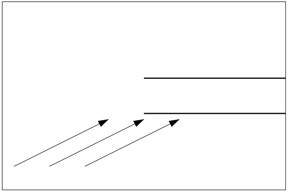
- Waves in plates are important for modelling non-destructive testing of thin panels (e.g. aeroplane wings).

Introduction

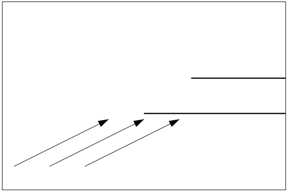
- Waves in plates are important for modelling non-destructive testing of thin panels (e.g. aeroplane wings).
- Consider any 2D diffraction/scattering problem in acoustics/electromagnetism/fluid mechanics.



Sommerfeld problem



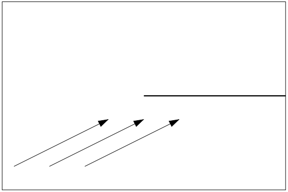
Semi-infinite waveguide
(Heins 1948)



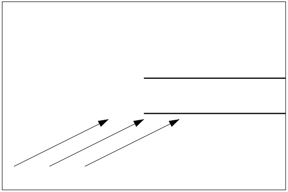
Staggered waveguide
(Abrahams & Wickham 1988 & 1990)

Introduction

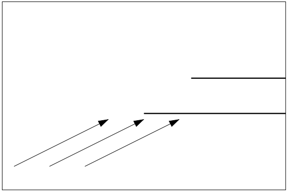
- Waves in plates are important for modelling non-destructive testing of thin panels (e.g. aeroplane wings).
- Consider any 2D diffraction/scattering problem in acoustics/electromagnetism/fluid mechanics.



Sommerfeld problem



Semi-infinite waveguide
(Heins 1948)



Staggered waveguide
(Abrahams & Wickham 1988 & 1990)

- Rather than the x, y plane representing a cross section of a 3D problem, it now represents a plate.

Kirchhoff (classical) plate theory

- Dates back to the 19th century.

Kirchhoff (classical) plate theory

- Dates back to the 19th century.
- Only transverse displacements ($w(x, y)$) are included:

$$w = w_1 + w_2, \quad \underbrace{\left(\nabla^2 + k^2\right) w_1 = 0}_{\text{propagating modes}}, \quad \underbrace{\left(\nabla^2 - k^2\right) w_2}_{\text{evanescent modes}} = 0.$$

Kirchhoff (classical) plate theory

- Dates back to the 19th century.
- Only transverse displacements ($w(x, y)$) are included:

$$w = w_1 + w_2, \quad \underbrace{\left(\nabla^2 + k^2\right) w_1 = 0,}_{\text{propagating modes}} \qquad \underbrace{\left(\nabla^2 - k^2\right) w_2}_{\text{evanescent modes}} = 0.$$

- Here, motion is assumed to be time-harmonic, so

$$W(x, y; t) = \mathrm{Re}[w(x, y)\mathrm{e}^{-\mathrm{i}\omega t}].$$

Kirchhoff (classical) plate theory

- Dates back to the 19th century.
- Only transverse displacements ($w(x,y)$) are included:

$$w = w_1 + w_2, \quad \underbrace{(\nabla^2 + k^2) w_1 = 0}_{\text{propagating modes}}, \qquad \underbrace{(\nabla^2 - k^2) w_2}_{\text{evanescent modes}} = 0.$$

- Here, motion is assumed to be time-harmonic, so

$$W(x,y;t) = \mathrm{Re}[w(x,y)\mathrm{e}^{-\mathrm{i}\omega t}].$$

- The flexural wavenumber is given by $k = (\rho h \omega^2 / D)^{1/4}$

  ρ: density, h: thickness, ω: frequency, D: stiffness.

Kirchhoff (classical) plate theory

- Dates back to the 19th century.
- Only transverse displacements ($w(x, y)$) are included:

$$w = w_1 + w_2, \quad \underbrace{\left(\nabla^2 + k^2\right) w_1 = 0,}_{\text{propagating modes}} \qquad \underbrace{\left(\nabla^2 - k^2\right) w_2}_{\text{evanescent modes}} = 0.$$

- Here, motion is assumed to be time-harmonic, so

$$W(x, y; t) = \mathrm{Re}[w(x, y)\mathrm{e}^{-\mathrm{i}\omega t}].$$

- The flexural wavenumber is given by $k = (\rho h \omega^2 / D)^{1/4}$

  ρ: density, h: thickness, ω: frequency, D: stiffness.

- Two bc's apply at an interface; e.g. at a fixed edge $w = \dfrac{\partial w}{\partial n} = 0$.

Kirchhoff (classical) plate theory

- Dates back to the 19th century.
- Only transverse displacements ($w(x, y)$) are included:

$$w = w_1 + w_2, \quad \underbrace{\left(\nabla^2 + k^2\right) w_1 = 0,}_{\text{propagating modes}} \qquad \underbrace{\left(\nabla^2 - k^2\right) w_2}_{\text{evanescent modes}} = 0.$$

- Here, motion is assumed to be time-harmonic, so

$$W(x, y; t) = \mathrm{Re}[w(x, y)\mathrm{e}^{-\mathrm{i}\omega t}].$$

- The flexural wavenumber is given by $k = (\rho h \omega^2 / D)^{1/4}$

  ρ: density, h: thickness, ω: frequency, D: stiffness.

- Two bc's apply at an interface; e.g. at a fixed edge $w = \dfrac{\partial w}{\partial n} = 0$.
- In addition, strain energy density must be integrable in all regions of the plate (Norris & Wang 1994).

Mindlin Theory

- Developed by Raymond D Mindlin in the 1950s.

Mindlin Theory

- Developed by Raymond D Mindlin in the 1950s.
- Includes in-plane rotations (ψ_x, ψ_y) in addition to transverse displacements (w).

Mindlin Theory

- Developed by Raymond D Mindlin in the 1950s.
- Includes in-plane rotations (ψ_x, ψ_y) in addition to transverse displacements (w).
- Three Helmholtz equations to solve:

$$(\nabla^2 + k_1^2)w_1 = 0, \quad (\nabla^2 + k_2^2)w_2 = 0, \quad (\nabla^2 + k_3^2)\phi = 0,$$

where k_1 is real, k_2 and k_3 are imaginary, with $|k_3| \gg |k_2|$.

Mindlin Theory

- Developed by Raymond D Mindlin in the 1950s.
- Includes in-plane rotations (ψ_x, ψ_y) in addition to transverse displacements (w).
- Three Helmholtz equations to solve:

$$(\nabla^2 + k_1^2)w_1 = 0, \quad (\nabla^2 + k_2^2)w_2 = 0, \quad (\nabla^2 + k_3^2)\phi = 0,$$

  where k_1 is real, k_2 and k_3 are imaginary, with $|k_3| \gg |k_2|$.
- Transverse displacement is still $w = w_1 + w_2$,

$$\psi_x = \frac{\partial}{\partial x}(A_1 w_1 + A_2 w_2) + \frac{\partial \phi}{\partial y} \quad \text{and} \quad \psi_y = \frac{\partial}{\partial y}(A_1 w_1 + A_2 w_2) - \frac{\partial \phi}{\partial x},$$

  where A_1 and A_2 are constants.

