THE TANGENT SPACE TO THE SPACE OF 0-CYCLES

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Abstract. Let S be a Noetherian scheme, and let X be a scheme over S, such that all relative symmetric powers of X over S exist. Assume that either S is of pure characteristic 0 or X is flat over S. Assume also that the structural morphism from X to S admits a section, and use it to construct the connected infinite symmetric power $\mathrm{Sym}^{\infty}(X/S)$ of the scheme X over S. This is a commutative monoid whose group completion $\mathrm{Sym}^{\infty}(X/S)^{+}$ is an abelian group object in the category of set valued sheaves on the Nisnevich site over S, which is known to be isomorphic, as a Nisnevich sheaf, to the sheaf of relative 0-cycles in Rydh's sense. Being restricted on seminormal schemes over $\mathbb{Q}$, it is also isomorphic to the sheaf of relative 0-cycles in the sense of Suslin-Voevodsky and Kollár. In the paper we construct a locally ringed Nisnevich-étale site of 0-cycles $\mathrm{Sym}^{\infty}(X/S)^{+}_{\text{Nis-ét}}$, such that the category of étale neighbourhoods, at each point P on it, is cofiltered. This yields the sheaf of Kähler differentials $\Omega^1_{\mathrm{Sym}^{\infty}(X/S)^{+}}$ and its dual, the tangent sheaf $T_{\mathrm{Sym}^{\infty}(X/S)^{+}}$ on the space $\mathrm{Sym}^{\infty}(X/S)^{+}$. Applying the stalk functor, we obtain the stalk $T_{\mathrm{Sym}^{\infty}(X/S)^{+},P}$ of the tangent sheaf at P, whose tensor product with the residue field $\kappa(P)$ is our tangent space to the space of 0-cycles at P.

Contents

1. Introduction

The aim of this paper is to make it precise the intuitive feeling that rational equivalence of 0-cycles on an algebraic variety is the same as rational connectedness of the corresponding points on the group completed infinite symmetric

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power of that variety. To be more precise, let X be a smooth projective variety over a field k, and assume for simplicity that k is algebraically closed of zero characteristic. Fix a point on X and use it to embed the d-th symmetric power in to the $(d+1)$-th symmetric power of X. Passing to colimit, we obtain the infinite connective symmetric power $\mathrm{Sym}^\infty(X)$ of the variety X over k. Looking at this infinite symmetric power as a commutative monoid, we can consider its group completion $\mathrm{Sym}^\infty(X)^+$ in the category of groups. If now P and Q are closed points on X, they can be also considered as elements of the group completed symmetric power $\mathrm{Sym}^\infty(X)^+$. Then P is rationally equivalent to Q on X if and only if one can draw a rational curve through P and Q on $\mathrm{Sym}^\infty(X)^+$.

This philosophy tracks back through the cult paper by Mumford, [15], to Francesco Severi and possibly earlier, but it does not give us too much, as the object $\mathrm{Sym}^\infty(X)^+$ is not a variety, and it is not clear what could be a rational curve on it and, more importantly, an appropriate deformation theory of rational curves on the object $\mathrm{Sym}^\infty(X)^+$ in the style of Kollár's book [14]. Though Roitman had managed working with the group $\mathrm{Sym}^\infty(X)^+$ as a geometrical object replacing it by the products $\mathrm{Sym}^d(X) \times \mathrm{Sym}^d(X)$, see [16] and [17], his approach seems to be a compromise, which is not amazing as the necessary technique to deform weird objects was not developed in the early seventies.

So, this is our aim in this paper to develop a technical foundation of deformation theory of rational curves on $\mathrm{Sym}^\infty(X)^+$, as we see it, and now we are going to explain and justify the concepts promoted in the paper. First of all, we should ask ourselves what is the broadest notion of a geometrical object nowadays? One possible answer might be that a geometrical object is a locally ringed site whose Grothendieck topology is of some geometric nature. On the other hand, whereas the monoid $\mathrm{Sym}^\infty(X)$ is an ind-scheme, so can be managed in terms of schemes, the group completion $\mathrm{Sym}^\infty(X)^+$ clearly requires a spacewalk in the category of sheaves on schemes with an appropriate topology, such as étale topology or maybe the better Nisnevich one. Therefore, we choose that our initial environment is the category of set valued Nisnevich sheaves on locally Noetherian schemes over a base scheme S, and the latter will be always Noetherian.

But sheaves on a site are still not geometrical enough. To produce geometry on a sheaf $\mathscr{X}$ we suggest to use the notion of an *atlas*, which roughly means that we have a collection of schemes X_i and morphisms of sheaves $X_i \to \mathscr{X}$, such that the induced morphism from the coproduct $\coprod_i X_i$ to $\mathscr{X}$ is an effective epimorphism (see `nLab`). Sheaves with atlases will be called *spaces*. The idea of an atlas gives us a possibility to speak about whether a morphism from a scheme to a Nisnevich sheaf $\mathscr{X}$ is étale with regard to a given atlas on $\mathscr{X}$. A Nisnevich-étale site $\mathscr{X}_{\text{Nis-ét}}$ is then the site whose underlying category is the category of morphisms from schemes to $\mathscr{X}$, which are étale with regard to the atlas on $\mathscr{X}$, and whose topology is the restriction of the Nisnevich topology on schemes.

For the local study, let P be a point on $\mathscr{X}$, i.e. a morphism from the spectrum of a field to $\mathscr{X}$, and let $\mathscr{N}_P$ be the category of étale neighbourhoods of the point P on the site $\mathscr{X}_{\text{Nis-ét}}$. If the category $\mathscr{N}_P$ is cofiltered, we obtain an honest stalk functor at P, which yields the corresponding point of the topos of sheaves on the site $\mathscr{X}_{\text{Nis-ét}}$. If now $\mathscr{O}_{\mathscr{X}}$ is the sheaf of rings on the site $\mathscr{X}_{\text{Nis-ét}}$, inherited from the regular functions on schemes, its stalk $\mathscr{O}_{\mathscr{X},P}$ is a local ring, for each point P on $\mathscr{X}$. Then $(\mathscr{X}_{\text{Nis-ét}}, \mathscr{O}_{\mathscr{X}})$ is a locally ringed site. The standard procedure then gives us the sheaf of Kähler differentials $\Omega^1_{\mathscr{X}/S}$ and its dual, the tangent sheaf $T_{\mathscr{X}/S}$ to the space $\mathscr{X}$. Applying the stalk at P functor to the latter, we obtain the stalk $T_{\mathscr{X},P}$, and tensoring by the residue field $\kappa(P)$ of the local ring $\mathscr{O}_{\mathscr{X},P}$ we obtain the tangent space

$$T_{\mathscr{X}}(P) = T_{\mathscr{X},P} \otimes \kappa(P)$$

to the space $\mathscr{X}$ at P, with regard to the atlas on $\mathscr{X}$. Thus, a geometrical object to us is a sheaf $\mathscr{X}$ with an atlas, such that $\mathscr{N}_P$ is cofiltered for each point P on $\mathscr{X}$, and hence the site $\mathscr{X}_{\text{Nis-ét}}$ is locally ringed by the ring $\mathscr{O}_{\mathscr{X}}$.

This approach works very well when we want to geometrize groups of 0-cycles. Indeed, let X be a locally Noetherian scheme over S, such that the relative symmetric power $\mathrm{Sym}^d(X/S)$ exists for each d (this is always the case if, say, X is quasi-affine or quasi-projective over S). Assume, moreover, that the structural morphism from X to S admits a section. Use this section to construct the monoid $\mathrm{Sym}^\infty(X/S)$, which is and ind-scheme over S. Then we look at the group completion $\mathrm{Sym}^\infty(X/S)^+$ in the category of Nisnevich sheaves on locally Noetherian schemes over S. The point here is that if S is either of pure characteristic 0 or flat over S, then $\mathrm{Sym}^\infty(X/S)^+$ is isomorphic to the sheaf of relative 0-cycles in the sense of Rydh, see [18]. If, moreover, S is seminormal over $\mathrm{Spec}(\mathbb{Q})$, then the restriction of the sheaf $\mathrm{Sym}^\infty(X/S)^+$ on schemes seminormal over S gives us a sheaf isomorphic to the sheaves of relative 0-cycles constructed by Suslin and Voevodsky, [21], and by Kollár, [14]. This is why the sheaf $\mathrm{Sym}^\infty(X/S)^+$ is really the best reincarnation of a sheaf of relative 0-cycles on X over S.

Now, the fibred squares $\mathrm{Sym}^d(X/S) \times_S \mathrm{Sym}^d(X/S)$ yield a natural atlas, the *Chow atlas*, on the sheaf $\mathrm{Sym}^\infty(X/S)^+$. The problem, however, is that we do not know a priori whether the category $\mathscr{N}_P$ of étale neigbourhoods of a point P on $\mathrm{Sym}^\infty(X/S)^+$, constructed with regard to the Chow atlas, is cofiltered. This is our main technical result in the paper (Theorem 6) which asserts that $\mathscr{N}_P$ is cofiltered indeed, for every point P on $\mathrm{Sym}^\infty(X/S)^+$. It follows that we obtain the locally ringed site $\mathrm{Sym}^\infty(X/S)^+_{\text{Nis-ét}}$ with the structural sheaf $\mathscr{O}_{\mathrm{Sym}^\infty(X/S)^+}$ on it. As a consequence of that, we also obtain the sheaf of Kähler differentials $\Omega^1_{\mathrm{Sym}^\infty(X/S)^+}$ and the tangent sheaf $T_{\mathrm{Sym}^\infty(X/S)^+}$ on $\mathrm{Sym}^\infty(X/S)^+$, as well as the tangent space

$$T_{\mathrm{Sym}^\infty(X/S)^+}(P)$$

to the space $\mathrm{Sym}^\infty(X/S)^+$ at a point P.

Assume now for simplicity that S is the spectrum of an algebraically closed field k of zero characteristic, such as $\mathbb{C}$ or $\bar{\mathbb{Q}}$, for example. Any k-rational point P on $\mathrm{Sym}^\infty(X/S)^+$ corresponds to a 0-cycle on X, which we denote by the same symbol P. Two points P and Q are rationally equivalent, as two 0-cycles on X, if and only if there exists a rational curve

$$f : \mathbb{P}^1 \to \mathrm{Sym}^\infty(X/S)^+$$

on the space of 0-cycles passing through P and Q. Suppose, for example, that X is a smooth projective surface of general type with trivial transcendental part in the second étale l-adic cohomology group $H^2_{\text{ét}}(X)$. Bloch's conjecture predicts that any two closed points on X are rationally equivalent to each other. Reformulating, the space of 0-cycles $\mathrm{Sym}^\infty(X)^+$ is rationally connected. The usual way of proving that a variety is rationally connected is that we first find a rational curve on it, and then prove that this curve is sufficiently free. As we have now Kähler differentials and the tangent sheaf with tangent spaces at points on the space of 0-cycles, one can try to do the same on $\mathrm{Sym}^\infty(X)^+$. The pullback of the tangent sheaf on 0-cycles to $\mathbb{P}^1$ by f is a coherent sheaf. Therefore,

$$f^*T_{\mathrm{Sym}^\infty(X)^+} = \mathscr{O}_{\mathbb{P}^1}(a_1) \oplus \ldots \oplus \mathscr{O}_{\mathbb{P}^1}(a_n) \oplus \mathscr{T} \ ,$$

where $\mathscr{T}$ is a torsion sheaf, and $\mathscr{O}(a_i)$ are Serre's twists. If the deformation theory of rational curves on $\mathrm{Sym}^\infty(X/S)^+$ would be properly developed, we could apply the "same" arguments as in deforming curves on varieties, to prove that $\mathrm{Sym}^\infty(X)^+$ is rationally connected, in case when X is a surface of general type with no transcendental part in the second cohomology group.

Another approach to the same subject had been developed by Green and Griffiths in the book [7], which contains a lot of new deep ideas, supported by masterly computations, towards infinitesimal study of 0-cycles on algebraic varieties. The problem to us with Green-Griffiths' approach is, however, that their tangent space is the stalk of a sheaf on the variety itself, but not on a space of 0-cycles, see, for example, the definition on page 90, or formula (8.1) on page 105 in [7], and, moreover, the space of 0-cycles, as a geometrical object, is missing in the book. Our standpoint here is that the concept of a space of 0-cycles should be taken seriously, and we believe that many of our constructions are implicitly there, in the Green-Griffiths' book. In a sense, the present paper can be also considered as an attempt to prepare a technical basis to rethink the approach by Green and Griffiths, and then try to put a "functorial order" upon the heuristic discoveries in [7].

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2. Kähler differentials on spaces with atlases

Throughout the paper we will systematically choose and fix Grothendieck universes, and then working with categories small with regard to these universes, but not mentioning this in the text explicitly. A discussion of the foundational aspects of category theory can be found, for example, in [19] or [22].

Let S be a topos, and let C be a full subcategory in S, which is closed under finite fibred products. For the purposes which will be clear later, objects in the smaller category C will be denoted by Latin letters X, Y, Z etc, whereas objects in the topos S will be denoted by the calligraphic letters, such as $\mathscr{X}$, $\mathscr{Y}$, $\mathscr{Z}$ etc.

Let τ be a topology on C, and let $\mathscr{O}$ be a sheaf of commutative rings on the site C_τ, which will be considered as the structural sheaf of the ringed site C_τ.

Then $\mathscr{O}$ is an object of the topos $\mathsf{Shv}(\mathsf{C}_\tau)$ of set valued sheaves on C_τ, so that the latter is a ringed topos with the structural sheaf $\mathscr{O}$.

Given an object $\mathscr{X}$ in S consider the category $\mathsf{C}/\mathscr{X}$ whose objects are morphisms $X \to \mathscr{X}$ in S, where X are objects of C, and morphisms are morphism $f : X \to Y$ in C over the object $\mathscr{X}$. Let $(\mathsf{C}/\mathscr{X})_\tau$ be the big site whose underlying category is $\mathsf{C}/\mathscr{X}$ and the topology on $\mathsf{C}/\mathscr{X}$ is induced by the topology τ on C. For short of notation, we denote this site by $\mathscr{X}_\tau$.

Let also $\mathscr{O}_{\mathscr{X}}$ be the restriction of the structural sheaf $\mathscr{O}$ on the site $\mathscr{X}_\tau$. We shall look at $\mathscr{O}_{\mathscr{X}}$ as the structural sheaf of the site $\mathscr{X}_\tau$. Naturally, $\mathscr{O}_{\mathscr{X}}$ is an object of the topos $\mathsf{Shv}(\mathscr{X}_\tau)$.

The following definitions are slightly extended versions of the definitions in stack theory. An *atlas* A on $\mathscr{X}$ is a collection of morphisms

$$A = \{X_i \to \mathscr{X}\}_{i \in I} ,$$

indexed by a set I, such that all the objects X_i are objects of the category C, the induced morphism

$$e_A : \coprod_{i \in I} X_i \to \mathscr{X}$$

is an epimorphism in S, and if

$$X \to \mathscr{X}$$

is in A and

$$X' \to X$$

is a morphism in C, the composition

$$X' \to X \to \mathscr{X}$$

is again in A. The epimorphism e_A will be called the *atlas epimorphism* of the atlas A.

Notice that since the category S is a topos, and in a topos every epimorphism is regular, for any atlas A on an object $\mathscr{X}$ in S the atlas epimorphism e_A is a regular epimorphism. Moreover, since every topos is a regular category, and in a regular category regular epimorphisms are preserved by pullbacks, every pullback of e_A is again an epimorphism.

If A is an atlas on $\mathscr{X}$ and B is a sunset in A, such that B is an atlas on $\mathscr{X}$, then we will say that B is a *subatlas* on $\mathscr{X}$. If A_0 is a collection of morphisms from objects of C whose coproduct gives an epimorphism onto $\mathscr{X}$, the set A of all possible precompositions of morphisms from A_0 with morphisms from C is an atlas on $\mathscr{X}$. We will say that A is generated by the collection A_0, and write

$$A = \langle A_0 \rangle .$$

If A consists of all morphisms from objects of C to $\mathscr{X}$, then we will say that the atlas A is *complete*. In contrast, if A is generated by A_0 and the latter collection consists of one morphism only, then we will be saying that A is a *monoatlas* on the object $\mathscr{X}$.

