Exercise 1.
Use the negative cost cycle algorithm to find the optimal solution of the following capacitated min-cost-network-flow problem, or determine that no feasible solution exists. The label \((c, u)\) on each arc denotes the cost \(c\) of the arc and its capacity upper bound \(u\).

Solution to Exercise 1.
To find an initial solution, first remove all sources and sinks using the two-step process.

The first step reduces to a single source and a single sink:

The second step removes the remaining source and sink:
We start with the zero flow as the initial solution:

The residual network without capacities, but with costs:
We use Bellman-Ford to find a negative cost cycle. For example, we start searching at vertex 4 (no particular reason). We list the updated costs of minimal cost paths through the iterations and highlight the precedence list (to reconstruct the cycle afterwards) via fat arcs.
A directed cycle of negative cost has been found: \((4, 5, 6, t, s, 4)\). The capacity constraint on arc \((4, 5)\) implies that \(\theta^* = 1\). The updated flow is the following:

The new residual network (with costs, without capacities) is the following:
Bellman-Ford (for example started at node 5) can again be used to find a directed cycle of negative cost (we omit the details): (5, 6, t, s, 1, 2, 5). The capacities allow 1 flow unit to be sent around the cycle. The new flow is the following:

The new residual network is the following:
Bellman-Ford can find the following negative cost cycle:

We can send 1 unit along this cycle. The resulting flow:
At this point the Bellman-Ford algorithm does not find a negative cost cycle anymore. Hence an optimal solution to the auxiliary problem is found. Since the flow on arc \((t,s)\) is only 3 and not 4, the original problem was infeasible.