Mindlin Theory

- Developed by Raymond D Mindlin in the 1950s.
- Includes in-plane rotations (ψ_x, ψ_y) in addition to transverse displacements (w).
- Three Helmholtz equations to solve:

$$(\nabla^2 + k_1^2)w_1 = 0, \quad (\nabla^2 + k_2^2)w_2 = 0, \quad (\nabla^2 + k_3^2)\phi = 0,$$

  where k_1 is real, k_2 and k_3 are imaginary, with $|k_3| \gg |k_2|$.
- Transverse displacement is still $w = w_1 + w_2$,

$$\psi_x = \frac{\partial}{\partial x}(A_1 w_1 + A_2 w_2) + \frac{\partial \phi}{\partial y} \quad \text{and} \quad \psi_y = \frac{\partial}{\partial y}(A_1 w_1 + A_2 w_2) - \frac{\partial \phi}{\partial x},$$

  where A_1 and A_2 are constants.
- Three boundary conditions at an interface, e.g. at a rigid edge,

$$w = \psi_x = \psi_y = 0.$$

Mindlin Theory

- Developed by Raymond D Mindlin in the 1950s.
- Includes in-plane rotations (ψ_x, ψ_y) in addition to transverse displacements (w).
- Three Helmholtz equations to solve:

$$(\nabla^2 + k_1^2)w_1 = 0, \quad (\nabla^2 + k_2^2)w_2 = 0, \quad (\nabla^2 + k_3^2)\phi = 0,$$

  where k_1 is real, k_2 and k_3 are imaginary, with $|k_3| \gg |k_2|$.
- Transverse displacement is still $w = w_1 + w_2$,

$$\psi_x = \frac{\partial}{\partial x}(A_1 w_1 + A_2 w_2) + \frac{\partial \phi}{\partial y} \quad \text{and} \quad \psi_y = \frac{\partial}{\partial y}(A_1 w_1 + A_2 w_2) - \frac{\partial \phi}{\partial x},$$

  where A_1 and A_2 are constants.
- Three boundary conditions at an interface, e.g. at a rigid edge,

$$w = \psi_x = \psi_y = 0.$$

- Strain energy density is integrable if all displacements are bounded.

Relationship between Kirchhoff & Mindlin

- As $\omega \to 0$, $k_1 \to k$ and $k_2 \to \mathrm{i}k$, so at the leading order we have

$$(\nabla^2 + k_1^2) \to (\nabla^2 + k^2) \quad \text{and} \quad (\nabla^2 + k_2^2) \to (\nabla^2 - k^2),$$

recovering the governing PDEs for Kirchhoff theory.

Relationship between Kirchhoff & Mindlin

- As $\omega \to 0$, $k_1 \to k$ and $k_2 \to \mathrm{i}k$, so at the leading order we have

$$(\nabla^2 + k_1^2) \to (\nabla^2 + k^2) \quad \text{and} \quad (\nabla^2 + k_2^2) \to (\nabla^2 - k^2),$$

  recovering the governing PDEs for Kirchhoff theory.

- Also, $A_1 \to -1$ and $A_2 \to -1$, so

$$\psi_x = \frac{\partial}{\partial x}(A_1 w_1 + A_2 w_2) + \frac{\partial \phi}{\partial y} \to -\frac{\partial w}{\partial x} + \frac{\partial \phi}{\partial y}$$
$$\psi_y = \frac{\partial}{\partial y}(A_1 w_1 + A_2 w_2) - \frac{\partial \phi}{\partial x} \to -\frac{\partial w}{\partial y} - \frac{\partial \phi}{\partial x}.$$

Relationship between Kirchhoff & Mindlin

- As $\omega \to 0$, $k_1 \to k$ and $k_2 \to \mathrm{i}k$, so at the leading order we have
$$(\nabla^2 + k_1^2) \to (\nabla^2 + k^2) \quad \text{and} \quad (\nabla^2 + k_2^2) \to (\nabla^2 - k^2),$$
recovering the governing PDEs for Kirchhoff theory.
- Also, $A_1 \to -1$ and $A_2 \to -1$, so
$$\psi_x = \frac{\partial}{\partial x}(A_1 w_1 + A_2 w_2) + \frac{\partial \phi}{\partial y} \to -\frac{\partial w}{\partial x} + \frac{\partial \phi}{\partial y}$$
$$\psi_y = \frac{\partial}{\partial y}(A_1 w_1 + A_2 w_2) - \frac{\partial \phi}{\partial x} \to -\frac{\partial w}{\partial y} - \frac{\partial \phi}{\partial x}.$$
- If $y = 0$ is a fixed edge, then $w(x, 0) = w_x(x, 0) = 0$.

Relationship between Kirchhoff & Mindlin

- As $\omega \to 0$, $k_1 \to k$ and $k_2 \to \mathrm{i}k$, so at the leading order we have
$$(\nabla^2 + k_1^2) \to (\nabla^2 + k^2) \quad \text{and} \quad (\nabla^2 + k_2^2) \to (\nabla^2 - k^2),$$
recovering the governing PDEs for Kirchhoff theory.
- Also, $A_1 \to -1$ and $A_2 \to -1$, so
$$\psi_x = \frac{\partial}{\partial x}(A_1 w_1 + A_2 w_2) + \frac{\partial \phi}{\partial y} \to -\frac{\partial w}{\partial x} + \frac{\partial \phi}{\partial y}$$
$$\psi_y = \frac{\partial}{\partial y}(A_1 w_1 + A_2 w_2) - \frac{\partial \phi}{\partial x} \to -\frac{\partial w}{\partial y} - \frac{\partial \phi}{\partial x}.$$
- If $y = 0$ is a fixed edge, then $w(x,0) = w_x(x,0) = 0$.
- All three fixed edge bc's are satisfied (at leading order) if $w(x,0) = w_y(x,0) = 0$ and $\phi(x,y) \equiv 0$.

Relationship between Kirchhoff & Mindlin

- As $\omega \to 0$, $k_1 \to k$ and $k_2 \to \mathrm{i}k$, so at the leading order we have

$$(\nabla^2 + k_1^2) \to (\nabla^2 + k^2) \quad \text{and} \quad (\nabla^2 + k_2^2) \to (\nabla^2 - k^2),$$

  recovering the governing PDEs for Kirchhoff theory.
- Also, $A_1 \to -1$ and $A_2 \to -1$, so

$$\psi_x = \frac{\partial}{\partial x}(A_1 w_1 + A_2 w_2) + \frac{\partial \phi}{\partial y} \to -\frac{\partial w}{\partial x} + \frac{\partial \phi}{\partial y}$$
$$\psi_y = \frac{\partial}{\partial y}(A_1 w_1 + A_2 w_2) - \frac{\partial \phi}{\partial x} \to -\frac{\partial w}{\partial y} - \frac{\partial \phi}{\partial x}.$$

- If $y = 0$ is a fixed edge, then $w(x,0) = w_x(x,0) = 0$.
- All three fixed edge bc's are satisfied (at leading order) if $w(x,0) = w_y(x,0) = 0$ and $\phi(x,y) \equiv 0$.
- A similar (albeit more complicated) reduction occurs in the case of a free edge.