Let

$$f : \mathscr{X} \to \mathscr{Y}$$

be a morphism in S, and assume that the object $\mathscr{Y}$ has an atlas B on it. We will be saying that f is *representable*, with regard to the atlas B, if for any morphism

$$Y \to \mathscr{Y}$$

from B the fibred product

$$\mathscr{X} \times_{\mathscr{Y}} Y$$

is an object in C.

Let $\mathbf{P}$ be a property of morphisms in C which is τ-local on the source and target, with regard to the topology τ and in the sense of Definitions 34.19.1 and 34.23.1 in [23]. We will say that the morphism $f : \mathscr{X} \to \mathscr{Y}$ possesses the property $\mathbf{P}$, with regard to the atlas B on $\mathscr{Y}$, if (i) f is representable with regard to B, and (ii) for any morphism $Y \to \mathscr{Y}$ from B the base change

$$\mathscr{X} \times_{\mathscr{Y}} Y \to Y$$

possesses $\mathbf{P}$. The stability of $\mathbf{P}$ under base change and compositions is then straightforward.

Let $\mathscr{X}$ and $\mathscr{Y}$ be objects in S and assume that $\mathscr{X}$ is endowed with an atlas A and $\mathscr{Y}$ with an atlas B on them. In such a case the product $\mathscr{X} \times \mathscr{Y}$ also admits an atlas $A \times B$ which consists of products of morphisms from the atlases on $\mathscr{X}$ and $\mathscr{Y}$. We will say the $A \times B$ is the *product atlas* on $\mathscr{X} \times \mathscr{Y}$.

For example, if $\mathscr{X}$ admits an atlas A, the product $\mathscr{X} \times \mathscr{X}$ admits the square $A \times A$ of the atlas A, which is an atlas on $\mathscr{X} \times \mathscr{X}$. For short, we will write A^2 instead of $A \times A$. The diagonal morphism

$$\Delta : \mathscr{X} \to \mathscr{X} \times \mathscr{X}$$

is representable with regard to A^2 if and only if for any two morphisms

$$X \to \mathscr{X} \quad \text{and} \quad Y \to \mathscr{X}$$

from A the fibred product

$$X \times_{\mathscr{X}} Y$$

is an object in C. In other words, Δ is representable with regard to A^2 if and only if any morphism from A is representable with regard to A. If Δ is representable with regard to A^2 then, for short, we will say that $\mathscr{X}$ is Δ-*representable* with regard to A.

Let $\mathscr{X}$ be an object in S with an atlas A on it. Let $(\mathsf{C}/\mathscr{X})_{\mathbf{P}}$ be the subcategory in $\mathsf{C}/\mathscr{X}$ generated by morphisms $X \to \mathscr{X}$ which are representable and possess the property $\mathbf{P}$ with regard to the atlas A on $\mathscr{X}$. Since the property $\mathbf{P}$ is τ-local on the source and target, the subcategory $(\mathsf{C}/\mathscr{X})_{\mathbf{P}}$ is closed under fibred products, and therefore we can restrict the topology τ from $\mathsf{C}/\mathscr{X}$ to $(\mathsf{C}/\mathscr{X})_{\mathbf{P}}$ to obtain a small site $\mathscr{X}_{\tau\text{-}\mathbf{P}}$. This site depends on the atlas on $\mathscr{X}$.

The site $\mathscr{X}_{\tau\text{-}\mathbf{P}}$ can be further tuned as follows. Let $\mathbf{T}$ be a type of objects in C, and let $\mathsf{C}_{\mathbf{T}}$ be the corresponding full subcategory in C. Assume that $\mathbf{T}$ is closed under fibred products in C, i.e. for any two morphisms $X \to Z$ and $Y \to Z$ in $\mathsf{C}_{\mathbf{T}}$ the fibred product $X \times_Z Y$ in C is again an object of type $\mathbf{T}$. Let $(\mathsf{C}_{\mathbf{T}}/\mathscr{X})_{\mathbf{P}}$ be the full subcategory in the category $(\mathsf{C}/\mathscr{X})_{\mathbf{P}}$ generated by morphisms $X \to \mathscr{X}$ possessing the property $\mathbf{P}$ and such that X is of type $\mathbf{T}$. Since $\mathbf{P}$ is τ-local on source and target and type $\mathbf{T}$ is closed under fibred products in C, the category $(\mathsf{C}_{\mathbf{T}}/\mathscr{X})_{\mathbf{P}}$ is closed under fibred products. Then we restrict the topology τ from the category $(\mathsf{C}/\mathscr{X})_{\mathbf{P}}$ to the category $(\mathsf{C}_{\mathbf{T}}/\mathscr{X})_{\mathbf{P}}$ and obtain a smaller site $\mathscr{X}_{\tau\text{-}\mathbf{P}\text{-}\mathbf{T}}$.

Let $\mathscr{X}$ and $\mathscr{Y}$ be two objects in S with atlases A and B respectively, and let

$$f : \mathscr{X} \to \mathscr{Y}$$

be a morphism in S. For any morphism

$$X \to \mathscr{X}$$

from $\mathscr{X}_{\tau\text{-}\mathbf{P}\text{-}\mathbf{T}}$ consider the category

$$X/(\mathsf{C}_{\mathbf{T}}/\mathscr{Y})_{\mathbf{P}}$$

of morphisms

$$X \to Y \to \mathscr{Y}$$

such that the square

(1)
$$\begin{tikzcd} X \arrow[r] \arrow[d] & Y \arrow[d] \\ \mathscr{X} \arrow[r, "f"] & \mathscr{Y} \end{tikzcd}$$

commutes, and the morphism $Y \to \mathscr{Y}$ is in $\mathscr{Y}_{\tau\text{-}\mathbf{P}\text{-}\mathbf{T}}$. If the category $X/(\mathsf{C}_{\mathbf{T}}/\mathscr{Y})_{\mathbf{P}}$ is nonempty, for any morphism $X \to \mathscr{X}$ from $\mathscr{X}_{\tau\text{-}\mathbf{P}\text{-}\mathbf{T}}$, the morphism f creates a functor

$$f^{-1} : \mathsf{Shv}(\mathscr{Y}_{\tau\text{-}\mathbf{P}\text{-}\mathbf{T}}) \to \mathsf{Shv}(\mathscr{X}_{\tau\text{-}\mathbf{P}\text{-}\mathbf{T}})$$

which associates, to any sheaf $\mathscr{F}$ on $\mathscr{Y}_{\tau\text{-}\mathbf{P}\text{-}\mathbf{T}}$, the sheaf $f^{-1}\mathscr{F}$ on $\mathscr{X}_{\tau\text{-}\mathbf{P}\text{-}\mathbf{T}}$, such that, by definition

$$f^{-1}\mathscr{G}(X \to \mathscr{X}) = \operatorname{colim} \mathscr{F}(Y \to \mathscr{Y}) \ ,$$

where the colimit is taken over the category $X/(\mathsf{C}_{\mathbf{T}}/\mathscr{Y})_{\mathbf{P}}$.

If $\mathscr{F}$ is a sheaf of rings[1] on $\mathscr{Y}_{\tau}$, it is *not* true in general that $f^{-1}\mathscr{F}$ is a sheaf of rings on $\mathscr{X}_{\tau}$. The reason for that is that the forgetful functor from rings to sets commutes with only filtered colimits, whereas the category $X/(\mathsf{C}_{\mathbf{T}}/\mathscr{Y})_{\mathbf{P}}$ might be well not filtered. But whenever the category $X/(\mathsf{C}_{\mathbf{T}}/\mathscr{Y})_{\mathbf{P}}$ is nonempty and filtered, the set $f^{-1}\mathscr{F}(X \to \mathscr{X})$ inherits the structure of a ring, and if, moreover, this category is nonempty and filtered for any morphism $X \to \mathscr{X}$ from $\mathscr{X}_{\tau\text{-}\mathbf{P}\text{-}\mathbf{T}}$ the sheaf $f^{-1}\mathscr{F}$ is a sheaf of rings on the site $\mathscr{X}_{\tau\text{-}\mathbf{P}\text{-}\mathbf{T}}$.

Let us apply the pullback functor f^{-1} to the structural sheaf of rings $\mathscr{O}_{\mathscr{Y}}$. For each pair of two morphisms

$$X \xrightarrow{g} Y \to \mathscr{Y} \ ,$$

such that the square (1) commutes and the second morphism possesses $\mathbf{P}$, we have a homomorphism of rings

$$\mathscr{O}_{\mathscr{Y}}(Y \to \mathscr{Y}) = \mathscr{O}(Y) \xrightarrow{\mathscr{O}(g)} \mathscr{O}(X) = \mathscr{O}_{\mathscr{X}}(X \to \mathscr{X}) \ .$$

[1]in the paper all rings are commutative rings, if otherwise is not mentioned explicitly

Such homomorphisms induce a morphism

$$f^{-1}\mathscr{O}_{\mathscr{Y}}(X \to \mathscr{X}) \to \mathscr{O}_{\mathscr{X}}(X \to \mathscr{X})\ ,$$

for all morphisms $X \to \mathscr{X}$, and hence a morphism of set valued sheaves

$$f^{-1}\mathscr{O}_{\mathscr{Y}} \to \mathscr{O}_{\mathscr{X}} \tag{2}$$

If we assume that the category $X/(\mathsf{C}_{\mathbf{T}}/\mathscr{Y})_{\mathbf{P}}$ is nonempty and filtered for every $X \to \mathscr{X}$ from $\mathscr{X}_{\tau\text{-}\mathbf{P}\text{-}\mathbf{T}}$, the morphism (2) is a morphism of ring valued sheaves on the site $\mathscr{X}_{\tau\text{-}\mathbf{P}\text{-}\mathbf{T}}$. In such a case, though f does not in general give us a morphism of ring topoi, still we can define the sheaf of Kähler differentials on $\mathscr{X}_{\tau\text{-}\mathbf{P}\text{-}\mathbf{T}}$ of the morphism f as

$$\Omega^1_{\mathscr{X}/\mathscr{Y}} = \Omega^1_{\mathscr{O}_{\mathscr{X}}/f^{-1}\mathscr{O}_{\mathscr{Y}}}\ ,$$

in terms of page 115 in the first part of [13] (see also the earlier book [10]).

Any Gothendieck topos is a cartesian closed category. In particular, the topos $\mathsf{Shv}(\mathscr{X}_{\tau\text{-}\mathbf{P}\text{-}\mathbf{T}})$ is a cartesian closed category, for each object $\mathscr{X}$ in S. The internal Hom-objects are given by the following formula. For any two set valued sheaves $\mathscr{F}$ and $\mathscr{G}$ on the site $\mathscr{X}_{\tau\text{-}\mathbf{P}\text{-}\mathbf{T}}$,

$$\mathscr{H}om(\mathscr{F},\mathscr{G})(X \to \mathscr{X}) = \mathrm{Hom}_{\mathsf{Shv}(\mathscr{X}_{\tau\text{-}\mathbf{P}\text{-}\mathbf{T}})}(\mathscr{F} \times X, \mathscr{G})\ ,$$

where X is considered as a sheaf on $\mathscr{X}_{\tau\text{-}\mathbf{P}\text{-}\mathbf{T}}$ via the Yoneda embedding. Notice also that, if

$$\mathrm{Hom}_X(\mathscr{F} \times X, \mathscr{G} \times X)$$

is a subset of morphisms from $\mathscr{F}$ to $\mathscr{G}$ over X, i.e. the set of morphisms in the slice category $\mathsf{Shv}(\mathscr{X}_{\tau\text{-}\mathbf{P}\text{-}\mathbf{T}})/X$, then

$$\mathrm{Hom}_X(\mathscr{F} \times X, \mathscr{G} \times X) = \mathrm{Hom}_{\mathsf{Shv}(\mathscr{X}_{\tau\text{-}\mathbf{P}\text{-}\mathbf{T}})}(\mathscr{F} \times X, \mathscr{G})$$

for elementary categorical reasons. Then the internal Hom can be equivalently defined by setting

$$\mathscr{H}om(\mathscr{F},\mathscr{G})(X \to \mathscr{X}) = \mathrm{Hom}_X(\mathscr{F} \times X, \mathscr{G} \times X)\ .$$

Now, if the category $X/(\mathsf{C}_{\mathbf{T}}/\mathscr{Y})_{\mathbf{P}}$ is nonempty and filtered, for every $X \to \mathscr{X}$ in $\mathscr{X}_{\tau\text{-}\mathbf{P}\text{-}\mathbf{T}}$, so that we have the sheaf of Kähler differentials $\Omega^1_{\mathscr{X}/\mathscr{Y}}$, then we can also define the tangent sheaf on $\mathscr{X}_{\tau\text{-}\mathbf{P}\text{-}\mathbf{T}}$ to be the dual sheaf

$$T_{\mathscr{X}/\mathscr{Y}} = \mathscr{H}om(\Omega^1_{\mathscr{X}/\mathscr{Y}}, \mathscr{O}_{\mathscr{X}})\ .$$

If

$$\mathscr{Y} = Z \in \mathrm{Ob}(\mathsf{C}_{\mathbf{T}})\ ,$$

the category $X/(\mathsf{C}_{\mathbf{T}}/\mathscr{Y})_{\mathbf{P}}$ has a terminal object

$$\begin{tikzcd} X \arrow[r] \arrow[d] & Z \arrow[d, "\mathrm{id}"] \\ \mathscr{X} \arrow[r, "f"] & Z \end{tikzcd}$$

And since every category with a terminal object is nonempty and filtered, the morphism (2) is a morphism of ring valued sheaves, and we obtain the sheaf of Kähler differentials

$$\Omega^1_{\mathscr{X}/Z} \in \mathrm{Ob}(\mathsf{Shv}(\mathscr{X}_{\tau\text{-}\mathbf{P}\text{-}\mathbf{T}}))$$

and the tangent sheaf

$$T_{\mathscr{X}/Z} \in \mathrm{Ob}(\mathsf{Shv}(\mathscr{X}_{\tau\text{-}\mathbf{P}\text{-}\mathbf{T}}))$$

The above constructions of Kähler differentials and tangent sheaves apply to all kinds of geometric setups, embracing smooth and complex-analytic manifolds in terms of synthetic differential geometry, algebraic varieties, schemes, algebraic spaces, stacks, etc. All we need is to choose an appropriate category C, a topology τ on C, a sheaf of rings $\mathscr{O}$ and then take S to be the category $\mathsf{PShv}(\mathsf{C})$ of set valued presheaves on C or, when the topology τ is subcanonical, the category $\mathsf{Shv}(\mathsf{C}_\tau)$ of sheaves on the site C_τ. If a set valued sheaf $\mathscr{X}$ on C_τ is endowed with an atlas A of morphisms from objects of the category C to $\mathscr{X}$, then we will say that $\mathscr{X}$ is a *space*, with regard to the atlas A. In other words, a space to us is a sheaf with a fixed atlas on it.

For the purposes of the present paper we need to work in terms of schemes. All schemes in this paper will be separated by default. If X is a scheme and P is a point of X then $\varkappa(P)$ will be the residue field of the scheme X at P.

Let Sch be the category of schemes. If S is a scheme, let Sch/S be the category of schemes over S. We will always assume that the base scheme S is Noetherian. Let Noe/S be the full subcategory in Sch/S generated by locally Noetherian schemes over S. We will also need the full subcategory Nor/S in Noe/S generated by locally Noetherian schemes which are locally of finite type over S whose structural morphism is normal in the sense of Definition 36.18.1 in [23], the full subcategory Reg/S in Nor/S generated by locally Noetherian schemes locally of finite type over S whose structural morphism is regular, in the sense of Definition 36.19.1 in [23] (since every regular local ring is integrally closed, every regular scheme is normal). Finally, let Sm/S be the full subcategory in Reg/S generated by locally Noetherian schemes locally of finite type over S whose structural morphism is smooth. Recall that every smooth scheme over a field is regular, this is why Sm/S is indeed a full subcategory in Reg/S. Since every regular scheme over a perfect field is smooth, if the residue fields of points on the base scheme S are perfect, the categories Sm/S and Reg/S coincide. Thus, we obtain the following chain of full embeddings

$$\mathsf{Sm}/S \subset \mathsf{Reg}/S \subset \mathsf{Nor}/S \subset \mathsf{Noe}/S \subset \mathsf{Sch}/S\ . \tag{3}$$

The category Sch possesses the following well-known topologies: the Zariski topology Zar, h-topology, the étale topology ét, the Nisnevich topology Nis and the completely decomposed h-topology denoted by cdh. Notice that only the topologies Zar, Nis and ét are subcanonical, the topologies cdh and h are not subcanonical. The relation between these topologies is given by the chains of inclusions

$$\mathrm{Zar} \subset \mathrm{Nis} \subset \text{ét} \subset \mathrm{h} \tag{4}$$

and

$$\text{Nis} \subset \text{cdh} \subset \text{h} \; . \tag{5}$$

The categories Sch/S and Noe/S are obviously closed under fibred products. Moreover, the categories Nor/S, Reg/S and Sm/S are also closed under fibred products by Propositions 6.8.2 and 6.8.3 in [8]. For simplicity of notation, the restrictions of all five topologies from (4) and (5) on the categories from (3) will be denoted by the same symbols.

