Thm 7.6: If f is not optimal, then there exists an unsaturated cycle with negative cost.

Proof: Let f+T be a feasible flow whose cost is less than that of f.

In particular, \( c^TF < 0 \).

Therefore there is a circulation in the residual network of f whose cost is \( c^TF < 0 \).

We use Lemma 7.1: \( F = \sum_{i=1}^{k} a_i f^i \) \( \leftarrow \) simple circulation

\[ 0 > c^Tf = \sum_{i=1}^{k} a_i c^T f^i, \] therefore there exists \( i \) with \( c^T f^i < 0 \). \( \square \)
Lemma 7.1 by example (greedy algorithm):

Start walking.
The flow conservation constraints ensure that every node with an incoming arc also has an outgoing arc.

If you reach a node for the second time, you found a cycle.

Push as many units as possible in the cycle's opposite direction. At least one arc capacity becomes zero. Remove these arcs.