Fourier representations

Fourier representations

- Applying a Fourier transform (in x) to the Helmholtz equation

$$(\nabla^2 + k^2)u(x,y) = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + k^2\right)u(x,y) = 0$$

leads to the ODE

$$\left(\frac{\partial^2}{\partial y^2} - \alpha^2 + k^2\right)\hat{u}(\alpha; y).$$

Fourier representations

- Applying a Fourier transform (in x) to the Helmholtz equation

$$(\nabla^2 + k^2)u(x,y) = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + k^2\right)u(x,y) = 0$$

leads to the ODE

$$\left(\frac{\partial^2}{\partial y^2} - \alpha^2 + k^2\right)\hat{u}(\alpha;y).$$

- Hence, $\hat{u}(\alpha;y) = B(\alpha)\mathrm{e}^{-\gamma(\alpha)y} + C(\alpha)\mathrm{e}^{\gamma(\alpha)y}$, with $\gamma(\alpha) = (\alpha^2 - k^2)^{1/2}$.

Fourier representations

- Applying a Fourier transform (in x) to the Helmholtz equation

$$(\nabla^2 + k^2)u(x,y) = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + k^2\right)u(x,y) = 0$$

leads to the ODE

$$\left(\frac{\partial^2}{\partial y^2} - \alpha^2 + k^2\right)\hat{u}(\alpha; y).$$

- Hence, $\hat{u}(\alpha; y) = B(\alpha)\mathrm{e}^{-\gamma(\alpha)y} + C(\alpha)\mathrm{e}^{\gamma(\alpha)y}$, with $\gamma(\alpha) = \left(\alpha^2 - k^2\right)^{1/2}$.
- By convention, $\gamma(0) = -\mathrm{i}k$ and $\gamma(\alpha) \to |\alpha|$ as $\alpha \to \infty \in \mathbb{R}$.

Fourier representations

- Applying a Fourier transform (in x) to the Helmholtz equation
$$(\nabla^2 + k^2)u(x,y) = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + k^2\right)u(x,y) = 0$$
leads to the ODE
$$\left(\frac{\partial^2}{\partial y^2} - \alpha^2 + k^2\right)\hat{u}(\alpha; y).$$
- Hence, $\hat{u}(\alpha; y) = B(\alpha)\mathrm{e}^{-\gamma(\alpha)y} + C(\alpha)\mathrm{e}^{\gamma(\alpha)y}$, with $\gamma(\alpha) = \left(\alpha^2 - k^2\right)^{1/2}$.
- By convention, $\gamma(0) = -\mathrm{i}k$ and $\gamma(\alpha) \to |\alpha|$ as $\alpha \to \infty \in \mathbb{R}$.
- There may be different forms for B and C in different regions.

Fourier representations

- Applying a Fourier transform (in x) to the Helmholtz equation

$$(\nabla^2 + k^2)u(x, y) = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + k^2\right)u(x, y) = 0$$

  leads to the ODE

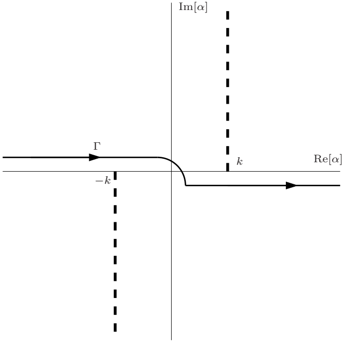
$$\left(\frac{\partial^2}{\partial y^2} - \alpha^2 + k^2\right)\hat{u}(\alpha; y).$$

- Hence, $\hat{u}(\alpha; y) = B(\alpha)\mathrm{e}^{-\gamma(\alpha)y} + C(\alpha)\mathrm{e}^{\gamma(\alpha)y}$, with $\gamma(\alpha) = \left(\alpha^2 - k^2\right)^{1/2}$.
- By convention, $\gamma(0) = -\mathrm{i}k$ and $\gamma(\alpha) \to |\alpha|$ as $\alpha \to \infty \in \mathbb{R}$.
- There may be different forms for B and C in different regions.
- In a Sommerfeld-type geometry, $C \equiv 0$ for $y > 0$ and $B \equiv 0$ for $y < 0$, to satisfy the radiation condition.

- A typical Fourier representation for a solution to the Helmholtz equation:

$$u = \frac{1}{2\pi} \int_\Gamma B(\alpha) \mathrm{e}^{-\gamma(\alpha)|y|} \mathrm{e}^{-\mathrm{i}\alpha x} \, \mathrm{d}\alpha.$$

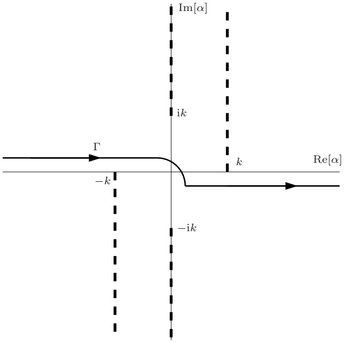
Branch points at $\alpha = \pm k$.



- A typical Fourier representation for a solution to the Helmholtz equation:

$$u = \frac{1}{2\pi}\int_\Gamma B(\alpha)\mathrm{e}^{-\gamma(\alpha)|y|}\mathrm{e}^{-\mathrm{i}\alpha x}\,\mathrm{d}\alpha.$$

Branch points at $\alpha = \pm k$.



- For a Kirchhoff plate

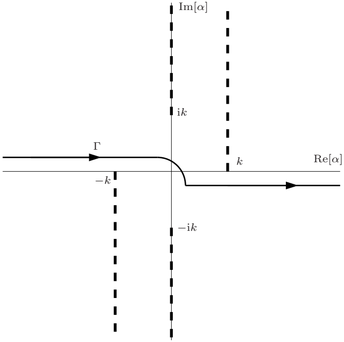
$$w = \frac{1}{2\pi}\int_\Gamma \left[B(\alpha)\mathrm{e}^{-\gamma(\alpha)|y|} + C(\alpha)\mathrm{e}^{-\lambda(\alpha)|y|}\right]\mathrm{e}^{-\mathrm{i}\alpha x}\,\mathrm{d}\alpha, \quad \lambda = (\alpha^2 + k^2)^{1/2}$$

(with $\lambda(0) = k$). Additional branch points at $\alpha = \pm \mathrm{i}k$.