For our purposes the most convenient setup is this:

$$\mathsf{C} = \mathsf{Noe}/S \; , \quad \tau = \text{Nis} \, , \quad \mathbf{P} = \text{ét}$$

and

$$\mathbf{T} \in \{\text{sm} \, , \; \text{reg} \, , \; \text{nor} \, , \; \text{Noe}\} \; ,$$

i.e.

$$\mathsf{C}_{\mathbf{T}} \in \{\mathsf{Sm}/S \, , \; \mathsf{Reg}/S \, , \; \mathsf{Nor}/S \, , \; \mathsf{Noe}/S\} \; .$$

Since the Nisnevich topology is subcanonical, we can choose

$$\mathsf{S} = \mathsf{Shv}((\mathsf{Noe}/S)_{\text{Nis}})$$

to be the category of set valued sheaves on the Nisnevich site $(\mathsf{Noe}/S)_{\text{Nis}}$. If a Nisnevich sheaf $\mathscr{X}$ is endowed with an atlas A on it, then we will say that $\mathscr{X}$ is a *Nisnevich space*, with regard to the atlas A. Accordingly, for any Nisnevich space $\mathscr{X}$ we have the site

$$\mathscr{X}_{\text{Nis-ét-}\mathbf{T}}$$

of morphisms from locally Noetherian schemes of type $\mathbf{T}$ over S to $\mathscr{X}$, étale with regard to the atlas on $\mathscr{X}$, endowed with the induced Nisnevich topology on it.

If

$$\mathbf{T} = \text{Noe} \; ,$$

i.e.

$$\mathsf{C}_{\mathbf{T}} = \mathsf{Noe}/S \; ,$$

then, for short of notation, we will write

$$\mathscr{X}_{\text{Nis-ét}}$$

for instead of $\mathscr{X}_{\text{Nis-ét-Noe}}$.

Notice also that S is a terminal object in the category Noe/S, and, since any sheaf in $\mathsf{Shv}((\mathsf{Noe}/S)_{\text{ét}})$ is the colimit of representable sheaves, S is also a terminal object in the category $\mathsf{Shv}((\mathsf{Noe}/S)_{\text{ét}})$.

Let $\mathscr{X}$ be a Nisnevich sheaf on Noe/S. A *point* P on $\mathscr{X}$ is an equivalence class of morphisms

$$\text{Spec}(K) \to \mathscr{X}$$

from spectra of fields to $\mathscr{X}$ in the category $\mathsf{Shv}_{\text{Nis}}(\mathsf{Noe}/S)$. Two morphisms

$$\text{Spec}(K) \to \mathscr{X} \quad \text{and} \quad \text{Spec}(K') \to \mathscr{X}$$

are said to be equivalent if there exists a third field K'', containing the fields K and K', such that the diagram

$$\begin{tikzcd} \mathrm{Spec}(K'') \arrow[r] \arrow[d] & \mathrm{Spec}(K') \arrow[d] \\ \mathrm{Spec}(K) \arrow[r] & \mathscr{X} \end{tikzcd}$$

commutes. If a morphism from $\mathrm{Spec}(K)$ to $\mathscr{X}$ represents P then, by abuse of notation, we will write

$$P : \mathrm{Spec}(K) \to \mathscr{X} \ .$$

The set of points on $\mathscr{X}$ will be denoted by $|\mathscr{X}|$. Certainly, if $\mathscr{X}$ is represented by a locally Noetherian scheme X over S, then $|\mathscr{X}|$ is the set of points of the scheme X. A geometric point on $\mathscr{X}$ is a morphism from $\mathrm{Spec}(K)$ to $\mathscr{X}$, where K is algebraically closed. Any geometric point on $\mathscr{X}$ represents a point on $\mathscr{X}$, and any point on $\mathscr{X}$ is represented by a geometric point.

Fix an atlas A on the sheaf $\mathscr{X}$. If a point P on $\mathscr{X}$ has a representative

$$\mathrm{Spec}(K) \to \mathscr{X} \ ,$$

and the latter factors through a morphism from A, then we will say that P factors through A.

Let P be a point of $\mathscr{X}$ which factors through A. Choose a representative

$$\mathrm{Spec}(K) \to \mathscr{X}$$

of the point P with K being algebraically closed. Define a functor

$$u_P : \mathscr{X}_{\text{Nis-ét-}\mathbf{T}} \to \mathsf{Set}$$

sending an étale morphism

$$X \to \mathscr{X} \ ,$$

where X is of type $\mathbf{T}$ over S, to the set

$$u_P(X \to \mathscr{X}) = |X_P|$$

of points on the fibre

$$X_P = X \times_{\mathscr{X}} \mathrm{Spec}(K)$$

of the morphism $X \to \mathscr{X}$ at P. Notice that since the morphism $X \to \mathscr{X}$ is étale, it is representable with regard to the atlas A on the sheaf $\mathscr{X}$. And since P factorizes through A, the fibre X_P is a locally Noetherian scheme over S.

If X and X' are two schemes of type $\mathbf{T}$ over S and endowed with two étale morphisms $X \to \mathscr{X}$ and $X' \to \mathscr{X}$, and if

$$f : X \to X'$$

is a morphism of schemes over S and over $\mathscr{X}$, i.e. a morphism in $\mathscr{X}_{\text{Nis-ét-}\mathbf{T}}$, then

$$u_P(f) : u_P(X) \to u_P(X')$$

is the map of sets

$$|X_P| \to |X'_P|$$

induced by the scheme-theoretical morphism

$$X_P \to X'_P \ ,$$

which is, in turn, induced by the morphism $X \to X'$.

Let X be a locally Noetherian scheme of type $\mathbf{T}$ over S, let

$$\{X_i \to X\}_{i \in I}$$

be a Nisnevich covering in Noe/S, and let

$$X \to \mathscr{X}$$

be a morphism in $\mathsf{Shv}((\mathsf{Noe}/S)_{\text{ét}})$, étale with regard to the atlas A on $\mathscr{X}$. Since every morphism $X_i \to X$ is smooth, and therefore of type $\mathbf{T}$ over X, the cover $\{X_i \to X\}$ is also a Nisnevich cover of the site $\mathscr{X}_{\text{Nis-ét-}\mathbf{T}}$. Applying the functor u_P we obtain the morphism

$$\coprod_{i \in I} u_P(X_i) \to u_P(X) \ ,$$

which is nothing else but the set-theoretical map

$$\coprod_{i \in I} |(X_i)_P| \to |X_P| \ .$$

Since P factors through A, the latter map is surjective.

If X' is another locally Noetherian scheme of type $\mathbf{T}$ over S and

$$X' \to X$$

is a morphism of schemes over S, such that the composition

$$X' \to X \to \mathscr{X}$$

is étale with regard to A, then we look at the morphism

$$u_P(X_i \times_X X') \to u_P(X_i) \times_{u_P(X)} u_P(X') \ ,$$

that is the map

$$|(X_i \times_X X')_P| \to |(X_i)_P| \times_{|X_P|} |X'_P| \ .$$

Now again, since P factors through A, the latter map is bijective.

In other words, the functor u_P satisfies the items (1) and (2) of Definition 7.31.2 in [23]. The last item (3) of the same definition is satisfied when, for example, the category of neighbourhoods of the point P is cofiltered. Let us discuss item (3) in some more detail.

An étale neighbourhood of P, in the sense of the site $\mathscr{X}_{\text{Nis-ét-}\mathbf{T}}$, is a pair

$$N = (X \to \mathscr{X}, T \in u_P = |X_P|) \ ,$$

where X is of type $\mathbf{T}$ over S, $X \to \mathscr{X}$ is a morphism over S, étale with regard to the atlas A on $\mathscr{X}$, and T is a point of the scheme X_P, represented by, say, the morphism

$$\mathrm{Spec}(\kappa(T)) \to X_P \ .$$

Equivalently, an étale neighbourhood of P is just a commutative diagram of type

$$\begin{tikzcd}
\mathrm{Spec}(K) \arrow[d] \arrow[dr] & \\
X \arrow[r] & \mathscr{X}
\end{tikzcd}$$

where the morphism $X \to \mathscr{X}$ is étale with regard to the atlas A on $\mathscr{X}$, the morphism $\mathrm{Spec}(K) \to \mathscr{X}$ represents the point P, and all morphisms are over the base scheme S.

If

$$N' = (X' \to \mathscr{X}, T' \in |X_P|)$$

is another neighbourhood of P, a morphism

$$N \to N'$$

is a morphism

$$X \to X'$$

over $\mathscr{X}$, and hence over S, such that, if

$$X_P \to X_{P'}$$

is the morphism induced on fibres, the composition

$$\mathrm{Spec}(\kappa(T)) \to X_P \to X_{P'}$$

represents the point T'.

Equivalently, if

$$\begin{tikzcd}
\mathrm{Spec}(K') \arrow[d] \arrow[dr, "P"] & \\
X' \arrow[r] & \mathscr{X}
\end{tikzcd}$$

is another neighbourhood of P, a morphism of neighbourhoods is a morphism

$$X \to X'$$

over $\mathscr{X}$, and hence over S, such that, there is a common field extension K'' of K and K', such that $\mathrm{Spec}(K'') \to \mathscr{X}$ represents P, and the diagram

$$\begin{tikzcd}
\mathrm{Spec}(K'') \arrow[r] \arrow[d] & X' \arrow[d] \\
X \arrow[r] \arrow[ur] & \mathscr{X}
\end{tikzcd}$$

commutes.

Notice that the above definition of a neighbourhood of a point P on $\mathscr{X}$ depends on the functor u_P, sending $X \to \mathscr{X}$ to $|X_P|$. If we change the functor u_P, the notion of neighbourhood will be different, see Section 7.31 in [23].

Let $\mathscr{N}_P$ be the category of neighbourhoods of the point P on $\mathscr{X}$, in the sense of the site $\mathscr{X}_{\text{Nis-ét-}\mathbf{T}}$. If $\mathscr{F}$ is a set valued sheaf on $\mathscr{X}_{\text{Nis-ét-}\mathbf{T}}$, it is, in particular, a set valued presheaf on the same category, and, as such, it induces a functor

$$\mathscr{F}|_{\mathscr{N}_P^{\mathrm{op}}} : \mathscr{N}_P^{\mathrm{op}} \to \mathsf{Set}$$

sending $N = (X \to \mathscr{X}, T \in |X_P|)$ to $\mathscr{F}(X)$ and a morphism $N \to N'$ to the obvious map

$$\mathscr{F}(X') \to \mathscr{F}(X) \; .$$

The stalk functor

$$\mathrm{st}_P : \mathsf{Shv}(\mathscr{X}_{\text{Nis-ét-}\mathbf{T}}) \to \mathsf{Set}$$

sends a sheaf $\mathscr{F}$ on $\mathscr{X}_{\text{Nis-ét-}\mathbf{T}}$ to the colimit

$$\mathrm{colim}\,(\mathscr{F}|_{\mathscr{N}_P^{\mathrm{op}}})$$

of the functor $\mathscr{F}|_{\mathscr{N}_P^{\mathrm{op}}}$.

Once again, we should not forget here that the stack functor st_P depends on the definition of a neighbourhood, and the latter depends on the choice of the functor u_P, see Section 7.31 in [23].

Now, as finite limits commute with filtered colimits, if the category $\mathscr{N}_P$ is cofiltered, the stalk functor st_P is left exact, and item (3) of Definition 7.31.2 in [23] holds true as well, and the stalk functor st_P gives rise to a point of the topos $\mathsf{Shv}(\mathscr{X}_{\text{Nis-ét-}\mathbf{T}})$, see Lemma 7.31.7 in [23]. If this is the case, it gives us the well-behaved stalks

$$\mathscr{O}_{\mathscr{X},P} = \mathrm{st}_P(\mathscr{O}_{\mathscr{X}}) \; ,$$
$$\Omega^1_{\mathscr{X}/S,P} = \mathrm{st}_P(\Omega^1_{\mathscr{X}/S})$$

and

$$T_{\mathscr{X}/S,P} = \mathrm{st}_P(T_{\mathscr{X}/S})$$

at the point P.

The latter stalk is not, however, a tangent space to $\mathscr{X}$ at P. To achieve an honest tangent space we need to observe that, whenever $\mathscr{N}_P$ is cofiltered for each P, the site $\mathscr{X}_{\text{Nis-ét-}\mathbf{T}}$ is locally ringed in the sense of the definition appearing in Exercise 13.9 on page 512 in [1] (see page 313 in the newly typeset version), as well as in the sense of a sightly different Definition 18.39.4 in [23]. Indeed, any scheme U is a locally ring site with enough points. Applying Lemma 18.39.2 in loc.cit we see that for any Zariski open subset V in U and for any function $\mathscr{O}_U(V)$ there exists an open covering $V = \cup V_i$ of the set V such that for each index i either $f|_{V_i}$ is invertible or $(1-f)|_{V_i}$ is invertible. If now $U \to \mathscr{X}$ is an étale morphism from a scheme U to $\mathscr{X}$ over S, with regard to the atlas on $\mathscr{X}$, since

$$\Gamma(U, \mathscr{O}_{\mathscr{X}}) = \Gamma(U, \mathscr{O}_U) \; ,$$

we obtain item (1) of Lemma 18.39.1 in [23], and the condition (18.39.2.1) in loc.cit. is obvious.

Now, since the site $\mathscr{X}_{\text{Nis-ét-}\mathbf{T}}$ is locally ringed, we consider the maximal ideal

$$\mathfrak{m}_{\mathscr{X},P} \subset \mathscr{O}_{\mathscr{X},P}$$

and let

$$\kappa(P) = \mathscr{O}_{\mathscr{X},P}/\mathfrak{m}_{\mathscr{X},P}$$

be the residue field of the locally ring site at the point P. Then we also have two vector spaces

$$\Omega^1_{\mathscr{X}/S}(P) = \Omega^1_{\mathscr{X}/S,P} \otimes_{\mathscr{O}_P} \kappa(P)$$

and

$$T_{\mathscr{X}/S}(P) = T_{\mathscr{X}/S,P} \otimes_{\mathscr{O}_P} \kappa(P)$$

over the residue field $\kappa(P)$. The latter is our *tangent space* to the space $\mathscr{X}$ at the point P.

3. Categorical monoids and group completions

Let S be a cartesian monoidal category, so that the terminal object $*$ is the monoidal unit in S. Denote by $\mathsf{Mon}(\mathsf{S})$ the full subcategory of monoids[2], and by $\mathsf{Ab}(\mathsf{S})$ the full subcategory of abelian group objects in the category S. Assume that S is closed under finite colimits and countable coproducts which are distributive with regard to the cartesian product in S. Then the forgetful functor from $\mathsf{Mon}(\mathsf{S})$ to S has left adjoint which can be constructed as follows.

