Assignment 6
discussed during the tutorials in the week Nov 4 – Nov 8, 2019

Exercise 1.
Write down a linear program and a non-optimal basic feasible solution at which the reduced costs of a least one non-basic variable are zero.

Solution to Exercise 1.

\[
\begin{align*}
\text{min} & \quad - x_2 \\
\text{s.t.} & \quad x_1 + x_3 = 1 \\
& \quad x_2 + x_4 = 1 \\
& \quad x_1, x_2, x_3, x_4 \geq 0
\end{align*}
\]

Choose the basis \((3, 4)\) with basic feasible solution \((0, 0, 1, 1)\). The objective function value is 0. As desired, this is not optimal, because \(x = (1, 1, 0, 0)\) satisfies all constraints and has objective function value \(-1\).

Reduced costs:
\[
\begin{align*}
\overline{c}_1 &= c_1 - c_B^T B^{-1} A_1 = c_1 = 0 \\
\overline{c}_2 &= c_2 - c_B^T B^{-1} A_2 = c_2 = -1
\end{align*}
\]

Exercise 2.
Consider the following LP:

\[
\begin{align*}
\text{min} & \quad - x_1 - 2x_2 - 3x_3 \\
\text{s.t.} & \quad x_1 + x_4 = 2 \\
& \quad x_2 + x_5 = 2 \\
& \quad x_3 + x_6 = 2 \\
& \quad x_1 + x_2 + x_3 + x_7 = 3 \\
& \quad \text{all } x_i \geq 0
\end{align*}
\]

Construct the perturbation of this LP, parametrized by \(\varepsilon\).

Solution to Exercise 2.

\[
A = \begin{pmatrix}
1 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 1 & 0 \\
1 & 1 & 1 & 0 & 0 & 0 & 1
\end{pmatrix}
\]

\[
\varepsilon = (1, 0, 0, 2, 0, 3, 0, 4, 0, 5, 0, 6, 0, 7)^T
\]
\[ A\varepsilon = \begin{pmatrix} \varepsilon + \varepsilon^4 \\ \varepsilon^2 + \varepsilon^5 \\ \varepsilon^3 + \varepsilon^6 \\ \varepsilon + \varepsilon^2 + \varepsilon^3 + \varepsilon^7 \end{pmatrix} \]

\[ b = \begin{pmatrix} 2 \\ 2 \\ 2 \\ 3 \end{pmatrix} \]

\[ \tilde{b} = b + A\varepsilon = \begin{pmatrix} 2 + \varepsilon + \varepsilon^4 \\ 2 + \varepsilon^2 + \varepsilon^5 \\ 2 + \varepsilon^3 + \varepsilon^6 \\ 3 + \varepsilon + \varepsilon^2 + \varepsilon^3 + \varepsilon^7 \end{pmatrix} \]

The perturbation is \( \min c^T x \text{ s.t. } Ax = \tilde{b} \)

**Exercise 3.**

Run the simplex algorithm on the perturbation from exercise 2 with \( \varepsilon = 0.01 \). Start at \( x_1 = x_2 = x_3 = 0 \). You are allowed to use computer support for the single steps.

**Solution to Exercise 3.**

Even though not basis is degenerate, different pivoting strategies are possible, so this solution is not unique.

\[ A = \begin{pmatrix} 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 1 \end{pmatrix} \]

\[ c^T = (-1, -2, -3, 0, 0, 0) \]

\[ b^T = (2 + \varepsilon + \varepsilon^4, 2 + \varepsilon^2 + \varepsilon^5, 2 + \varepsilon^3 + \varepsilon^6, 3 + \varepsilon + \varepsilon^2 + \varepsilon^3 + \varepsilon^7) \]

\[ = (2.01, 2.001000010000001, 2.0001000001000001, 2.000001000000100000001) \]

The algorithm now works in the same way as Exercise 2 on sheet 2.