- A typical Fourier representation for a solution to the Helmholtz equation:

$$u = \frac{1}{2\pi}\int_\Gamma B(\alpha)\mathrm{e}^{-\gamma(\alpha)|y|}\mathrm{e}^{-\mathrm{i}\alpha x}\,\mathrm{d}\alpha.$$

Branch points at $\alpha = \pm k$.



- For a Kirchhoff plate

$$w = \frac{1}{2\pi}\int_\Gamma \left[B(\alpha)\mathrm{e}^{-\gamma(\alpha)|y|} + C(\alpha)\mathrm{e}^{-\lambda(\alpha)|y|}\right]\mathrm{e}^{-\mathrm{i}\alpha x}\,\mathrm{d}\alpha, \quad \lambda = (\alpha^2 + k^2)^{1/2}$$

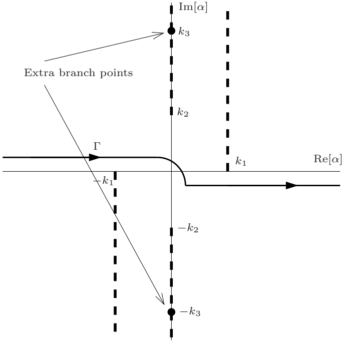
(with $\lambda(0) = k$). Additional branch points at $\alpha = \pm\mathrm{i}k$.

- λ in exponent $\Rightarrow$ evanescent modes; growing solutions are forbidden.

- A typical Fourier representation for a solution to the Helmholtz equation:

$$u = \frac{1}{2\pi}\int_\Gamma B(\alpha)\mathrm{e}^{-\gamma(\alpha)|y|}\mathrm{e}^{-\mathrm{i}\alpha x}\,\mathrm{d}\alpha.$$

  Branch points at $\alpha = \pm k$.



- For a Kirchhoff plate

$$w = \frac{1}{2\pi}\int_\Gamma \Big[B(\alpha)\mathrm{e}^{-\gamma(\alpha)|y|}+C(\alpha)\mathrm{e}^{-\lambda(\alpha)|y|}\Big]\mathrm{e}^{-\mathrm{i}\alpha x}\,\mathrm{d}\alpha, \quad \lambda = (\alpha^2+k^2)^{1/2}$$

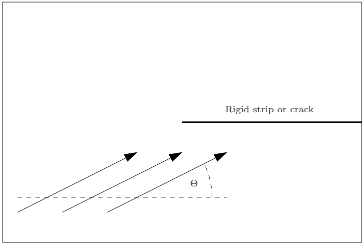
  (with $\lambda(0) = k$). Additional branch points at $\alpha = \pm\mathrm{i}k$.
- λ in exponent $\Rightarrow$ evanescent modes; growing solutions are forbidden.
- For a Mindlin plate, we write $\gamma_j(\alpha) = (\alpha^2 - k_j^2)^{1/2}$

$$w_j = \frac{1}{2\pi}\int_\Gamma B_j(\alpha)\mathrm{e}^{-\gamma_j(\alpha)|y|-\mathrm{i}\alpha x}\,\mathrm{d}\alpha, \quad \phi = \frac{1}{2\pi}\int_\Gamma R(\alpha)\mathrm{e}^{-\gamma_3(\alpha)|y|-\mathrm{i}\alpha x}\,\mathrm{d}\alpha.$$

  Three pairs of branch points: $\pm k_1$ (real), $\pm k_2$, $\pm k_3$ (imaginary).

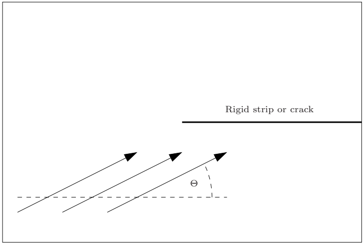
Flexural wave diffraction

- Norris & Wang (1994) considered diffraction by semi-infinite rigid strips and cracks in Kirchhoff plates.



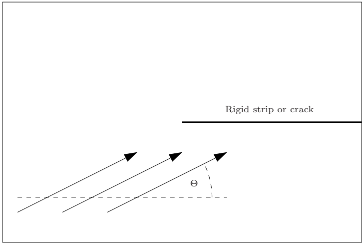
Flexural wave diffraction

- Norris & Wang (1994) considered diffraction by semi-infinite rigid strips and cracks in Kirchhoff plates.
- Since there are two boundary conditions, this geometry leads to two Wiener–Hopf equations.



Flexural wave diffraction

- Norris & Wang (1994) considered diffraction by semi-infinite rigid strips and cracks in Kirchhoff plates.
- Since there are two boundary conditions, this geometry leads to two Wiener–Hopf equations.



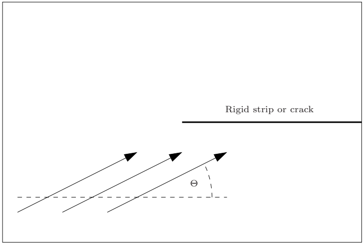
- However, N & W split the incident wave $w^{\mathrm{i}} = \mathrm{e}^{\mathrm{i}k(x\cos\Theta + y\sin\Theta)}$:

$$w^{\mathrm{i}}_{\mathsf{sym}} = \mathrm{e}^{\mathrm{i}kx\cos\Theta}\cos(ky\sin\Theta), \qquad w^{\mathrm{i}}_{\mathsf{asym}} = \mathrm{e}^{\mathrm{i}kx\cos\Theta}\,\mathrm{i}\sin(ky\sin\Theta).$$

Then $w^{\mathrm{i}} = w^{\mathrm{i}}_{\mathsf{sym}} + w^{\mathrm{i}}_{\mathsf{asym}}$ and $\partial w_{\mathsf{sym}}/\partial y = w_{\mathsf{asym}} = 0$ on $y = 0$.

Flexural wave diffraction

- Norris & Wang (1994) considered diffraction by semi-infinite rigid strips and cracks in Kirchhoff plates.
- Since there are two boundary conditions, this geometry leads to two Wiener–Hopf equations.



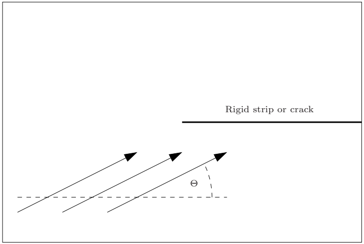
- However, N & W split the incident wave $w^{\mathrm{i}} = \mathrm{e}^{\mathrm{i}k(x\cos\Theta + y\sin\Theta)}$:

$$w^{\mathrm{i}}_{\mathrm{sym}} = \mathrm{e}^{\mathrm{i}kx\cos\Theta}\cos(ky\sin\Theta), \quad w^{\mathrm{i}}_{\mathrm{asym}} = \mathrm{e}^{\mathrm{i}kx\cos\Theta}\,\mathrm{i}\sin(ky\sin\Theta).$$

  Then $w^{\mathrm{i}} = w^{\mathrm{i}}_{\mathrm{sym}} + w^{\mathrm{i}}_{\mathrm{asym}}$ and $\partial w_{\mathrm{sym}}/\partial y = w_{\mathrm{asym}} = 0$ on $y = 0$.
- The result is four Sommerfeld-type problems (two for the strip and two for the crack) that can be solved in $y \geq 0$ only.