For any object $\mathscr{X}$ in S and for any natural number d let $\mathscr{X}^d$ be the d-fold monoidal product of $\mathscr{X}$. Consider the d-th symmetric power

$$\mathrm{Sym}^d(\mathscr{X}) ,$$

i.e. the quotient of the object $\mathscr{X}^d$ by the natural action of the d-th symmetric group Σ_d in the category S. In particular,

$$\mathrm{Sym}^0(\mathscr{X}) = * \quad \text{and} \quad \mathrm{Sym}^1(\mathscr{X}) = \mathscr{X} .$$

The coproduct

$$\coprod_{d=0}^{\infty} \mathrm{Sym}^d(\mathscr{X})$$

is a monoid, whose concatenation product

$$\coprod_{d=0}^{\infty} \mathrm{Sym}^d(\mathscr{X}) \times \coprod_{d=0}^{\infty} \mathrm{Sym}^d(\mathscr{X}) \to \coprod_{d=0}^{\infty} \mathrm{Sym}^d(\mathscr{X})$$

is induced by the obvious morphism

$$\coprod_{d=0}^{\infty} \mathscr{X}^d \times \coprod_{d=0}^{\infty} \mathscr{X}^d \to \coprod_{d=0}^{\infty} \mathscr{X}^d$$

and the embeddings of $\Sigma_i \times \Sigma_j$ in to Σ_{i+j}. The unit

$$* \to \coprod_{d=0}^{\infty} \mathrm{Sym}^d(\mathscr{X})$$

identifies $*$ with $\mathscr{X}^{(0)}$. This monoid will be called the *free monoid* generated by $\mathscr{X}$ and denoted by $\mathbb{N}(\mathscr{X})$. Thus,

$$\mathbb{N}(\mathscr{X}) = \coprod_{d=0}^{\infty} \mathrm{Sym}^d(\mathscr{X}) .$$

For example,

$$\mathbb{N}(*) = \mathbb{N} .$$

[2]all monoids in this paper will be commutative by default

It is easy to verify that the functor

$$\mathbb{N} : \mathsf{S} \to \mathsf{Mon}(\mathsf{S})$$

is left adjoint to the forgetful functor from $\mathsf{Mon}(\mathsf{S})$ to S.

The full embedding of $\mathsf{Ab}(\mathsf{S})$ in to $\mathsf{Mon}(\mathsf{S})$ admits left adjoint, if we impose some extra assumption on the category S. Namely, let $\mathscr{X}$ be a monoid in S, and look at the obvious diagonal morphism

$$\Delta : \mathscr{X} \to \mathscr{X} \times \mathscr{X} \tag{6}$$

in the category S, which is also a morphism in the category $\mathsf{Mon}(\mathsf{S})$. The terminal object $*$ in the category S is a trivial monoid, i.e. a terminal object in the category $\mathsf{Mon}(\mathsf{S})$.

Assume there exists a co-Cartesian square

$$\begin{tikzcd} \mathscr{X} \arrow[r, "\Delta"] \arrow[d] & \mathscr{X} \times \mathscr{X} \arrow[d] \\ * \arrow[r] & \mathscr{X}^+ \end{tikzcd} \tag{7}$$

in the category of monoids $\mathsf{Mon}(\mathsf{S})$. Then $\mathscr{X}^+$ is an abelian group object in the category S.

Let

$$\iota_{\mathscr{X}} : \mathscr{X} \to \mathscr{X}^+$$

be the composition of the canonical embedding

$$\iota_1 : \mathscr{X} \to \mathscr{X} \times \mathscr{X} \ ,$$

$$x \mapsto (x, 0)$$

with the projection

$$\pi_{\mathscr{X}} : \mathscr{X} \times \mathscr{X} \to \mathscr{X}^+ \ .$$

If

$$f : \mathscr{X} \to \mathscr{Y}$$

is a morphism of monoids and $\mathscr{Y}$ is an abelian group object in S, the precomposition of the homomorphism

$$(f, -f) : \mathscr{X} \times \mathscr{X} \to \mathscr{Y} \ ,$$

sending (x_1, x_2) to $f(x_1) - f(x_2)$ with the diagonal embedding is 0, whence there exists a unique group homomorphism h making the diagram

$$\begin{tikzcd} \mathscr{X} \arrow[r, "\iota_{\mathscr{X}}"] \arrow[dr, "f"'] & \mathscr{X}^+ \arrow[d, dotted, "\exists ! h"] \\ & \mathscr{Y} \end{tikzcd}$$

commutative.

This all shows that $\mathscr{X}^+$ is nothing else but the the *group completion* of the monoid $\mathscr{X}$, and the group completion functor

$$-^+ : \mathsf{Mon}(\mathsf{S}) \to \mathsf{Ab}(\mathsf{S})$$

is left adjoint to the forgetful functor from $\mathsf{Ab}(\mathsf{S})$ to $\mathsf{Mon}(\mathsf{S})$.

For example,

$$\mathbb{Z} = \mathbb{N}^+$$

is the group completion of the free monoid $\mathbb{N}$, generated by the terminal object $*$ in the category S.

Notice that, as the categories $\mathsf{Mon}(\mathsf{S})$ and $\mathsf{Ab}(\mathsf{S})$ are pointed, one can show the existence of the canonical isomorphism of monoids

$$(\mathscr{X} \times \mathscr{X})^+ \stackrel{\sim}{\to} \mathscr{X}^+ \times \mathscr{X}^+ \ .$$

In other words, the group completion functor is monoidal.

It is useful to understand how all these constructions work for set-theoretical monoids. Since monoids are not groups, some care is in place here.

Let M be a monoid in the category of sets Set, written additively, and assume first that we are given with a submonoid N in M. To understand what would be the quotient monoid of M by N, we define a relation

$$R \subset M \times M$$

saying that, for any two elements $m, m' \in M$,

$$mRm' \ \Leftrightarrow \ \exists n, n' \in N \text{ with } m + n = m' + n' \ . \tag{8}$$

Then R is a congruence relation on M, i.e. an equivalence relation compatible with the operation in M. Indeed, the reflexivity and symmetry are obvious. Suppose that we have three elements

$$m, m', m'' \in M \ ,$$

and

$$\exists n, n' \in N, \text{ such that } m + n = m' + n' \ .$$

and

$$\exists l', l'' \in N, \text{ such that } m' + l' = m'' + l'' \ .$$

Then

$$m + n + l' = m' + n' + l' = m'' + l'' + n' \ .$$

Clearly,

$$n + l', \ l'' + n' \in N \ ,$$

and we get transitivity. Thus, R is an equivalence relation.

Let M/N be the corresponding quotient set, and let

$$\pi : M \to M/N$$

$$m \mapsto [m]$$

be the quotient map. The structure of a monoid on M/N is obvious,

$$[m] + [\tilde{m}] = [m + \tilde{m}] \ ,$$

and since M is a commutative monoid[3], it follows easily that the map π is a homomorphism of monoids. In other terms, the above relation R on M is a congruence relation.

Moreover, the quotient homomorphism

$$M \to M/N$$

enjoys the standard universal property, loc.cit. To be more precise, for any homomorphism of monoids

$$f : M \to T \ ,$$

such that

$$N \subset \ker(f) = \{m \in M \mid f(m) = 0\} \ ,$$

there exists a commutative diagram of type

$$\begin{tikzcd}
M \arrow[r] \arrow[dr] & M/N \arrow[d, dotted, "\exists !"] \\
 & T
\end{tikzcd}$$

Now, let $M \times M$ be the product monoid, let

$$\Delta : M \to M \times M$$

be the diagonal homomorphism, and let

$$\Delta(M)$$

be the set-theoretical image of the homomorphism Δ. Trivially, $\Delta(M)$ is a submonoid in the product monoid $M \times M$, and we can construct the quotient monoid

$$M^+ = (M \times M)/\Delta(M) \ ,$$

using the procedure explained above. The universal property of the quotient monoid gives us that the diagram

$$\begin{tikzcd}
M \arrow[r, "\Delta"] \arrow[d] & M \times M \arrow[d] \\
* \arrow[r] & M^+
\end{tikzcd} \tag{9}$$

is pushout in the category $\mathsf{Mon}(\mathsf{Set})$. It follows that M^+ is the group completion of M in the sense of our definition given for the general category S.

[3]recall that, within this paper, all monids are commutative by defaul

Clearly, the composition

$$\begin{tikzcd} M \arrow[r, "m\mapsto(m,0)"] \arrow[dr, "\iota_M"'] & M\times M \arrow[d] \\ & M^+ \end{tikzcd}$$

is a homomorphism of monoids. If

$$f : M \to A$$

is a homomorphism from the monoid M to an abelian group A, then we define a homomorphism of monoids

$$M \times M \to A$$

sending

$$(m_1, m_2) \mapsto f(m_1) - f(m_2) \ ,$$

and the universal property of the diagram (9) gives us the needed commutative diagram

$$\begin{tikzcd} M \arrow[r, "\iota_M"] \arrow[dr] & M^+ \arrow[d, dotted, "\exists !"] \\ & A \end{tikzcd}$$

Moreover, if M is cancellative, the diagram (9) is not only a pushout square in $\mathsf{Mon}(\mathsf{Set})$ but also a pullback square in Set.

Indeed, if

$$(m_1, m_2),\ (m_1', m_2') \in M \times M \ ,$$

then, according to (8),

$$\exists n, n' \in M$$

such that

$$(m_1, m_2) + (n, n) = (m_1', m_2') + (n', n')$$

in $M \times M$, or, equivalently,

(10) $$m_1 + n = m_1' + n' \quad \text{and} \quad m_2 + n = m_2' + n' \ .$$

Now, suppose we want to find h completing a commutative diagram of type

(11) $$\begin{tikzcd} T \arrow[dr, dotted, "\exists ! h"] \arrow[drr, bend left, "f"] \arrow[ddr, bend right] & & \\ & M \arrow[r] \arrow[d] & M\times M \arrow[d, "\pi"] \\ & * \arrow[r] & M^+ \end{tikzcd}$$

in the category Set. If (m_1, m_2) is an element of $M \times M$, the equivalence class $[m_1, m_2]$ is 0 in M^+, i.e. the ordered pair (m_1, m_2) is equivalent to $(0, 0)$ in $M \times M$ modulo the subtractive submonoid $\Delta(M)$, if and only if, by (10),

$$m_1 + n = n' \quad \text{and} \quad m_2 + n = n' \ ,$$

whence

$$m_1 + n = m_2 + n \ .$$

Since M is a cancellation monoid, the latter equality gives us that $m_1 = m_2$, i.e. (m_1, m_2) is in $\Delta(M)$. In other words, $[m_1, m_2] = 0$ in M^+ if and only if (m_1, m_2) is in $\Delta(M)$. And as the diagram (11) is commutative without h, it follows that the set-theoretical image of the map f is in $\Delta(M)$. It follows that f factorizes through Δ, i.e. the needed map h exists.

Thus, we see that the abstract constructions relevant to group completions are generalizations of the standard constructions in terms of set-theoretical monoids.

All the same arguments apply when S is the category $\mathsf{PShv}(\mathsf{C})$ of set valued presheaves on a category C, as all limits and colimits in $\mathsf{PShv}(\mathsf{C})$ are sectionwise. Thus, for any monoid $\mathscr{X}$ in $\mathsf{PShv}(\mathsf{C})$ the group completion $\mathscr{X}^+$ exists and it is a section wise group completion. If $\mathscr{X}$ is cancellative, and this is equivalent to saying that $\mathscr{X}$ is section wise cancellative, then the diagram (7) is Cartesian in $\mathsf{PShv}(\mathsf{C})$.

Now let us come back to the general setting. Let again $\mathscr{X}$ be a monoid in S. The notion of a cancellative monoid can be categorified as follows. A morphism

$$\iota : \mathbb{N} \to \mathscr{X}$$

in the category $\mathsf{Mon}(\mathsf{S})$, that is a homomorphism of monoids from $\mathbb{N}$ to $\mathscr{X}$, is uniquely defined by the restriction

$$\iota(1) : * \to \mathscr{X}$$

of α on to the subobject $* = \mathrm{Sym}^1(*)$ of the object $\mathbb{N} = \coprod_{d=0}^{\infty} \mathrm{Sym}^d(*)$ in S. Vice versa, as soon as we have a morphism $* \to \mathscr{X}$ in the category S, it uniquely defines the obvious morphism $\iota : \mathbb{N} \to \mathscr{X}$ in the category $\mathsf{Mon}(\mathsf{S})$. The homomorphism of monoids ι will be said to be *cancellative* if the composition

$$\mathrm{ad}_{\iota(1)} : \mathscr{X} \simeq \mathscr{X} \times * \stackrel{\mathrm{id} \times \iota(1)}{\longrightarrow} \mathscr{X} \times \mathscr{X} \to \mathscr{X} \ ,$$

that is the addition of $\iota(1)$ on $\mathscr{X}$, is a monomorphism in S. The monoid $\mathscr{X}$ is a *cancellation monoiod* if any homomorphism $\iota : \mathbb{N} \to \mathscr{X}$ is cancellative.

Clearly, if $\mathscr{X}$ is a monoid in $\mathsf{PShv}(\mathsf{C})$, then $\mathscr{X}$ is cancellative if and only if it is section wise cancellative.

A *pointed monoid* in S is a pair $(\mathscr{X}, \iota)$, where $\mathscr{X}$ is a monoid in S and ι is a morphism of monoids from $\mathbb{N}$ to $\mathscr{X}$. A *graded pointed monoid* is a triple $(\mathscr{X}, \iota, \sigma)$, where $(\mathscr{X}, \iota)$ is a pointed monoid and σ is a morphism of monoids from $\mathscr{X}$ to $\mathbb{N}$, such that

$$\sigma \circ \iota = \mathrm{id}_{\mathbb{N}} \ .$$

If $\mathscr{X}$ is a pointed graded monoid in S, for any natural number $d \in \mathbb{N}$ one can consider the cartesian square

$$\begin{tikzcd} \mathscr{X}_d \arrow[r] \arrow[d] & * \arrow[d, "d"] \\ \mathscr{X} \arrow[r, "\sigma"] & \mathbb{N} \end{tikzcd}$$

in the category S. The addition of $\iota(1)$ in $\mathscr{X}$ induces morphisms

$$\mathscr{X}_d \to \mathscr{X}_{d+1}$$

for all $d \geq 0$. Let $\mathscr{X}_\infty$ be the colimit

$$\mathscr{X}_\infty = \operatorname{colim}(\mathscr{X}_0 \to \mathscr{X}_1 \to \mathscr{X}_2 \to \dots)$$

in S. Equivalently, $\mathscr{X}_\infty$ is the coequalizer of the addition of $\iota(1)$ in $\mathscr{X}$ and the identity automorphism of $\mathscr{X}$. Since filtered colimits commute with finite products, there is a canonical isomorphism between the colimit of the obvious diagram composed by the objects $\mathscr{X}_d \times \mathscr{X}_{d'}$, for all $d, d' \geq 0$, and the product $\mathscr{X}_\infty \times \mathscr{X}_\infty$. Since the colimit of that diagram is the colimit of its diagonal, this gives the canonical morphism from $\mathscr{X}_\infty \times \mathscr{X}_\infty$ to $\mathscr{X}_\infty$. The latter defines the structure of a monoid on $\mathscr{X}_\infty$, such that the canonical morphism

$$\pi : \mathscr{X} = \coprod_{d \geq 0} \mathscr{X}_d \to \mathscr{X}_\infty$$

is a homomorphism of monoids in S. We call $\mathscr{X}_\infty$ the *connective* monoid associated to the pointed graded monoid $\mathscr{X}$.

Notice that if the category S is exhaustive[4], monomorphicity of the morphisms $\mathscr{X}_d \to \mathscr{X}_{d+1}$ yields that the transfinite compositions $\mathscr{X}_d \to \mathscr{X}_\infty$ are monomorphisms too. The morphisms $\mathscr{X}_d \to \mathscr{X}_{d+1}$ are monomorphic, for example, if $\mathscr{X}$ is a cancelation monoid.

Now assume that the colimit $\mathscr{X}_\infty^+$ exists in the category $\mathsf{Mon}(\mathsf{S})$. Since $\mathscr{X}_\infty$ is the coequalizer of $\mathrm{ad}_{\iota(1)}$ and $\mathrm{id}_{\mathscr{X}}$, the group completion $\mathscr{X}_\infty^+$ is the coequalizer of the corresponding homomorphism $\mathrm{ad}_{\iota(1)}^+ : \mathscr{X}^+ \to \mathscr{X}^+$ and $\mathrm{id}_{\mathscr{X}^+}$. It follows that the sequence

$$0 \to \mathbb{Z} \xrightarrow{\iota^+} \mathscr{X}^+ \to \mathscr{X}_\infty^+ \to 0$$

is short exact. Moreover, this sequens splits by the morphism σ^+. This gives us that

$$\mathscr{X}^+ = \mathbb{Z} \oplus \mathscr{X}_\infty^+$$

in the abelian category $\mathsf{Ab}(\mathsf{S})$.

