Extreme point $\Rightarrow$ basic feasible solution.

Without loss of generality, all inequalities are of the form $a_i^T x \geq b_i$.

Contrapositive:
Suppose $x^* \notin P$ is not a basic feasible solution.

Let $I = \{ i \mid a_i^T x^* = b_i \}$

Since $x^*$ is not a basic feasible solution, there don't exist $n$ linearly independent vectors in $\{a_i \mid i \in I\}$.

Thus the $a_i$ span a proper linear subspace of $\mathbb{R}^n$.

We choose $d \in \mathbb{R}^n$ from its orthogonal complement: $a_i^T d = 0$ for all $i \in I$.

Let $\varepsilon > 0$ be a small number.

Let $y = x^* + \varepsilon d$, $z = x^* - \varepsilon d$.

Observe $a_i^T y = a_i^T x^* = b_i$ for $i \notin I$.

Thus $y \in P$. Analogously $z \in P$.

Since $x^* = \frac{1}{2} y + \frac{1}{2} z$, $x^*$ is not an extreme point.
and hence \( x^* \) is a vertex of \( P \).

It follows that \( x^* \) is the unique minimizer of \( c \cdot x \) over \( P \).

(Thm 2.2)

Solve the system of equations \( \mathbf{A} \mathbf{x} = \mathbf{b}^* \). \( \mathbf{x} \) and \( x^* \) is the unique solution to the system. Since \( x^* \) is a basic feasible solution, there are linearly independent constraints, and equality holds in (5) if and only if \( a_i' \cdot x^* = 0 \), formally.

Thus \( x^* \) is an optimal solution to minimize \( c \cdot x \) over \( P \).

\[
\begin{align*}
\text{at } x^* & \\
\mathbf{D}' & \mathbf{\leq} \mathbf{b}^* & \mathbf{c}^* \mathbf{x} = \mathbf{0} & \mathbf{s} = \mathbf{0}
\end{align*}
\]

For any \( \mathbf{x} \in P \):

\[
\begin{align*}
\sum_{i=1}^{n} x^*_i & = \mathbf{1}^* \mathbf{x} = \mathbf{1}^* \mathbf{s} = \mathbf{0} & \mathbf{s} \text{ is defined for } \mathbf{x} \end{align*}
\]

We then have \( c^* \mathbf{x} = \mathbf{c}^* \mathbf{s} = \mathbf{0} \).

\[
\begin{align*}
\text{let } c^* & = \mathbf{1}^* \mathbf{a} & \text{for the above}\end{align*}
\]

Basic feasible solution is a vertex.

From the above in the general case, \( x^*_i \) is the largest of \( \frac{b_i}{a_{ij}} \) in the \( j \)-th column of the matrix.