Flexural wave diffraction

- Norris & Wang (1994) considered diffraction by semi-infinite rigid strips and cracks in Kirchhoff plates.
- Since there are two boundary conditions, this geometry leads to two Wiener–Hopf equations.



- However, N & W split the incident wave $w^{\mathrm{i}} = \mathrm{e}^{\mathrm{i}k(x\cos\Theta + y\sin\Theta)}$:

$$w^{\mathrm{i}}_{\mathsf{sym}} = \mathrm{e}^{\mathrm{i}kx\cos\Theta}\cos(ky\sin\Theta), \quad w^{\mathrm{i}}_{\mathsf{asym}} = \mathrm{e}^{\mathrm{i}kx\cos\Theta}\,\mathrm{i}\sin(ky\sin\Theta).$$

  Then $w^{\mathrm{i}} = w^{\mathrm{i}}_{\mathsf{sym}} + w^{\mathrm{i}}_{\mathsf{asym}}$ and $\partial w_{\mathsf{sym}}/\partial y = w_{\mathsf{asym}} = 0$ on $y = 0$.
- The result is four Sommerfeld-type problems (two for the strip and two for the crack) that can be solved in $y \geq 0$ only.
- The equivalent Mindlin problems partially decouple. Each requires three bc's, so the result is two scalar problems and two 2×2 matrix problems.

Some functions that need to be factorised

Kirchhoff

- Rigid strip symmetric: $K_S(\alpha) = \lambda(\alpha) - \gamma(\alpha)$.
- Rigid strip antisymmetric: $K_A(\alpha) = \lambda(\alpha) + \gamma(\alpha) = 2k^2/K_S(\alpha)$.

Some functions that need to be factorised

Kirchhoff

- Rigid strip symmetric: $K_S(\alpha) = \lambda(\alpha) - \gamma(\alpha)$.
- Rigid strip antisymmetric: $K_A(\alpha) = \lambda(\alpha) + \gamma(\alpha) = 2k^2/K_S(\alpha)$.

Mindlin

- Rigid strip scalar:

$$K(\alpha) = A_1\gamma_1(\alpha)\gamma_3(\alpha) - A_2\gamma_2(\alpha)\gamma_3(\alpha) + \alpha^2(A_2 - A_1).$$

- Rigid strip matrix:

$$\mathsf{T}(\alpha) = \begin{bmatrix} \dfrac{1}{\gamma_1(\alpha)} & \dfrac{1}{\gamma_2(\alpha)} \\ \mathrm{i}A_1\left(\dfrac{\gamma_3(\alpha)}{\alpha} - \dfrac{\alpha}{\gamma_1(\alpha)}\right) & \mathrm{i}A_2\left(\dfrac{\gamma_3(\alpha)}{\alpha} - \dfrac{\alpha}{\gamma_2(\alpha)}\right) \end{bmatrix}.$$

- It turns out that $\det \mathsf{T}(\alpha) = -\dfrac{\mathrm{i}}{\alpha\gamma_1(\alpha)\gamma_2(\alpha)} K(\alpha)$.

Scalar kernel factorisation

- Consider a Kirchhoff problem (easier algebra!):

$$K(\alpha)Q^{+}(\alpha) = w^{+}(\alpha) + w^{-}(\alpha),$$

where

$$K(\alpha) = \underbrace{(\alpha^2 + k^2)^{1/2}}_{\lambda(\alpha)} - \underbrace{(\alpha^2 - k^2)^{1/2}}_{\gamma(\alpha)} \quad \text{and} \quad w^{+}(\alpha) = \frac{-\mathrm{i}}{\alpha - \alpha_0}$$

with $\alpha_0 = -k\cos\Theta$. The functions Q^{+} and w^{-} are unknown.

Scalar kernel factorisation

- Consider a Kirchhoff problem (easier algebra!):

$$K(\alpha)Q^+(\alpha) = w^+(\alpha) + w^-(\alpha),$$

where

$$K(\alpha) = \underbrace{(\alpha^2 + k^2)^{1/2}}_{\lambda(\alpha)} - \underbrace{(\alpha^2 - k^2)^{1/2}}_{\gamma(\alpha)} \quad \text{and} \quad w^+(\alpha) = \frac{-\mathrm{i}}{\alpha - \alpha_0}$$

with $\alpha_0 = -k\cos\Theta$. The functions Q^+ and w^- are unknown.

Observations

1. $K(\alpha) \to k^2/|\alpha|$ as $\alpha \to \infty \in \mathbb{R}$.
2. $K(\alpha)$ changes sign if α winds once around k and $\mathrm{i}k$ (or $-k$ and $-\mathrm{i}k$).

Scalar kernel factorisation

- Consider a Kirchhoff problem (easier algebra!):

$$K(\alpha)Q^+(\alpha) = w^+(\alpha) + w^-(\alpha),$$

where

$$K(\alpha) = \underbrace{(\alpha^2 + k^2)^{1/2}}_{\lambda(\alpha)} - \underbrace{(\alpha^2 - k^2)^{1/2}}_{\gamma(\alpha)} \quad \text{and} \quad w^+(\alpha) = \frac{-\mathrm{i}}{\alpha - \alpha_0}$$

with $\alpha_0 = -k\cos\Theta$. The functions Q^+ and w^- are unknown.

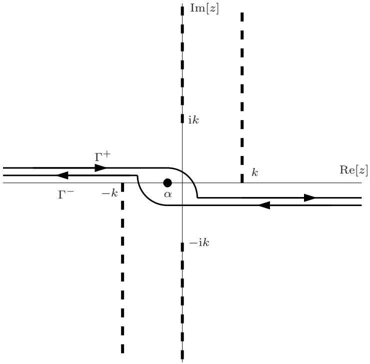
Observations

1. $K(\alpha) \to k^2/|\alpha|$ as $\alpha \to \infty \in \mathbb{R}$.
2. $K(\alpha)$ changes sign if α winds once around k and $\mathrm{i}k$ (or $-k$ and $-\mathrm{i}k$).

- If we write $\tilde{K}(\alpha) = k^{-2}\gamma(\alpha)K(\alpha)$ then $\tilde{K}(\alpha) \to 1$ as $\alpha \to \infty \in \mathbb{R}$ and has no branch point at infinity. $\gamma(\alpha)$ is easy to factorise.