A typical example of a pointed graded monoid in S is the free monoid

$$\mathbb{N}(\mathscr{X}) = \coprod_{d=0}^{\infty} \mathrm{Sym}^d(\mathscr{X}) ,$$

[4]see `https://ncatlab.org/nlab/show/exhaustive+category`

where $\mathscr{X}$ is a pointed object in S, i.e. the morphism from $\mathscr{X}$ to the terminal object $*$ has a section. For this pointed graded monoid we have that

$$\mathbb{N}(\mathscr{X})_d = \mathrm{Sym}^d(\mathscr{X}) ,$$

for all natural numbers d, and the pointing of each symmetric power $\mathrm{Sym}^d(\mathscr{X})$ is induced by the pointing of $\mathscr{X}$ in the obvious way. The section gives embeddings

$$\mathrm{Sym}^d(\mathscr{X}) \to \mathrm{Sym}^{d+1}(\mathscr{X}) ,$$

and the corresponding connective monoid

$$\mathbb{N}(\mathscr{X})_\infty = \mathrm{colim}_d \, \mathrm{Sym}^d(\mathscr{X})$$

will be denoted by $\mathrm{Sym}^\infty(\mathscr{X})$ and called the *free connective monoid* of the object $\mathscr{X}$. Then, of course,

$$\mathrm{Sym}^\infty(\mathscr{X})^+ = \mathbb{N}(\mathscr{X})^+_\infty .$$

Moreover, both free monoids $\mathbb{N}(\mathscr{X})$ and $\mathbb{N}(\mathscr{X})_\infty$ are cancellative monoids in S.

Now, let C be a cartesian monoidal category with a terminal object $*$, closed under finite fibred products and equipped with a subcanonical topology τ. Let $\mathsf{PShv}(\mathsf{C})$ be the category of set valued presheaves on C, and let $\mathsf{Shv}(\mathsf{C}_\tau)$ be the full subcategory in $\mathsf{PShv}(\mathsf{C})$ of sheaves on C with regard to the topology τ. Since the category C is cartesian, so are the categories $\mathsf{PShv}(\mathsf{C})$ and $\mathsf{Shv}(\mathsf{C}_\tau)$, and therefore we can consider the monoids in the categories of sheaves and pre-sheaves. Our aim is now to apply the constructions above in the case when

$$\mathsf{S} = \mathsf{PShv}(\mathsf{C}) \quad \text{or} \quad \mathsf{S} = \mathsf{Shv}(\mathsf{C}_\tau) .$$

The Yoneda embedding

$$h : \mathsf{C} \to \mathsf{PShv}(\mathsf{C})$$

is a continuous functor, i.e. it preserves limits. It follows that, if $\mathscr{X}$ is a monoid in $\mathsf{PShv}(\mathsf{C})$, then it is equivalent to saying that $\mathscr{X}$ is a section wise monoid, and the two diagrams

$$\begin{tikzcd} \mathsf{Mon}(\mathsf{C}) \arrow[r] \arrow[d, "h"] & \mathsf{C} \arrow[d, "h"] \\ \mathsf{Mon}(\mathsf{PShv}(\mathsf{C})) \arrow[r] & \mathsf{PShv}(\mathsf{C}) \end{tikzcd}$$

and

$$\begin{tikzcd} \mathsf{Ab}(\mathsf{C}) \arrow[r] \arrow[d] & \mathsf{Mon}(\mathsf{C}) \arrow[d] \\ \mathsf{Ab}(\mathsf{PShv}(\mathsf{C})) \arrow[r] & \mathsf{Mon}(\mathsf{PShv}(\mathsf{C})) \end{tikzcd}$$

are commutative. Moreover, the diagonal morphism (6) for a presheaf $\mathscr{X}$ is diagonal section-wise. It follows that the colimit diagram (7) exists in $\mathsf{Mon}(\mathsf{PShv}(\mathsf{C}))$

and, accordingly, the group completion $\mathscr{X}^+$ is then the section wise group completion of $\mathscr{X}$. In particular, the group completion $\mathscr{X}^+$ of the presheaf monoid $\mathscr{X}$ is topology free.

Let $\mathsf{PShv(C)_s}$ be the full subcategory of separated presheaves in $\mathsf{PShv(C)}$, and let

$$-^{\mathrm{s}} : \mathsf{PShv(C)} \to \mathsf{PShv(C)_s}$$

$$\mathscr{F} \mapsto \mathscr{F}^{\mathrm{s}}$$

be the left adjoint to the forgetful functor from $\mathsf{PShv(C)_s}$ to $\mathsf{PShv(C)}$, as constructed on page 40 in [5]. Let also

$$-^{\mathrm{g}} : \mathsf{PShv(C)_s} \to \mathsf{Shv(C_\tau)}$$

$$\mathscr{F} \mapsto \mathscr{F}^{\mathrm{g}}$$

be the second stage of sheafification, i.e. the gluing of sections as described on the same page of the same book, or, in other words, the left adjoint to the forgetful functor from $\mathsf{Shv(C_\tau)}$ to $\mathsf{PShv(C)_s}$. The composition

$$-^{\mathrm{a}} : \mathsf{PShv(Sch}/S) \to \mathsf{Shv(C_\tau)}$$

of these two functors $-^{\mathrm{s}}$ and $-^{\mathrm{g}}$ is the left adjoint to the forgetful functor from $\mathsf{Shv(C_\tau)}$ to $\mathsf{PShv(C)}$, i.e. the functor which associates to any presheaf the corresponding associated sheaf in the topology τ, see pp 39 - 40 in [5].

Now, since the sheafification functor $-^{\mathrm{a}}$ is left adjoint to the forgetful functor from sheaves to presheaves, the latter is right adjoint, and hence it commutes with limits. In particular, the forgetful functor from sheaves to presheaves commutes with products. It follows that the diagrams

$$\begin{tikzcd} \mathsf{Mon(PShv(C))} \arrow[r] & \mathsf{PShv(C)} \\ \mathsf{Mon(Shv(C_\tau))} \arrow[r] \arrow[u] & \mathsf{Shv(C_\tau)} \arrow[u] \end{tikzcd}$$

and

$$\begin{tikzcd} \mathsf{Ab(PShv(C))} \arrow[r] & \mathsf{Mon(PShv(C))} \\ \mathsf{Ab(Shv(C_\tau))} \arrow[r] \arrow[u] & \mathsf{Mon(Shv(C_\tau))} \arrow[u] \end{tikzcd}$$

are commutative.

Next, it is well-known that the functor $-^{\mathrm{a}}$ is exact too, and hence it commutes with products. It follows that $-^{\mathrm{a}}$ takes monoids to monoids, and abelian groups to abelian groups, and therefore we have the commutative diagrams

$$\begin{tikzcd}
\mathsf{Mon}(\mathsf{PShv}(\mathsf{C})) \arrow[r] \arrow[d, "-^{\mathrm{a}}"'] & \mathsf{PShv}(\mathsf{C}) \arrow[d, "-^{\mathrm{a}}"'] \\
\mathsf{Mon}(\mathsf{Shv}(\mathsf{C}_\tau)) \arrow[r] & \mathsf{Shv}(\mathsf{C}_\tau)
\end{tikzcd}$$

and

$$\begin{tikzcd}
\mathsf{Ab}(\mathsf{PShv}(\mathsf{C})) \arrow[r] \arrow[d, "-^{\mathrm{a}}"'] & \mathsf{Mon}(\mathsf{PShv}(\mathsf{C})) \arrow[d, "-^{\mathrm{a}}"'] \\
\mathsf{Ab}(\mathsf{Shv}(\mathsf{C}_\tau)) \arrow[r] & \mathsf{Mon}(\mathsf{Shv}(\mathsf{C}_\tau))
\end{tikzcd}$$

Now, the functor $\mathbb{N}$ exists for set valued presheave monoids and it is given section wise. Moreover, as we mentioned above, the group completion functor exists for set valued presheaf monoids, and it is also given section wise. It follows that the functors $\mathbb{N}$ and $-^+$ exist also for sheaves on the site C_τ, and can be constructed by means of composing of the corresponding functors for presheaves with the sheafification functor.

To be more precise, since the sheafification $-^{\mathrm{a}}$ is left adjoint, it also commutes with all colimits. And as the functors $\mathbb{N}$ and $-^+$ are constructed merely by means of products and colimits, we conclude that that these two functors are preserved by sheafification. In other words, the diagrams

$$\begin{tikzcd}
\mathsf{Mon}(\mathsf{PShv}(\mathsf{C})) \arrow[d, "-^{\mathrm{a}}"'] & \mathsf{PShv}(\mathsf{C}) \arrow[l, "\mathbb{N}"'] \arrow[d, "-^{\mathrm{a}}"'] \\
\mathsf{Mon}(\mathsf{Shv}(\mathsf{C}_\tau)) & \mathsf{Shv}(\mathsf{C}_\tau) \arrow[l, "\mathbb{N}"']
\end{tikzcd}$$

and

$$\begin{tikzcd}
\mathsf{Ab}(\mathsf{PShv}(\mathsf{C})) \arrow[d, "-^{\mathrm{a}}"'] & \mathsf{Mon}(\mathsf{PShv}(\mathsf{C})) \arrow[l, "-^+"'] \arrow[d, "-^{\mathrm{a}}"'] \\
\mathsf{Ab}(\mathsf{Shv}(\mathsf{C}_\tau)) & \mathsf{Mon}(\mathsf{Shv}(\mathsf{C}_\tau)) \arrow[l, "-^+"']
\end{tikzcd}$$

both commute. As a consequence of that, the diagrams

$$\begin{tikzcd}
\mathsf{Mon}(\mathsf{PShv}(\mathsf{C})) \arrow[d, "_^{\mathrm{a}}"'] & \mathsf{PShv}(\mathsf{C}) \arrow[l, "\mathbb{N}"'] \\
\mathsf{Mon}(\mathsf{Shv}(\mathsf{C}_\tau)) & \mathsf{Shv}(\mathsf{C}_\tau) \arrow[l, "\mathbb{N}"'] \arrow[u]
\end{tikzcd}$$

and

$$\begin{tikzcd}
\mathsf{Ab}(\mathsf{PShv}(\mathsf{C})) \arrow[d, "_^{\mathrm{a}}"'] & \mathsf{Mon}(\mathsf{PShv}(\mathsf{C})) \arrow[l, "_^{+}"'] \\
\mathsf{Ab}(\mathsf{Shv}(\mathsf{C}_\tau)) & \mathsf{Mon}(\mathsf{Shv}(\mathsf{C}_\tau)) \arrow[l, "_^{+}"'] \arrow[u]
\end{tikzcd}$$

are also both commutative.

The latter two commutative diagrams mean the following. If $\mathscr{X}$ is a set valued sheaf on C_τ, then, in order to construct the free monoid $\mathbb{N}(\mathscr{X})$ in the category $\mathsf{Mon}(\mathsf{Shv}(\mathsf{C}_\tau))$ we first forget the sheaf property on $\mathscr{X}$ and construct $\mathbb{N}(\mathscr{X})$ in the category $\mathsf{Mon}(\mathsf{PShv}(\mathsf{C}))$, looking at $\mathscr{X}$ as a presheaf, and then sheafify to get an object in $\mathsf{Mon}(\mathsf{Shv}(\mathsf{C}_\tau))$. Similarly, if $\mathscr{X}$ is a set valued sheaf monoid, i.e. an object of the category $\mathsf{Mon}(\mathsf{Shv}(\mathsf{C}_\tau))$, then, in order to construct its group completion in the category $\mathsf{Ab}(\mathsf{Shv}(\mathsf{C}_\tau))$ we forget the sheaf property on $\mathscr{X}$ and construct $\mathscr{X}^+$ in the category $\mathsf{Ab}(\mathsf{PShv}(\mathsf{C}))$, looking at $\mathscr{X}$ as a presheaf monoid, and then sheafify to get an object in $\mathsf{Ab}(\mathsf{Shv}(\mathsf{C}_\tau))$.

Similarly, if $\mathscr{X}$ is a pointed graded monoid in presheaves, then it is a pointed graded monoid section wise. The construction of the connective monoid $\mathscr{X}_\infty$, as an object in the category $\mathsf{Mon}(\mathsf{PShv}(\mathsf{C}))$, is then section wise and topology free. But if $\mathscr{X}$ is a pointed graded monoid in sheaves, the construction of $\mathscr{X}_\infty$ follows the rule above. Namely, we first forget the sheaf property of $\mathscr{X}$ and construct $\mathscr{X}_\infty$ section wise, i.e. in the category $\mathsf{Mon}(\mathsf{PShv}(\mathsf{C}))$, and then sheafify to get the object $\mathscr{X}_\infty$ in the category $\mathsf{Ab}(\mathsf{PShv}(\mathsf{C}))$.

As in the previous section, for simplicity of notation, we will write X instead of the sheaf h_X, for any object X in C, and denote objects in $\mathsf{PShv}(\mathsf{C})$ and $\mathsf{Shv}(\mathsf{C}_\tau)$ by calligraphic letters $\mathscr{X}$, $\mathscr{Y}$, etc.

Notice also that if X is a pointed object of C and for any d the d-th symmetric power $\mathrm{Sym}^d(X)$ exists already in C, then $\mathbb{N}(X)_\infty$ is an ind-object of C. Recall that an ind-object in C is the colimit of the composition of a functor

$$I \to \mathsf{C}$$

with the embedding of C in to $\mathsf{PShv}(\mathsf{C})$, taken in the category $\mathsf{PShv}(\mathsf{C})$, such that the category I is filtered. Such a colimit is section-wise. Since C is equipped with a topology, one can also give the definition of a sheaf-theoretical ind-object. An ind-object in C_τ is the colimit of the same composition, but now taken in the category $\mathsf{Shv}(\mathsf{C}_\tau)$. The latter is obviously the sheafification of the previous ind-object, and therefore it depends on the topology τ. Let $\mathsf{Ind}(\mathsf{C})$ be the full subcategory in

$\mathsf{PShv}(\mathsf{C})$ of ind-objects in C, and let $\mathsf{Ind}(\mathsf{C}_\tau)$ be the full subcategory in $\mathsf{Shv}(\mathsf{C}_\tau)$ of ind-objects of C_τ.

Our aim will be to apply these abstract constructions to the case when

$$\mathsf{C} = \mathsf{Noe}/S$$

and

$$\tau = \mathrm{Nis}\ .$$

The choice of the topology will be explained in the next section. Now we need to recall relative symmetric powers of locally Noetherian schemes X over S.

Assume that the structural morphism

$$X \to S$$

satisfies the following property:

(AF) for any point $s \in S$ and for any finite collection $\{x_1, \dots, x_l\}$ of points in the fibre X_s of the structural morphism $X \to S$ at s there exists a Zariski open subset U in X, such that

$$\{x_1, \dots, x_l\} \subset U$$

and the composition

$$U \to X \to S$$

is a quasi-affine morphism of schemes.

Quasi-affine morphisms possess various nice properties, see Section 28.12 in [23], which can be used to prove that if $U \to S$ is a morphism of locally Noetherian schemes and X is AF over S then $X \times_S U$ is AF over U. If, moreover, U is AF over S the $X \times_S U$ is AF over S.

The property AF is satisfied if, for example, $X \to S$ is a quasi-affine or quasi-projective morphism of schemes, see Prop. (A.1.3) in Paper I in [18].

As we now assume that AF holds true for X over S, the d-th symmetric group Σ_d acts admissibly on the d-th fibred product

$$(X/S)^d = X \times_S \dots \times_S X$$

over S in the sense of [9], Exposé V, and the relative symmetric power

$$\mathrm{Sym}^d(X/S)$$

exists in the category Noe/S.

Then, according to the abstract constructions above, we obtain the free monoid $\mathbb{N}(X/S)$ generated by the scheme X over S in the category $\mathsf{Shv}((\mathsf{Noe}/S)_{\mathrm{Nis}})$. For every integer $d \geq 0$ the object $\mathbb{N}(X/S)_d$ is the relative d-th symmetric power $\mathrm{Sym}^d(X/S)$ of X over S, and as such it is an object of the category Noe/S. The free monoid of the scheme X over S is nothing else but the coproduct

$$\mathbb{N}(X/S) = \coprod_{i=0}^{\infty} \mathrm{Sym}^d(X/S)$$

taken in the category $\mathsf{Shv}((\mathsf{Noe}/S)_{\mathrm{Nis}})$.

