Exercise 1.
Let \( P = \{ x \in \mathbb{R}^n \mid Ax = b, \ x \geq 0 \} \) be a polyhedron in standard form. Prove that for every point \( x \in P \) and any nonzero direction vector \( d \in \mathbb{R}^n \) there exists \( \alpha \in \mathbb{R} \) such that \( x + \alpha d \notin P \).

Exercise 2.
Give a description of a nonempty polyhedron \( P \) in which for every point \( x \in P \) there are at least two active constraints.

Exercise 3.
This is the adjusted exercise 3 with solution provided.
Choose \( n \) and two matrices \( A, A' \) and two vectors \( b, b' \) such that
\[
P := \{ x \in \mathbb{R}^n \mid Ax \geq b \} = \{ x \in \mathbb{R}^n \mid A'x \geq b' \}
\]
and provide a vector \( x \) that is a basic solution for \( \{ Ax \geq b \} \) but not a basic solution for \( \{ A'x \geq b' \} \).
This does not work if \( x \in P \). Explain why.

Solution:
For example, choose
\[
A = \begin{pmatrix}
1 & 0 \\
0 & 1 \\
-2 & -1
\end{pmatrix}, \quad b = \begin{pmatrix}
0 \\
0 \\
-2
\end{pmatrix}
\]
and
\[
A' = \begin{pmatrix}
1 & 0 \\
0 & 1 \\
-1 & -1
\end{pmatrix}, \quad b' = \begin{pmatrix}
0 \\
0 \\
-1
\end{pmatrix}
\]
The additional inequality \(-2x_1 - x_2 \geq -2\) is implied as follows: \(-x_1 - x_2 \geq -1\) and \(x_2 \geq 0\) imply \(-x_1 \geq -1\). Adding \(-x_1 - x_2 \geq -1\) and \(-x_1 \geq -1\) gives \(-2x_1 - x_2 \geq -2\). Therefore both sets \( \{ x \in \mathbb{R}^n \mid Ax \geq b \} \) and \( \{ x \in \mathbb{R}^n \mid A'x \geq b' \} \) describe the same polyhedron.
The point \((x_1, x_2) = (0, 2)\) is a basic solution to the first system (2 linearly independent active constraints), while the same point only has 1 active constrains in the second system.
"This does not work if \( x \in P \). Explain why."
If \( x \in P \) and \( x \) is a basic solution of \( \{ Ax \geq b \} \), then \( x \) is a basic feasible solution (by definition). We know from Theorem 2.3 that this is equivalent to \( x \) being an extreme point of \( P \). Again by Theorem 2.3 this is equivalent to \( x \) being a basic feasible solution of \( \{ A'x \geq b' \} \). In particular, \( x \) is a basic solution for \( \{ A'x \geq b' \} \).