Scalar kernel factorisation (ctd)

- The standard factorisation formula is $f^{\pm}(\alpha) = \exp[J^{\pm}(\alpha)]$, where
$$J^{\pm}(\alpha) = -\frac{1}{2\pi \mathrm{i}} \int_{\Gamma^{\mp}} \frac{\log[f(z)]}{z-\alpha} \, \mathrm{d}z.$$



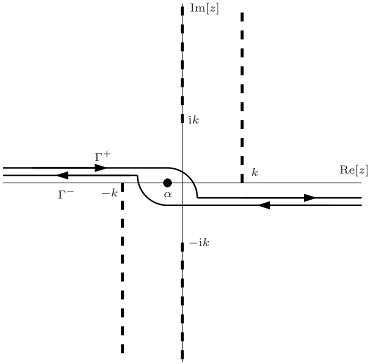
Scalar kernel factorisation (ctd)

- The standard factorisation formula is $f^{\pm}(\alpha) = \exp[J^{\pm}(\alpha)]$, where

$$J^{\pm}(\alpha) = -\frac{1}{2\pi \mathrm{i}} \int_{\Gamma^{\mp}} \frac{\log[f(z)]}{z-\alpha} \, \mathrm{d}z.$$

- By the residue theorem,

$$J^{+}(\alpha) + J^{-}(\alpha) = \log[f(\alpha)].$$



Scalar kernel factorisation (ctd)

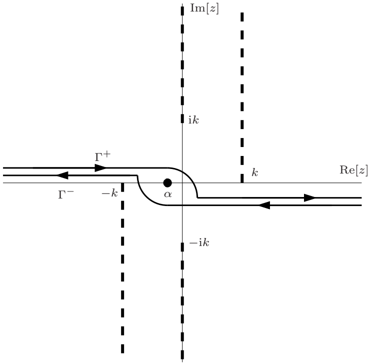
- The standard factorisation formula is $f^{\pm}(\alpha) = \exp[J^{\pm}(\alpha)]$, where

$$J^{\pm}(\alpha) = -\frac{1}{2\pi \mathrm{i}} \int_{\Gamma^{\mp}} \frac{\log[f(z)]}{z-\alpha}\, \mathrm{d}z.$$

- By the residue theorem,

$$J^{+}(\alpha) + J^{-}(\alpha) = \log[f(\alpha)].$$

- Convergence relies on the fact that $f(z) \to 1$ as $z \to \infty \in \Gamma^{\pm}$ — slow!



Scalar kernel factorisation (ctd)

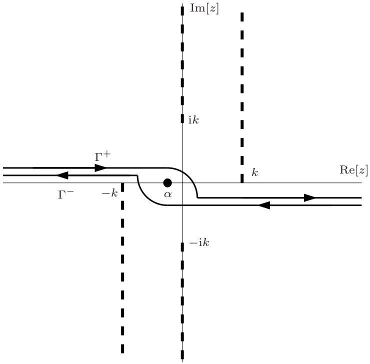
- The standard factorisation formula is $f^{\pm}(\alpha) = \exp[J^{\pm}(\alpha)]$, where

$$J^{\pm}(\alpha) = -\frac{1}{2\pi\mathrm{i}}\int_{\Gamma^{\mp}}\frac{\log[f(z)]}{z-\alpha}\,\mathrm{d}z.$$

- By the residue theorem,

$$J^{+}(\alpha) + J^{-}(\alpha) = \log[f(\alpha)].$$

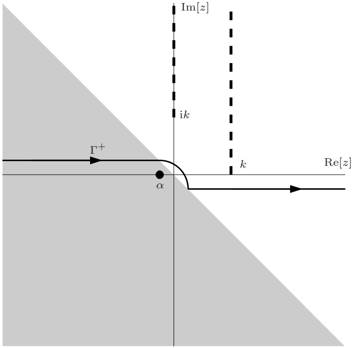
- Convergence relies on the fact that $f(z) \to 1$ as $z \to \infty \in \Gamma^{\pm}$ — slow!



- Norris & Wang took advantage of the fact that $\tilde{K}(z)$ has no branch point at infinity.

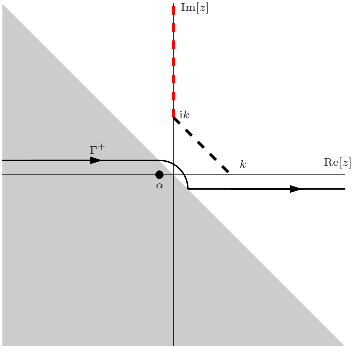
Computing $\tilde{K}^{-}(\alpha)$

- Since $\tilde{K}^{-}(\alpha) = \tilde{K}(\alpha)/\tilde{K}^{+}(\alpha)$, we need only compute $\tilde{K}^{-}(\alpha)$ directly in the half plane $\mathrm{Re}[\alpha] < -\mathrm{Im}[\alpha]$.



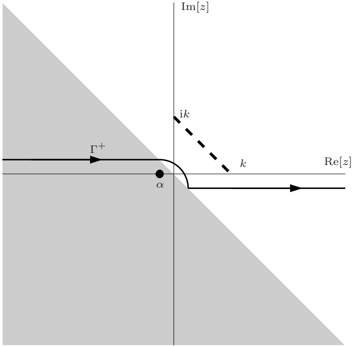
Computing $\tilde{K}^{-}(\alpha)$

- Since $\tilde{K}^{-}(\alpha) = \tilde{K}(\alpha)/\tilde{K}^{+}(\alpha)$, we need only compute $\tilde{K}^{-}(\alpha)$ directly in the half plane $\mathrm{Re}[\alpha] < -\mathrm{Im}[\alpha]$.
- Rotate the cut emanating from $\alpha = k$ so that it connects to $\alpha = \mathrm{i}k$.



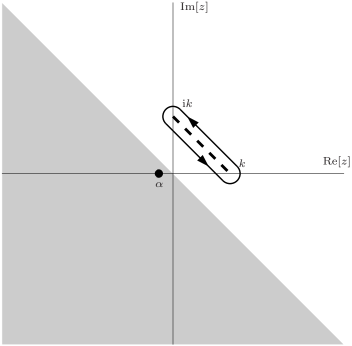
Computing $\tilde{K}^{-}(\alpha)$

- Since $\tilde{K}^{-}(\alpha) = \tilde{K}(\alpha)/\tilde{K}^{+}(\alpha)$, we need only compute $\tilde{K}^{-}(\alpha)$ directly in the half plane $\text{Re}[\alpha] < -\text{Im}[\alpha]$.
- Rotate the cut emanating from $\alpha = k$ so that it connects to $\alpha = \mathrm{i}k$.
- The cut above $\alpha = \mathrm{i}k$ disappears.



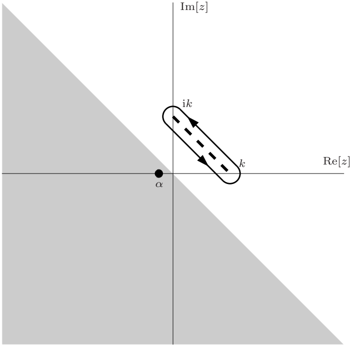
Computing $\tilde{K}^{-}(\alpha)$

- Since $\tilde{K}^{-}(\alpha) = \tilde{K}(\alpha)/\tilde{K}^{+}(\alpha)$, we need only compute $\tilde{K}^{-}(\alpha)$ directly in the half plane $\mathrm{Re}[\alpha] < -\mathrm{Im}[\alpha]$.
- Rotate the cut emanating from $\alpha = k$ so that it connects to $\alpha = \mathrm{i}k$.
- The cut above $\alpha = \mathrm{i}k$ disappears.
- Deform Γ^{+} upwards so that it wraps around the remaining cut.



