Assignment 4
discussed during the tutorials in the week Oct 21 – Oct 25, 2019

Exercise 1.
Let
\[ A = \begin{pmatrix} 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 1 & 1 & 0 \end{pmatrix}, \quad b = \begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix}, \]

Let \( P := \{ x \in \mathbb{R}^4 \mid Ax = b, x \geq 0 \} \).
Determine how many distinct sets (the order does not matter) of basic indices exist, i.e., for how many cardinality 3 subsets \( B \subseteq \{1, 2, 3, 4\} \) do we have \( \det(A_B) \neq 0 \)?

Solution to Exercise 1.

\[ B = (1, 2, 3), \quad A_B = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}, \quad \det A_B = -1, \]
\[ B = (1, 2, 4), \quad A_B = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \quad \det A_B = 1, \]
\[ B = (1, 3, 4), \quad A_B = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \quad \det A_B = 1, \]
\[ B = (2, 3, 4), \quad A_B = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad \det A_B = 0 \]

The top three cases have a nonzero determinant, so the answer to the counting problem is three.

Exercise 2.
Consider the same situation as in Exercise 1.
Determine how many distinct basic feasible solutions exist.

Solution to Exercise 2.
All basic solutions can be obtained by choosing 3 linearly independent columns:

\[ B = (1, 2, 3), \quad A_B = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}, \quad A_B^{-1} = \begin{pmatrix} 1 & 0 & -1 \\ 0 & -1 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad \det A_B = -1, \quad A_B^{-1}b = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \geq 0 \]
The 1st and 3rd row have nonnegative solutions, so there are two distinct basic feasible solutions.

Exercise 3.
The set

\[ P := \{ x \in \mathbb{R}^3 \mid x_1 + x_2 + x_3 \leq 1, x \geq 0 \} \]

is called a tetrahedron. Give a presentation of the tetrahedron for which each vertex is degenerate.

Solution to Exercise 3.
A first solution: Repeat all inequalities, so that the polyhedron is described by 8 inequalities instead of 4.

A second solution: The vertices of the tetrahedron are \((0, 0, 0), (1, 0, 0), (0, 1, 0), (0, 0, 1)\). Add these inequalities:

- \(x_1 + x_2 + x_3 \geq 0\)
- \(x_1 \leq 1\)
- \(x_2 \leq 1\)
- \(x_3 \leq 1\)

At each of the 4 vertices, one of these new inequalities is active.