Assume, in addition, that the structural morphism $X \to S$ has a section

$$S \to X\ .$$

Notice that the terminal object $*$ in the category Noe/S is the identity morphism of the scheme S, and therefore the splitting of the structural morphism $X \to S$ by the section $S \to X$ induces the splitting

$$\begin{tikzcd} & \mathbb{N}(X/S) \arrow[dr, "\sigma"] & \\ \mathbb{N} \arrow[ur, "\iota"] \arrow[rr, "\mathrm{id}"] & & \mathbb{N} \end{tikzcd}$$

in the category $\mathsf{Shv}((\mathsf{Noe}/S)_{\mathrm{Nis}})$. The corresponding connective monoid

$$\mathrm{Sym}^{\infty}(X/S) = \mathbb{N}(X/S)_{\infty} = \mathrm{colim}_{\,d}\, \mathrm{Sym}^d(X/S)$$

is an ind-scheme over S. As such it can be considered as an object of the category $\mathsf{Ind}((\mathsf{Noe}/S)_{\mathrm{Nis}})$.

The colimit

$$\mathrm{Sym}^{\infty}(X/S)^{+}$$

in the category $\mathsf{Mon}(\mathsf{Shv}((\mathsf{Noe}/S)_{\mathrm{Nis}}))$ is the group completion of the monoid $\mathrm{Sym}^{\infty}(X/S)$, and, according to what we discussed above, this colimit is nothing else but the Nisnevich sheafification of the corresponding section wise colimit.

4. Nisnevich spaces of 0-cycles over locally Noetherian schemes

The purpose of this section is define what exactly do we mean when we speak about spaces of 0-cycles. First we will discuss the latest approach presented in [18]. Rydh's construction of a sheaf of relative 0-cycles is compatible with the earlier approaches due to Suslin-Voevodsky, [21], and Kollár, [14], if we restrict all the sheaves on seminormal schemes. We think it is important to understand these two earlier approaches, but for the purpose of not enraging the manuscript unreasonably, we will discuss the necessary definitions and results from Suslin-Voevodsky's paper [21] only.

Rydh's approach

So let again X be AF over S, and for any nonnegative integer d let $\Gamma^d(X/S)$ be the d-divided power of X over S, as explained in Paper I in [18]. The infinite coproduct

$$\coprod_{d=0}^{\infty} \Gamma^d(X/S)$$

is a monoid in $\mathsf{Shv}((\mathsf{Noe}/S)_{\mathrm{Nis}})$.

The canonical morphism

$$(X/S)^d \to \Gamma^d(X/S)$$

is Σ_d-equivariant on the source, see Prop. 4.1.5 in loc.cit, so that there exists also a canonical morphism

$$\mathrm{Sym}^d(X/S) \to \Gamma^d(X/S)\ .$$

If the base scheme S is of pure characteristic 0, or if X is flat over S, the latter morphism is an isomorphism of schemes by Corollary 4.2.5 in Paper I in [18].

In other words, the divided power $\Gamma^d(X/S)$ differs from the symmetric power $\mathrm{Sym}^d(X/S)$ only if the residue fields $\kappa(s)$ can have positive characteristic for points $s \in S$ and, at the same time, X is not flat over S. From the point of view of the applications which we have in mind, this is quite a bizarre situations, so that the difference between divided and symmetric powers can be ignored in practice, and we introduce it merely for completeness of the theory.

Now, let U be a locally Noetherian scheme over S. According to Paper IV in [18], a *relative* 0*-cycle of degree* d on $X \times_S U$ over U is the equivalence class of ordered pairs (Z, α), where Z is a closed subscheme in $X \times_S U$, such that the composition

$$Z \to X \times_S U \to U$$

is a finite, and

$$\alpha : U \to \Gamma^d(Z/U)$$

is a morphism of schemes over U. Notice that since the morphism $Z \to U$ is finite, it is AF, and therefore the scheme $\Gamma^d(Z/U)$ does exist. Two such pairs (Z_1, α_1) and (Z_2, α_2) are said to be equivalent if there is a scheme Z and two closed embeddings $Z \to Z_1$ and $Z \to Z_2$, and a morphism of schemes $\alpha : U \to \Gamma^d(Z/U)$ over U, such that the obvious composition

$$U \xrightarrow{\alpha} \Gamma^d(Z/U) \to \Gamma^d(Z_i/U)$$

is α_i for $i = 1, 2$, see page 9 in Paper IV in [18]. If a relative cycle is represented by a pair (Z, α), we will denote it by $[Z, \alpha]$.

An important property of divided powers is that if

$$g : U' \to U$$

is a morphism of locally Noetherian schemes over S, the natural map

$$\mathrm{Hom}_{U'}(U', \Gamma^d(X \times_S U/U) \times_U U') \to \mathrm{Hom}_{U'}(U', \Gamma^d(X \times_S U'/U')) \tag{12}$$

is a bijection, see page 12 in paper I in [18]. This allows us to define pullbacks of relative 0-cycles. Indeed, let $[Z, \alpha]$ be a relative cycle on $X \times_S U$ over U. Define Z' and a closed embedding of Z' in to $X \times_S U'$ by the Cartesian square

$$\begin{tikzcd} Z' \arrow[r] \arrow[d] & X \times_S U' \arrow[d] \\ Z \arrow[r] & X \times_S U \end{tikzcd}$$

The composition

$$U' \to U \to \Gamma^d(Z/U)$$

induces the unique morphism

$$U' \to \Gamma^d(Z/U) \times_U U' \tag{13}$$

over U' whose composition with the projection onto $\Gamma^d(Z/U)$ is the initial composition. A particular case of the bijection (12) is the bijection

$$\mathrm{Hom}_{U'}(U', \Gamma^d(Z/U) \times_U U') \xrightarrow{\sim} \mathrm{Hom}_{U'}(U', \Gamma^d(Z'/U')) \tag{14}$$

Applying (14) to (13) we obtain the uniquely defined morphism

$$\alpha' : U' \to \Gamma^d(Z'/U') \, .$$

Then

$$g^*[Z, \alpha] = [Z', \alpha']$$

is, by definition, the pullback of the relative 0-cycle $[Z, \alpha]$ along the morphism g.

It is easy to verify that such defined pullback is functorial, and we obtain the corresponding set valued presheaf

$$\mathscr{Y}_{0,d}(X/S) : (\mathsf{Noe}/S)^{\mathrm{op}} \to \mathsf{Set}$$

sending any locally Noetherian scheme U over S to the set of all relative 0-cycles of degree d on $X \times_S U$ over U. Let also

$$\mathscr{Y}_0(X/S) = \coprod_{d=0}^{\infty} \mathscr{Y}_{0,d}(X/S)$$

be the total presheaf of relative 0-cycles of all degrees.

An important thing here is that the presheaf $\mathscr{Y}_{0,d}(X/S)$ is represented by the scheme $\Gamma^d(X/S)$, see Paper I and Paper II in [18]. And as the Nisnevich topology is subcanonical, it follows that $\mathscr{Y}_{0,d}(X/S)$ is a sheaf in Nisnevich topology, i.e. an object of the category $\mathsf{Shv}((\mathsf{Noe}/S)_{\mathrm{Nis}})$, and the same is true with regard to the presheaf $\mathscr{Y}_0(X/S)$.

Since each sheaf $\mathscr{Y}_{0,d}(X/S)$ is represented by the divided power $\Gamma^d(X/S)$, the sheaf $\mathscr{Y}_0(X/S)$ is represented by the infinite coproduct $\coprod_{d=0}^{\infty} \Gamma^d(X/S)$, the sheaf $\mathscr{Y}_0(X/S)$ is a graded monoid in $\mathsf{Shv}((\mathsf{Noe}/S)_{\mathrm{Nis}})$, and hence we also have its group completion

$$\mathscr{Z}_0(X/S) = \mathscr{Y}_0(X/S)^+ \, .$$

Moreover, if the structural morphism $X \to S$ admits a section, the graded monoid $\mathscr{Y}_0(X/S)$ is pointed, and we can also construct the connective monoid

$$\mathscr{Y}_0^{\infty}(X/S) = \mathrm{colim}_{\,d}\, \mathscr{Y}_{0,d}(X/S)$$

and its group completion

$$\mathscr{Z}_0^{\infty}(X/S) = \mathscr{Y}_0^{\infty}(X/S)^+ \, .$$

Suslin-Voevodsky's approach

For any scheme X let $t(X)$ be the topological space of the scheme X, and let $c(X)$ be the set of closed subschemes in X. Then we have a map

$$t(X) \to c(X)$$

sending any point $\zeta \in X$ to its closure $\overline{\{\zeta\}}$ with the induced reduced structure of a closed subscheme on it. Let

$$Cycl^{\mathrm{eff}}(X) = \mathbb{N}(t(X))$$

be the free monoid generated by points on X. Elements of $Cycl^{\mathrm{eff}}(X)$ are the *effective algebraic cycles*, or simply *effective cycles* on the scheme X. Let also

$$C^{\mathrm{eff}}(X) = \mathbb{N}(c(X))$$

free monoid generated by closed subschemes of X. For any closed subscheme

$$Z \to X \;\in\; c(X)$$

let $\zeta_1, \dots, \zeta_n$ be the generic points of the irreducible components of the scheme Z, let

$$m_i = \mathrm{length}(\mathscr{O}_{\zeta_i, Z})$$

be the multiplicity of the component $Z_i = \overline{\zeta_i}$ in Z, and let

$$cycl_X(Z) = \sum_i m_i Z_i$$

be the fundamental class of the closed subscheme Z of the scheme X. Then we obtain the standard map

$$cycl_X : c(X) \to Cycl^{\mathrm{eff}}(X) \; ,$$

$$Z \mapsto cycl_X(Z) \; .$$

The map $cycl_X$ extends to the homomorphism of monoids

$$cycl_X : C^{\mathrm{eff}}(X) \to Cycl^{\mathrm{eff}}(X) \; ,$$

If

$$C(X) = C^{\mathrm{eff}}(X)^+$$

and

$$Cycl(X) = Cycl^{\mathrm{eff}}(X)^+$$

then we also have the corresponding homomorphism of abelian groups

$$cycl_X : C(X) \to Cycl(X) \; .$$

Elements of the free abelian group $Cycl(X)$ will be called *algebraic cycles*, or simply *cycles* on the scheme X. Points

$$\zeta \in t(X) \; ,$$

or, equivalently, their closures

$$Z = \overline{\{\zeta\}} \; ,$$

considered as closed subschemes in X with the induced reduced closed subscheme structure, can be also considered as *prime cycles* on X. If

$$Z = \sum_i m_i Z_i \in Cycl(X)$$

is a cycle on X, where Z_i are prime cycles, define its support $\mathrm{Supp}(Z)$ to be the union

$$\mathrm{Supp}(Z) = \cup_i Z_i \in c(X)$$

with the induced reduced structure of a closed subscheme of X.

Let S be a Noetherian scheme. A point on S can be understood as a morphism

$$P : \mathrm{Spec}(k) \to S$$

from the spectrum of a field k to S. A *fat point* of S over P is then two morphisms of schemes

$$P_0 : \mathrm{Spec}(k) \to \mathrm{Spec}(R) \quad \text{and} \quad P_1 : \mathrm{Spec}(R) \to S \; ,$$

where R is a DVR whose residue field is k, such that

$$P_1 \circ P_0 = P \ ,$$

the image of P_0 is the closed point of $\mathrm{Spec}(R)$, and P_1 sends the generic point $\mathrm{Spec}(R_{(0)})$ to the generic point of the scheme S.

Let now

$$f : X \to S$$

be a scheme of finite type over S, and let

$$Z \to X$$

be a closed subscheme in X. Let R be a discrete valuation ring,

$$D = \mathrm{Spec}(R) \ ,$$

and let

$$g : D \to S$$

be a morphism of schemes from D to S. Let also

$$\eta = \mathrm{Spec}(R_{(0)})$$

be the generic point of D,

$$X_D = X \times_S D \ , \quad Z_D = Z \times_S D \quad \text{and} \quad Z_\eta = Z \times_S \eta \ .$$

Then there exists a unique closed embedding

$$Z'_D \to Z_D \ ,$$

such that its pull-back

$$Z'_\eta \to Z_\eta$$

along the morphism $Z_\eta \to Z_D$, is an isomorphism, and the composition

$$Z'_D \to Z_D \to D$$

is a flat morphism of schemes, see Proposition 2.8.5 in [8].

In particular, one can apply this "platification" process to a fat point (P_0, P_1) over a point $P \in S$ with $g = P_1$. Let X_P be the fibre of the morphism $X_D \to D$ over the point P_0,

$$Z_P = Z_D \times_{X_D} X_P \quad \text{and} \quad Z'_P = Z'_D \times_{Z_D} Z_P \ .$$

Since the closed subscheme Z'_D of X_D is flat over D, we define the pull-back $(P_0, P_1)^*(Z)$ of the closed subscheme Z to the fibre X_P by the formula

$$(P_0, P_1)^*(Z) = cycl_{X_P}(Z'_P) \ .$$

This gives the definition of a pullback along (P_0, P_1) for primes cycles and, by linearity, extends to a homomorphism

$$(P_0, P_1)^* : Cycl(X) \to Cycl(X_P) \ .$$

The following definition of Suslin and Voevodsky is of crucial importance, see pp 23 - 24 in [21].

Let

$$Z = \sum m_i Z_i \in Cycl(X)$$

be a cycle on X, and let ζ_i be the generic point of the prime cycle Z_i for each index i. Then Z is said to be a *relative cycle* on X over S if:

- for any generic point η of the scheme S there exists i, such that
$$f(\zeta_i) = \eta \ ,$$
- for any point P on S, and for any two fat points (P_0, P_1) and (P'_0, P'_1) over P,
$$(P_0, P_1)^*(Z) = (P'_0, P'_1)^*(Z)$$
in $Cycl(X_P)$.

The sum of relative cycles is a relative cycle again, and the same for taking the opposite cycle in $Cycl(X)$. The 0 in $Cycl(X)$ is relative by convention. Then we see that relative cycles form a subgroup
$$Cycl(X/S) = \{Z \in Cycl(X) \mid Z \text{ is relative over } S\} \ .$$
in $Cycl(X)$. Let also
$$Cycl^{\mathrm{eff}}(X/S) = \{Z = \sum m_i Z_i \in Cycl(X/S) \mid m_i \geq 0 \ \forall i \ \}$$
be a monoid of effective relative cycles in X over S.

In general the monoid $Cycl(X/S)$ is *not* a free monoid generated by prime relative cycles, and the group $Cycl(X/S)$ is *not* a free abelian group generated by prime relative cycles.

If $\zeta \in t(X)$, the dimension of ζ in X,
$$\dim(\zeta, X) \ ,$$
is, by definition, the dimension of the closure
$$Z = \overline{\{\zeta\}}$$
inside X. A relative cycle
$$Z = \sum m_i Z_i \in Cycl(X/S)$$
is said to be of *relative dimension* r if the generic point ζ_i of each prime cycle Z_i has dimension r in its fibre over S. In other words, if
$$\eta_i = f(\zeta_i) \ ,$$
we look at the fibre X_{η_i} of the morphism f at η_i. The cycle Z is of relative dimension r over S if
$$\dim(\zeta_i, X_{\eta_i}) = r$$
for each index i. If Z is a relative cycle of relative dimension r on X, then we write
$$\dim_S(Z) = r \ .$$
Following [21], p 24, we define
$$Cycl(X/S, r) = \{Z \in Cycl(X/S) \mid \dim_S(Z) = r\}$$
to be the subset of relative algebraic cycles of relative dimension r on X, which is obviously a subgroup in $Cycl(X/S)$. The definition of
$$Cycl^{\mathrm{eff}}(X/S, r) = \{Z = \sum m_i Z_i \in Cycl(X/S, r) \mid m_i \geq 0 \ \forall i \ \}$$
is straightforward.