Computing $\tilde{K}^{-}(\alpha)$

- Since $\tilde{K}^{-}(\alpha) = \tilde{K}(\alpha)/\tilde{K}^{+}(\alpha)$, we need only compute $\tilde{K}^{-}(\alpha)$ directly in the half plane $\text{Re}[\alpha] < -\text{Im}[\alpha]$.
- Rotate the cut emanating from $\alpha = k$ so that it connects to $\alpha = \mathrm{i}k$.
- The cut above $\alpha = \mathrm{i}k$ disappears.
- Deform Γ^{+} upwards so that it wraps around the remaining cut.



- We obtain an integral over a finite path:

$$J^{-}(\alpha) = -\frac{1}{2\pi\mathrm{i}} \int_{k}^{\mathrm{i}k} \frac{\log[\tilde{K}_R(z)] - \log[\tilde{K}_L(z)]}{z - \alpha}\, \mathrm{d}z,$$

where 'R' ('L') means evaluate on the right (left) face.

Implicit quadrature method

- Consider a Wiener–Hopf equation:

$$A(\alpha)f^+(\alpha) + B(\alpha)g^-(\alpha) = C(\alpha).$$

A, B and C are known. f^+ and g^- are analytic except for finite branch cuts.

Implicit quadrature method

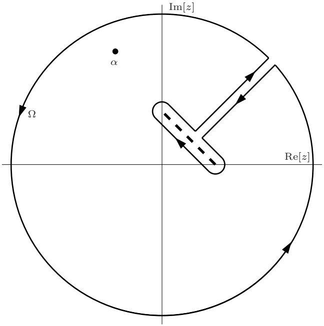
- Consider a Wiener–Hopf equation:

$$A(\alpha)f^{+}(\alpha) + B(\alpha)g^{-}(\alpha) = C(\alpha).$$

  A, B and C are known. f^{+} and g^{-} are analytic except for finite branch cuts.

- Cauchy's integral formula:

$$g^{-}(\alpha) = \frac{1}{2\pi \mathrm{i}} \int_{\Omega_b^{+}} \frac{g^{-}(z)}{z-\alpha}\,\mathrm{d}z.$$

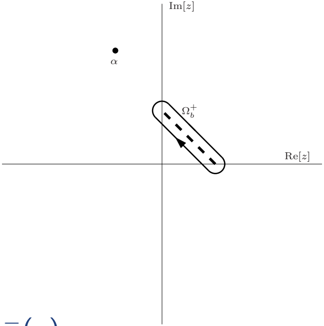


Implicit quadrature method

- Consider a Wiener–Hopf equation:

$$A(\alpha)f^{+}(\alpha) + B(\alpha)g^{-}(\alpha) = C(\alpha).$$

A, B and C are known. f^{+} and g^{-} are analytic except for finite branch cuts.



- Cauchy's integral formula:

$$g^{-}(\alpha) = \frac{1}{2\pi\mathrm{i}} \int_{\Omega_b^+} \frac{g^{-}(z)}{z-\alpha}\,\mathrm{d}z.$$

- If $g^{-}(\alpha) \to 0$ as $\alpha \to \infty$, then

$$g^{-}(\alpha) = \frac{1}{2\pi\mathrm{i}} \int_{\Omega_b^+} \frac{g^{-}(z)}{z-\alpha}\,\mathrm{d}z.$$

Similarly if $f^{+}(\alpha) \to 0$ as $\alpha \to \infty$, then

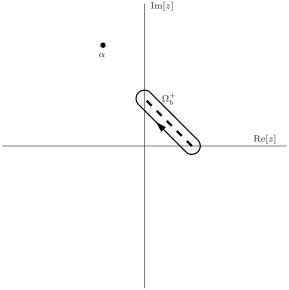
$$f^{+}(\alpha) = \frac{1}{2\pi\mathrm{i}} \int_{\Omega_b^-} \frac{f^{+}(z)}{z-\alpha}\,\mathrm{d}z,$$

where Ω_b^- encircles the finite cut of f^{+} in the lower half plane.

- Suppose the Cauchy integrals are evaluated by quadrature. That is,

$$g^{-}(\alpha) \approx \frac{1}{2\pi \mathrm{i}} \sum_{j=1}^{n} w_j^{+} \frac{G_{j,L} - G_{j,R}}{z_j^{+} - \alpha},$$

$$f^{+}(\alpha) \approx \frac{1}{2\pi \mathrm{i}} \sum_{j=1}^{n} w_j^{-} \frac{F_{j,L} - F_{j,R}}{z_j^{-} - \alpha},$$



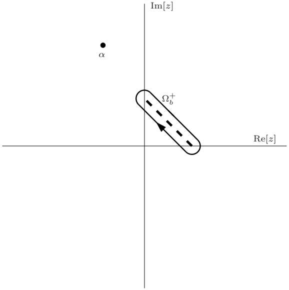
- Here, w_j are quadrature weights, 'L' and 'R' mean 'left' and 'right',

$$F_j = f^{+}(z_j^{-}) \quad \text{and} \quad G_j = g^{-}(z_j^{+}).$$

- Suppose the Cauchy integrals are evaluated by quadrature. That is,

$$g^-(\alpha) \approx \frac{1}{2\pi \mathrm{i}} \sum_{j=1}^{n} w_j^+ \frac{G_{j,L} - G_{j,R}}{z_j^+ - \alpha},$$

$$f^+(\alpha) \approx \frac{1}{2\pi \mathrm{i}} \sum_{j=1}^{n} w_j^- \frac{F_{j,L} - F_{j,R}}{z_j^- - \alpha},$$



- Here, w_j are quadrature weights, 'L' and 'R' mean 'left' and 'right',

$$F_j = f^+(z_j^-) \quad \text{and} \quad G_j = g^-(z_j^+).$$

- Return to the W–H equation: $A(\alpha)f^+(\alpha) + B(\alpha)g^-(\alpha) = C(\alpha)$ and evaluate at z_p^+:

$$A_S(z_p^+)f^+(z_p^+) + B_S(z_p^+)G_{p,S} = C_S(z_p^+).$$

- 'S' can be either 'L' or 'R' (two equations).

- No approximations yet! Insert quadrature form for f^+ ...

$$\frac{A_S(z_p^+)}{2\pi \mathrm{i}} \sum_{j=1}^{n} w_j^- \frac{F_{j,L} - F_{j,R}}{z_j^- - z_p^+} + B_S(z_p^+) G_{p,S} = C_S(z_p^+).$$

- The standard method requires one quadrature per α value (to split the kernel).
- Implicit quadrature requires one linear system solve per set of physical parameters (k etc.) and one quadrature per α value.
- The implicit quadrature method works for matrix W–H equations, provided the unknowns have finite branch cuts.

- No approximations yet! Insert quadrature form for f^+ ...

$$\frac{A_S(z_p^+)}{2\pi\mathrm{i}} \sum_{j=1}^{n} w_j^- \frac{F_{j,L} - F_{j,R}}{z_j^- - z_p^+} + B_S(z_p^+) G_{p,S} = C_S(z_p^+).$$

- Repeat for z_j^- ... a system of $4n$ linear, algebraic equations for the $4n$ unknowns.

- The standard method requires one quadrature per α value (to split the kernel).
- Implicit quadrature requires one linear system solve per set of physical parameters (k etc.) and one quadrature per α value.
- The implicit quadrature method works for matrix W–H equations, provided the unknowns have finite branch cuts.

- No approximations yet! Insert quadrature form for f^{+} ...

$$\frac{A_S(z_p^+)}{2\pi\mathrm{i}}\sum_{j=1}^{n} w_j^- \frac{F_{j,L}-F_{j,R}}{z_j^- - z_p^+} + B_S(z_p^+)G_{p,S} = C_S(z_p^+).$$

- Repeat for z_j^- ... a system of $4n$ linear, algebraic equations for the $4n$ unknowns.

Comparison of methods

- No approximations yet! Insert quadrature form for f^+ ...

$$\frac{A_S(z_p^+)}{2\pi \mathrm{i}} \sum_{j=1}^{n} w_j^- \frac{F_{j,L} - F_{j,R}}{z_j^- - z_p^+} + B_S(z_p^+) G_{p,S} = C_S(z_p^+).$$

- Repeat for z_j^- ... a system of $4n$ linear, algebraic equations for the $4n$ unknowns.

Comparison of methods

- The standard method requires one quadrature per α value (to split the kernel).

- No approximations yet! Insert quadrature form for f^+ ...

$$\frac{A_S(z_p^+)}{2\pi\mathrm{i}} \sum_{j=1}^{n} w_j^- \frac{F_{j,L} - F_{j,R}}{z_j^- - z_p^+} + B_S(z_p^+) G_{p,S} = C_S(z_p^+).$$

- Repeat for z_j^- ... a system of $4n$ linear, algebraic equations for the $4n$ unknowns.

Comparison of methods

- The standard method requires one quadrature per α value (to split the kernel).
- Implicit quadrature requires one linear system solve per set of physical parameters (k etc.) and one quadrature per α value.

- No approximations yet! Insert quadrature form for f^+ ...

$$\frac{A_S(z_p^+)}{2\pi \mathrm{i}} \sum_{j=1}^{n} w_j^- \frac{F_{j,L} - F_{j,R}}{z_j^- - z_p^+} + B_S(z_p^+) G_{p,S} = C_S(z_p^+).$$

- Repeat for z_j^- ... a system of $4n$ linear, algebraic equations for the $4n$ unknowns.

Comparison of methods

- The standard method requires one quadrature per α value (to split the kernel).
- Implicit quadrature requires one linear system solve per set of physical parameters (k etc.) and one quadrature per α value.
- The implicit quadrature method works for matrix W–H equations, provided the unknowns have finite branch cuts.

Concluding remarks

- The rigid strip problem is almost solved — it remains to complete the numerical code and analyse the diffraction pattern.

Concluding remarks

- The rigid strip problem is almost solved — it remains to complete the numerical code and analyse the diffraction pattern.
- The crack problem is next.

Concluding remarks

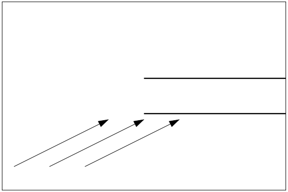
- The rigid strip problem is almost solved — it remains to complete the numerical code and analyse the diffraction pattern.
- The crack problem is next.
- Questions: do all diffraction problems of this type lead to kernels with finite cuts? Why (or why not)?

Concluding remarks

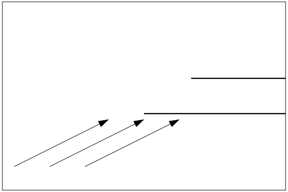
- The rigid strip problem is almost solved — it remains to complete the numerical code and analyse the diffraction pattern.
- The crack problem is next.
- Questions: do all diffraction problems of this type lead to kernels with finite cuts? Why (or why not)?
- Some discrete problems involving arrays of pins have also been considered.

Concluding remarks

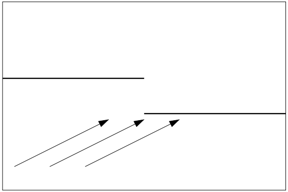
- The rigid strip problem is almost solved — it remains to complete the numerical code and analyse the diffraction pattern.
- The crack problem is next.
- Questions: do all diffraction problems of this type lead to kernels with finite cuts? Why (or why not)?
- Some discrete problems involving arrays of pins have also been considered.
- Many other 'classical' W–H geometries are untouched.



Parallel strips/cracks



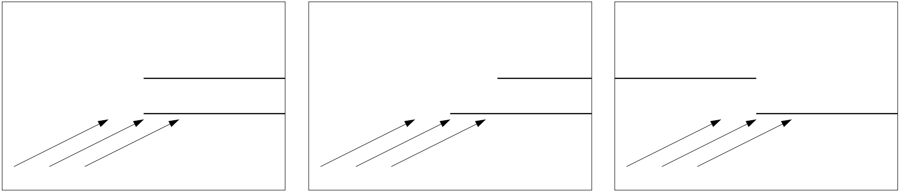
Staggered parallel strips/cracks



Offset strips/cracks

Concluding remarks

- The rigid strip problem is almost solved — it remains to complete the numerical code and analyse the diffraction pattern.
- The crack problem is next.
- Questions: do all diffraction problems of this type lead to kernels with finite cuts? Why (or why not)?
- Some discrete problems involving arrays of pins have also been considered.
- Many other ‘classical’ W–H geometries are untouched.



Parallel strips/cracks — Staggered parallel strips/cracks — Offset strips/cracks

Thanks for your attention.

References

- Kirchhoff theory
  - S. P. Timoshenko and S. Woinowsky–Krieger *'Theory of plates and shells'* (1959)
  - K. F. Graff *'Wave motion in elastic solids'* (1991)
- Mindlin theory
  - R. D. Mindlin *'Influence of rotary inertia and shear on flexural motion of isotropic, elastic plates'* (1951)
  - Graff chapter 8.
- Diffraction in plates
  - A. N. Norris & Z. Wang *'Bending-wave diffraction from strips and cracks on thin plates'* (1994)
  - I. Thompson & I. D. Abrahams *'Diffraction of flexural waves by cracks in orthotropic thin elastic plates. I Formal solution'* (2007)
  - D. V. Evans & R. Porter *'Penetration of flexural waves through a periodically constrained thin elastic plate floating in vacuo and floating on water'* (2007)
  - S. G. Haslinger et al. *'Dynamic interfacial trapping of flexural waves in structured plates'* (2016)