Notice that if Z is a relative cycle of relative dimension r, it does not mean that all the components Z_i are of the same dimension r. To pick up equidimensional cycles, we need the following definition. For any point $\zeta \in t(X)$ let

$$\dim(X/S)(x) = \dim_{\zeta}(f^{-1}(f(\zeta)))$$

be the dimension of the fibre $f^{-1}(f(\zeta))$ of the morphism f at ζ. The morphism f is said to be *equidimensional* of dimension r if every irreducible component of X dominates an irreducible component of S and the function

$$\dim(X/S) : t(X) \to \mathbb{Z}$$

is constant and equals r for every point ζ on the scheme X. A cycle $Z \in \mathit{Cycl}(X/S)$ is equidimensional of dimension r over S if so is the composition

$$\mathrm{Supp}(Z) \to X \to S\ .$$

Let then

$$\mathit{Cycl}_{\mathrm{equi}}(X/S, r) = \{Z \in \mathit{Cycl}(X/S, r) \mid Z \text{ is equidim. of dim. } r\}\ .$$

Accordingly,

$$\mathit{Cycl}^{\mathrm{eff}}_{\mathrm{equi}}(X/S, r) = \{Z = \sum_i m_i Z_i \in \mathit{Cycl}_{\mathrm{equi}}(X/S, r) \mid m_i \geq 0\ \forall i\ \}\ .$$

Next, let

$$U \to S$$

be a locally Noetherian scheme over S (not necessarily of finite type over S). In [21], for any cycle

$$Z \in \mathit{Cycl}(X/S, r)$$

Suslin and Voevodsky constructed a uniquely defined cycle

$$Z_U \in \mathit{Cycl}(X \times_S U/U, r)_{\mathbb{Q}}\ ,$$

a pullback of Z along $U \to S$, such that it is compatible with pullbacks long fat points. Here and below, for any abelian group A we denote by $A_{\mathbb{Q}}$ the tensor product $A \otimes_{\mathbb{Z}} \mathbb{Q}$.

Thus, following Suslin and Voevodsky, we obtain the obvious presheaf

$$\mathit{Cycl}(X/S, r)_{\mathbb{Q}}$$

on the category Noe/S, such that for any morphism

$$U \to S$$

in Noe/S,

$$\mathit{Cycl}(X/S, r)_{\mathbb{Q}}(U) = \mathit{Cycl}(X \times_S U/U, r)_{\mathbb{Q}}\ ,$$

and the restriction morphisms are induced by the Suslin-Voevodsky's pullbacks of relative cycles.

Following [21], we will say that the pullback Z_U of a cycle $Z \in \mathit{Cycl}(X/S, r)$ is *integral* if it lies in the image of the canonical homomorphism

$$\mathit{Cycl}(X \times_S U/U, r) \to \mathit{Cycl}(X \times_S U/U, r)_{\mathbb{Q}}$$

for all schemes U in Noe/S, and define the subgroup

$$z(X/S, r) = \{Z \in \mathit{Cycl}(X/S, r) \mid Z_U \text{ is integral}\}\ .$$

Then $z(X/S,r)$ is an abelian subpresheaf in the presheaf $Cycl(X/S,r)_{\mathbb{Q}}$ on the category Noe/S.

Let also

$$z^{\mathrm{eff}}(X/S,r) = \{Z = \sum m_i Z_i \in z(X/S,r) \mid m_i \geq 0 \ \forall i\}$$

and

$$z_{\mathrm{equi}}(X/S,r) = \{Z \in z(X/S,r) \mid Z \text{ is equidim. of dim. } r \text{ over } S\} .$$

Clearly, $z^{\mathrm{eff}}(X/S,r)$ is a subpresheaf of monoids and $z^{\mathrm{equi}}(X/S,r)$ is a presheaf of abelian groups in $z(X/S,r)$.

For any morphism

$$U \to S ,$$

which is an object of Noe/S, set

$$PrimeCycl(X \times_S U/U,r) = \{Z \in Cycl(X \times_S U/U,r) \mid Z \text{ is prime}\}$$

and

$$PrimeCycl_{\mathrm{equi}}(X \times_S U/U,r) = \{Z \in PrimeCycl(X \times_S U/U,r) \mid Z \text{ is equidim.}\}$$

If S is regular, and if the morphism $U \to S$ is an object of Reg/S, then

$$z^{\mathrm{eff}}(X/S,r) = \mathbb{N}(PrimeCycl_{\mathrm{equi}}(X \times_S U/U,r)) ,$$

and

$$z_{\mathrm{equi}}(X/S,r) = \mathbb{N}(PrimeCycl_{\mathrm{equi}}(X \times_S U/U,r))^+ ,$$

see Corollary 3.4.5 in [21].

It does not mean, however, that $z^{\mathrm{eff}}(X/S,r)$ is a free monoid in the category of set valued presheaves freely generated by a set valued "presheaf of relative prime cycles of dimension r" on the category Reg/S, as the Suslin-Voevodsky pullback of a relative prime cycle is not necessarily a prime cycle, so that the needed set valued presheaf does not exist. But $z_{\mathrm{equi}}(X/S,r)$ is certainly the group completion of $z^{\mathrm{eff}}(X/S,r)$ as a presheaf on Reg/S.

Theorem 1. *Let S be a Noetherian scheme, and let X be a scheme of finite type over S. Then the presheaves $z(X/S,r)$ and $z^{\mathrm{eff}}(X/S,r)$ are sheaves in* cdh*-topology and, as a consequence, in the Nisnevich topology on the category* Noe/S.

Proof. See Theorem 4.2.9(1) on page 65 in [21]. □

Relative cycles can be classified by their degrees, provided there exists a projective embedding of X over S. Indeed, assume that X is projective over S, i.e. there is a closed embedding

$$i : X \to \mathbb{P}^n_S$$

over S. For each cycle

$$Z = \sum m_j Z_j \in Cycl(X/S)$$

one can define its degree

$$\deg(Z,i) = \sum \deg(i(Z_j))$$

with regard to the embedding i. Let also

$$z^{\mathrm{eff}}_d((X,i)/S,r) = \{Z \in z_{\mathrm{equi}}(X/S,r) \mid \deg(Z,i) = d\} .$$

The set valued presheaf

$$z_d^{\mathrm{eff}}((X,i)/S,r) : \mathsf{Noe}/S \to \mathsf{Set}$$

is given by the formula

$$z_d^{\mathrm{eff}}((X,i)/S,r)(U) = \{Z \in z_{\mathrm{equi}}(X \times_S U/U,r) \mid \deg(Z, i \times_S \mathrm{id}_U) = d\} \ ,$$

for any locally Noetherian scheme U over S.

Now recall that if $\mathscr{F}$ is a set-valued presheaf on Noe/S then $\mathscr{F}$ is said to be h-representable if there is a scheme Y over S, such that the h-sheafification $\mathscr{F}_{\mathrm{h}}$ of the sheaf $\mathscr{F}$ is isomorphic to the h-sheafification $\mathrm{Hom}_S(-X)_{\mathrm{h}}$ of the representable presheaf $\mathrm{Hom}_S(-X)$, see Definition 4.4.1 in [21].

Theorem 2. *Let X be a projective scheme of finite type over S and fix a projective embedding $i : X \to \mathbb{P}^n_S$ over S. Then, for any two nonnegative integers r and d, the presheaf $z_d^{\mathrm{eff}}((X,i)/S,r)$ is* h*-representable by a scheme $C_{r,d}(X/S,i)$ projective over S, i.e. there is an isomorphism*

$$z_d^{\mathrm{eff}}((X,i)/S,r)_{\mathrm{h}} \simeq \mathrm{Hom}_S(-, C_{r,d}(X/S,i))_{\mathrm{h}}$$

of set valued sheaves in h*-topology on Noe/S. Moreover,*

$$z^{\mathrm{eff}}(X/S,r) = \coprod_{d=0}^{\infty} z_d^{\mathrm{eff}}((X,i)/S,r) \ ,$$

and then $z^{\mathrm{eff}}(X/S,r)$ is h*-representable by the scheme*

$$C_r(X/S) = \coprod_{d=0}^{\infty} C_{r,d}(X/S,i) \ .$$

Proof. See Section 4.2 in [21]. □

A disadvantage of Theorem 2 is in the presence of h-sheafification. The latter is a retribution for the generality of the representability result. For relative 0-cycles this obstacle can be avoided as follows.

Recall that we have already defined the category Nor/S, a full subcategory in Sch/S generated by schemes over S whose structural morphism is normal, i.e. the fibre at every point is a normal scheme, see Definition 36.18.1 in [23]. Similarly, one can define the notion of a seminormal morphism and introduce a full subcategory sNor/S generated by locally Noetherian schemes over S whose structural morphisms are seminormal, so that we have a chain of subcategories

$$\mathsf{Nor}/S \subset \mathsf{sNor}/S \subset \mathsf{Noe}/S \ .$$

For any presheaf $\mathscr{F}$ on Noe/S let $\mathscr{F}|_{\mathsf{sNor}/S}$ be the restriction of $\mathscr{F}$ on the subcategory sNor/S.

To avoid divided powers, suppose that either the base scheme S is of pure characteristic 0 or X is flat over S. Recall that it follows that

$$\Gamma^d(X/S) = \mathrm{Sym}^d(X/S)$$

by Corollary 4.2.5 in Paper I in [18], and hence one can work with symmetric powers instead of divided ones. By Theorem 3.1.11 on page 30 of the same paper, we have the canonical identifications

$$\mathscr{Y}_{0,d}(X/S) = \mathrm{Sym}^d(X/S)\ , \tag{15}$$

$$\mathscr{Y}_0(X/S) = \left(\coprod_{d=0}^{\infty} \mathrm{Sym}^d(X/S)\right)\ ,$$

$$\mathscr{Y}_0^{\infty}(X/S) = \mathrm{Sym}^{\infty}(X/S)\ ,$$

$$\mathscr{Z}_0(X/S) = \left(\coprod_{d=0}^{\infty} \mathrm{Sym}^d(X/S)\right)^+$$

and

$$\mathscr{Z}_0^{\infty}(X/S) = \mathrm{Sym}^{\infty}(X/S)^+\ .$$

In other words, we do not need h-sheafification to prove representability of sheaves of 0-cycles in Rydh's terms.

The point here is that, assuming that S is semi-normal over $\mathrm{Spec}(\mathbb{Q})$, after restricting of these five sheaves on the category sNor/S, we also have the corresponding canonical isomorphisms

$$\mathscr{Y}_{0,d}(X/S)|_{\mathsf{sNor}/S} \simeq z_d^{\mathrm{eff}}((X,i)/S,0)|_{\mathsf{sNor}/S}\ , \tag{16}$$

$$\mathscr{Y}_0(X/S)|_{\mathsf{sNor}/S} \simeq z^{\mathrm{eff}}(X/S,0)|_{\mathsf{sNor}/S}\ , \tag{17}$$

$$\mathscr{Y}_0^{\infty}(X/S)|_{\mathsf{sNor}/S} \simeq z^{\mathrm{eff}}(X/S,0)_{\infty}|_{\mathsf{sNor}/S}\ , \tag{18}$$

$$\mathscr{Z}_0(X/S)|_{\mathsf{sNor}/S} \simeq z(X/S,0)|_{\mathsf{sNor}/S} \tag{19}$$

and

$$\mathscr{Z}_0^{\infty}(X/S)|_{\mathsf{sNor}/S} \simeq z(X/S,0)_{\infty}|_{\mathsf{sNor}/S}\ . \tag{20}$$

Moreover, the same result holds true when we compare Rydh's sheaves of 0-cycles with Kollár's sheaves constructed in Chapter I of the book [14]. These important comparison results are proven in Section 10 of Paper IV in [18].

Thus, since now we will always assume that either the base scheme S is of pure characteristic 0 or X is flat over S, to work with symmetric powers, and in all cases when S will be semi-normal over $\mathbb{Q}$, we will systematically identify the restrictions of Suslin-Voevodsky's and Rydh's sheaves of 0-cycles on semi-normal schemes via the isomorphisms (16), (17), (18), (19) and (20).

The Nisnevich sheaf $\mathrm{Sym}^{\infty}(X/S)^+$ will be now used to construct what then will be the most preferable reincarnation of the space of 0-cycles on X over the base scheme S.

5. Chow atlases on the Nisnevich spaces of 0-cycles

To consider the sheaf $\mathrm{Sym}^{\infty}(X/S)^+$ as a geometrical object, we need to endow it with an atlas, in the line of the definitions in Section 2. The aim of this section is to present a natural atlas, the Chow atlas, on the sheaf of 0-cycles $\mathrm{Sym}^{\infty}(X/S)^+$.

First of all, the sheaf of 0-cycles possesses a natural inductive structure on it. For each non-negative integer d let

$$\iota_d : \mathrm{Sym}^d(X/S) \to \mathrm{Sym}^{\infty}(X/S)$$

be the canonical morphism in to the colimit. For short of notation, let also

$$\mathrm{Sym}^{d,d}(X/S) = \mathrm{Sym}^d(X/S) \times_S \mathrm{Sym}^d(X/S)\ ,$$

$$\mathrm{Sym}^{\infty,\infty}(X/S) = \mathrm{Sym}^\infty(X/S) \times_S \mathrm{Sym}^\infty(X/S)$$

and let

$$\iota_{d,d} : \mathrm{Sym}^{d,d}(X/S) \to \mathrm{Sym}^{\infty,\infty}(X/S)$$

be the fibred product of ι_d with itself over S. Recall that $\mathrm{Sym}^\infty(X/S)^+$ is the group completion of the monoid $\mathrm{Sym}^\infty(X/S)$ in the category $\mathsf{Shv}((\mathsf{Noe}/S)_{\mathrm{Nis}})$. It means that the we have a pushout square

$$\begin{tikzcd}
\mathrm{Sym}^\infty(X/S) \arrow[r, "\Delta"] \arrow[d] & \mathrm{Sym}^{\infty,\infty}(X/S) \arrow[d, "\sigma_\infty"] \\
S \arrow[r] & \mathrm{Sym}(X/S)^+
\end{tikzcd}$$

in the category $\mathsf{Mon}(\mathsf{Shv}((\mathsf{Noe}/S)_{\mathrm{Nis}}))$. In particular, the quotient morphism σ_∞ is a morphism of monoids, i.e. it respects the monoidal operations in the source and target. Let

$$\sigma_d : \mathrm{Sym}^{d,d}(X/S) \to \mathrm{Sym}^\infty(X/S)^+$$

be the composition of the morphisms $\iota_{d,d}$ and σ_∞ in the category $\mathsf{Shv}((\mathsf{Noe}/S)_{\mathrm{Nis}})$, and let

$$\mathrm{Sym}^d(X/S)^+$$

be the sheaf-theoretical image of the morphism σ_d, i.e. the image of σ_d in the category $\mathsf{Shv}((\mathsf{Noe}/S)_{\mathrm{Nis}})$.

Some explanation is in place here. A priori, for any nonnegative integer d, one can compute the d-th symmetric power

$$S^d(X/S)$$

in the category of presheaves $\mathsf{PShv}(\mathsf{Noe}/S)$, and the d-th symmetric power

$$\mathrm{Sym}^d(X/S)\ ,$$

computed in the category of sheaves $\mathsf{Shv}((\mathsf{Noe}/S)_{\mathrm{Nis}})$, is the Nisnevich sheafification of the presheaf $S^d(X/S)$. But since the symmetric power $S^d(X/S)$ exists already as a scheme in the category Noe/S, and since the Nisnevich topology is subcanonical, we have that

$$S^d(X/S) = \mathrm{Sym}^d(X/S)\ ,$$

for any $d \geq 0$.

Let

$$\coprod_{d=0}^{\infty} S^d(X/S)$$

be the free monoid $\mathbb{N}(X/S)$ of X over S computed in the category of presheaves $\mathsf{PShv}(\mathsf{Noe}/S)$. Since the category Noe/S is a Noetherian category, one can show that this infinite coproduct is a Nisnevich sheaf, and hence it coincides with the free monoid $\mathbb{N}(X/S)$ of X over S computed in the category of sheaves

$\mathsf{Shv}((\mathsf{Noe}/S)_{\mathrm{Nis}})$. In other words, there is no difference between $\mathbb{N}(X/S)$ in $\mathsf{PShv}(\mathsf{Noe}/S)$ and $\mathbb{N}(X/S)$ in $\mathsf{Shv}((\mathsf{Noe}/S)_{\mathrm{Nis}})$, and we write

$$\mathbb{N}(X/S) = \coprod_{d=0}^{\infty} \mathrm{Sym}^d(X/S) = \coprod_{d=0}^{\infty} S^d(X/S) \; .$$

Similarly, let

$$S^{\infty}(X/S)$$

be the free connective monoid $\mathbb{N}(X/S)_{\infty}$ of X over S computed in the category of presheaves $\mathsf{PShv}(\mathsf{Noe}/S)$, so that the free connective monoid $\mathrm{Sym}^{\infty}(X/S)$ of X over S, computed in the category of sheaves $\mathsf{Shv}((\mathsf{Noe}/S)_{\mathrm{Nis}})$, is nothing else but the Nisnevich sheafification of $S^{\infty}(X/S)$. Again, as the category Noe/S is a Noetherian category, one can show that $S^{\infty}(X/S)$ is a sheaf in Nisnevich topology, and hence

$$S^{\infty}(X/S) = \mathrm{Sym}^{\infty}(X/S) \; .$$

This gives us that, if

$$S^{\infty}(X/S)^+$$

is the group completion of the presheaf free monoid $S^{\infty}(X/S)$ in the category $\mathsf{Mon}(\mathsf{PShv}(\mathsf{Noe}/S))$, i.e. the square

$$\begin{tikzcd} S^{\infty}(X/S) \arrow[r, "\Delta"] \arrow[d] & S^{\infty,\infty}(X/S) \arrow[d, "\sigma_{\infty}"] \\ S \arrow[r] & S(X/S)^+ \end{tikzcd} \tag{21}$$

is co-Cartesian, the sheaf group completion $\mathrm{Sym}^{\infty}(X/S)^+$ of $\mathrm{Sym}^{\infty}(X/S)$ in the category $\mathsf{Mon}(\mathsf{Shv}((\mathsf{Noe}/S)_{\mathrm{Nis}}))$ is the sheafification of $S^{\infty}(X/S)^+$, i.e.

$$\mathrm{Sym}^{\infty}(X/S)^+ = S^{\infty}(X/S)^+_{\mathrm{a}} \; .$$

Lemma 3. *The presheaf $S^{\infty}(X/S)^+$ is separated. Equivalently, the canonical morphism*

$$S^{\infty}(X/S)^+ \to \mathrm{Sym}^{\infty}(X/S)^+$$

is a monomorphism in $\mathsf{PShv}(\mathsf{Noe}/S)$.

Proof. Since $S^{\infty}(X/S)^+$ is an abelian group object in the category $\mathsf{PShv}(\mathsf{Noe}/S)$, to prove the lemma it is enough to show that, if

$$F \in S^{\infty}(X/S)^+(U)$$

is a section of the presheaf $S^{\infty}(X/S)^+$ on some locally Noetherian scheme U over S, and if there exists a Nisnevich covering

$$\{f_i : U_i \to U\}_{i \in I} \; ,$$

such that the pullback F_i of the section F to U_i along each morphism $U_i \to U$ is 0 in the abelian group $S^{\infty}(X/S)^+(U_i)$, then F is 0 in the abelian group $S^{\infty}(X/S)^+(U)$.

The section F can be interpreted as a morphism

$$F : U \to S^{\infty}(X/S)^{+} .$$

For short of notation, let

$$S^{\infty,\infty}(X/S) = S^{\infty}(X/S) \times_S S^{\infty}(X/S) ,$$

and, for any nonnegative integer d let

$$S^{d,d}(X/S) = S^{d}(X/S) \times_S S^{d}(X/S) .$$

In these terms, the morphism F is the composition of a certain morphism

$$(f_1, f_2) : U \to S^{\infty,\infty}(X/S) ,$$

induced by two morphisms of presheaves

$$f_1 : U \to S^{\infty}(X/S) \quad \text{and} \quad f_1 : U \to S^{\infty}(X/S) ,$$

and the quotient morphism

$$\sigma_{\infty} : S^{\infty,\infty}(X/S) \to S^{\infty}(X/S)^{+} .$$

Moreover, there exists d, such that both morphisms f_1 and f_2 factorize through $S^d(X/S)$, and then F is the composition

$$U \stackrel{(f_1,f_2)}{\longrightarrow} S^{d,d}(X/S) \stackrel{\iota_{d,d}}{\longrightarrow} S^{\infty,\infty}(X/S) \stackrel{\sigma_{\infty}}{\longrightarrow} S^{\infty}(X/S)^{+} . \tag{22}$$

and the morphisms f_1 and f_2 are morphisms of locally Noetherian schemes over the base scheme S.

Now, since $S^{\infty}(X/S)$ is a cancellative monoid in $\mathsf{PShv}(\mathsf{Noe}/S)$, the commutative square (21) is a Cartesian square in $\mathsf{PShv}(\mathsf{Noe}/S)$. It follows that, since $F_i = 0$ for all $i \in I$, the images of the compositions

$$U_i \to U \stackrel{(f_1,f_2)}{\longrightarrow} S^{d,d}(X/S) \stackrel{\iota_{d,d}}{\longrightarrow} S^{\infty,\infty}(X/S) \stackrel{\sigma_{\infty}}{\longrightarrow} S^{\infty}(X/S)^{+}$$

are all in the image of the diagonal morphism

$$\Delta : S^{\infty}(X/S) \to S^{\infty,\infty}(X/S) .$$

And since the morphism

$$\coprod_{i \in I} U_i \to U$$

is a scheme-theoretical epimorphism, we see that the image of the morphism (22) is also in the image of the diagonal morphism Δ. The latter means that the section F equals 0. $\square$

Let

$$\sigma_d : S^{d,d}(X/S) \to S^{\infty}(X/S)^{+}$$

be the composition of the morphisms $\iota_{d,d}$ and σ_{∞} in the category $\mathsf{PShv}(\mathsf{Noe}/S)$, and let

$$S^d(X/S)^{+}$$

be the image of the morphism σ_d in the category $\mathsf{PShv}(\mathsf{Noe}/S)$. Then $S^d(X/S)^{+}$ is a sub-presheaf in $\mathrm{Sym}^{\infty}(X/S)^{+}$. As the sheafification functor is exact, it preserves monomorphisms. It follows that

$$\mathrm{Sym}^d(X/S)^{+} = (S^d(X/S)^{+})^{\mathrm{a}} ,$$

i.e. $\mathrm{Sym}^d(X/S)^+$ is the Nisnevich sheafification of the preshaef $S^d(X/S)^+$. And, once again, the sheaf-theoretical image $\mathrm{Sym}^d(X/S)^+$ of the morphism σ_d comes together with the epimorphism

$$\sigma_d : \mathrm{Sym}^{d,d}(X/S) \to \mathrm{Sym}^d(X/S)^+ \tag{23}$$

in the category $\mathsf{Shv}((\mathsf{Noe}/S)_{\mathrm{Nis}})$.

Next, the section $S \to X$ of the structural morphism $X \to S$ induces the closed embeddings

$$\mathrm{Sym}^d(X/S) \to \mathrm{Sym}^{d+1}(X/S) \ ,$$

which, in turn, induce the closed embeddings

$$\mathrm{Sym}^{d,d}(X/S) \to \mathrm{Sym}^{d+1,d+1}(X/S) \ .$$

The latter morphisms induce the corresponding morphisms

$$\mathrm{Sym}^d(X/S)^+ \to \mathrm{Sym}^{d+1}(X/S)^+$$

in the category $\mathsf{Shv}((\mathsf{Noe}/S)_{\mathrm{Nis}})$. Then

$$\mathrm{Sym}^d(X/S)^+ = \mathrm{colim}_{\,d}\, \mathrm{Sym}^d(X/S)^+ \ , \tag{24}$$

i.e. the space $\mathrm{Sym}^d(X/S)^+$ is naturally the colimit of the spaces $\mathrm{Sym}^d(X/S)^+$.

Remark 4. The sheaf $\mathrm{Sym}^d(X/S)^+$ is *not* a group completion of any monoid.

The constructions above allow us to consider a natural atlas for the

$$CA_0(X/S, 0) = \{\sigma_d \mid d \in \mathbb{Z} \ , \ d \geq 0\}$$

be the set of all morphisms σ_d, and let

$$CA(X/S, 0) = \langle CA_0(X/S, 0) \rangle$$

be the *Chow atlas* on the Nisnevich connective space $\mathrm{Sym}^\infty(X/S)^+$. According to Section 2, the sheaf $\mathrm{Sym}^\infty(X/S)^+$ is now the Nisnevich space of relative 0-cycles on X over S, with regard to the Chow atlas

$$CA = CA(X/S, 0) \ .$$

For short, we will say that $\mathrm{Sym}^\infty(X/S)^+$ is the *space of* 0-*cycles* on X over S

Hilbert schemes allow us to consider a natural subatlas in the Chow atlas CA. Indeed, let U be a locally Noetherian scheme over S, and let

$$Z \to X \times_S U$$

be a closed subscheme in $X \times_S U$. Suppose the composition

$$g : Z \to U$$

of the closed embedding of Z into $X \times_S U$ with the projection onto U is flat. Then, if V is an irreducible component of Z, the closure $\overline{g(V)}$ is an irreducible component of U. Therefore, if U is irreducible, $\overline{g(V)} = U$. If, moreover, g is proper, then $\overline{g(V)} = g(V)$, and hence g is a surjection.

Since X is embedded in to $\mathbb{P}^n_S$ over S via the closed embedding i, the scheme $X \times_S U$ embeds into $\mathbb{P}^m_U$ over U, and the morphism $g : Z \to U$ factorizes through the embedding of Z into $\mathbb{P}^m_U$ followed by the projection from $\mathbb{P}^m_U$ onto U. Therefore, if $u \in U$ and Z_u is the fibre of g at u, the Hilbert polynomial of the structural sheaf $\mathscr{O}_{Z_u}$ does not depend on u, see Theorem 9.9 on page 261 in [12].

This fact allows us to consider, for every polynomial

$$P \in \mathbb{Q}[x]$$

the standard Hilbert set valued presheaf

$$\mathit{Hilb}_P(X/S) : \mathsf{Noe}/S \to \mathsf{Set}$$

sending a locally Noetherian S-scheme U to the set of closed subschemes Z in the product $X \times_S U$, which are flat and proper over U, and such that the Hilbert polynomial of $\mathscr{O}_{Z_u}$ is P. Let also

$$\mathit{Hilb}(X/S) = \coprod_{P \in \mathbb{Q}[x]} \mathit{Hilb}_P(X/S) : \mathsf{Noe}/S \to \mathsf{Set}$$

be the total Hilbert functor on locally Noetherian schemes over S.

Since X is projective over S, the Hilbert functors $\mathit{Hilb}_P(X/S)$ are representable. This result is due to Grothendieck, see Chapter 5 in [5] or Chapter I.1 in [14]. For each polynomial P in $\mathbb{Q}[x]$ there exists a scheme, called the Hilbert scheme,

$$\mathrm{Hilb}_P(X/S)$$

over S representing the functor $\mathit{Hilb}_P(X/S)$. Moreover, this scheme is projective over S.

Within this paper we are interested in the case when $P = d$ is a non-negative integer. In that case the Hilbert scheme

$$\mathrm{Hilb}^d(X/S) = \mathrm{Hilb}_P(X/S)|_{P=d}$$

is a scheme over the d-th relative symmetric power, and we have the so-called Hilbert-Chow morphism of schemes

$$\mathrm{hc}_d : \mathrm{Hilb}^d(X/S) \to \mathrm{Sym}^d(X/S)\ . \tag{25}$$

For any nonnegative integer d let

$$\mathrm{Hilb}^{d,d}(X/S) = \mathrm{Hilb}^d(X/S) \times_S \mathrm{Hilb}^d(X/S)\ ,$$

and let

$$HA_0(X/S, 0) = \{a_d \circ (\mathrm{hc}_{d,d}) \mid d \in \mathbb{Z}\ ,\ d \geq 0\}\ ,$$

where

$$\mathrm{hc}_{d,d} : \mathrm{Hilb}^{d,d}(X/S) \to \mathrm{Sym}^{d,d}(X/S)$$

is the fibred self-product over S of the d-th Hilbert-Chow morphism hc_d. Let also

$$HA(X/S, 0) = \langle HA_0(X/S, 0)\rangle$$

be the *Hilbert atlas* on the space $\mathrm{Sym}^\infty(X/S)^+$. Obviously, the Hilbert atlas is a subatlas of the Chow atlas on $\mathrm{Sym}^\infty(X/S)^+$.

Now, let

$$\mathscr{O}_{\mathrm{Sym}^\infty(X/S)^+}$$

be the sheaf of regular functions on the site $\mathrm{Sym}^\infty(X/S)^+_{\text{Nis-ét}}$, constructed with regard to the Chow atlas CA on the sheaf $\mathrm{Sym}^\infty(X/S)^+$, as explained in Section 2. In particular, if $U \to \mathrm{Sym}^\infty(X/S)^+$ is a morphism from a scheme

U to $\mathrm{Sym}^\infty(X/S)^+$ over S, which is étale with regard to the Chow atlas on $\mathrm{Sym}^\infty(X/S)^+$, then since

$$\Gamma(U \to \mathrm{Sym}^\infty(X/S)^+, \mathscr{O}_{\mathrm{Sym}^\infty(X/S)^+}) = \Gamma(U, \mathscr{O}_U) \ .$$

As soon as the sheaf $\mathscr{O}_{\mathrm{Sym}^\infty(X/S)^+}$ is defined, we can also define the sheaf of Kähler differentials

$$\Omega^1_{\mathrm{Sym}^\infty(X/S)^+} = \Omega^1_{\mathrm{Sym}^\infty(X/S)^+/S}$$

on the site $\mathrm{Sym}^\infty(X/S)^+_{\text{Nis-ét}}$, see Section 2. Let also

$$T_{\mathrm{Sym}^\infty(X/S)^+} = T_{\mathrm{Sym}^\infty(X/S)^+/S}$$

be the tangent sheaf, i.e. the dual to the sheaf of Kähler differentials on the site $\mathrm{Sym}^\infty(X/S)^+_{\text{Nis-ét}}$.

Since now the sheaf of Kähler differentials and the tangent sheaf on the site $\mathrm{Sym}^\infty(X/S)^+_{\text{Nis-ét}}$ will be considered as the sheaf of Kähler differentials and the tangent sheaf on the space of 0-cycles $\mathrm{Sym}^\infty(X/S)^+$.

Notice that both sheaves are given in terms of the Chow atlas on $\mathrm{Sym}^\infty(X/S)^+$. Similar sheaves can be also defined in terms of the Hilbert atlas on the same space, and the connection between two types is an interesting question, also considered in [7], but in different terms.

Next, recall that a point P on $\mathrm{Sym}^\infty(X/S)^+$ is an equivalence class of morphisms from spectra of fields to $\mathrm{Sym}^\infty(X/S)^+$, as explained in Section 2. By abuse of notation, we write

$$P : \mathrm{Spec}(K) \to \mathrm{Sym}^\infty(X/S)^+ \ .$$

We will always assume that P factorizes through the Chow atlas CA on the space $\mathrm{Sym}^\infty(X/S)^+$.

As in Section 2, consider the functor

$$u_P : \mathrm{Sym}^\infty(X/S)^+_{\text{Nis-ét}} \to \mathsf{Set}$$

sending an étale morphism

$$U \to \mathrm{Sym}^\infty(X/S)^+ \ ,$$

where U is a locally Noetherian scheme over S, to the set

$$u_P(U) = |U_P|$$

of points on the fibre

$$U_P = U \times_{\mathrm{Sym}^\infty(X/S)^+} \mathrm{Spec}(K)$$

of the morphism $U \to \mathrm{Sym}^\infty(X/S)^+$ at P.

As soon as the functor u_P is introduced, we also define the notion of a neighbourhood of P, with regard to the functor u_P, as we did it in Section 2. Namely, an étale neighbourhood of P on $\mathrm{Sym}^\infty(X/S)^+$ is a pair

$$N = (U \to \mathrm{Sym}^\infty(X/S)^+, T \in u_P = |U_P|) \ ,$$

where the morphism $U \to \mathrm{Sym}^\infty(X/S)^+$ is over S and étale with regard to the Chow atlas CA on $\mathrm{Sym}^\infty(X/S)^+$, and T is a point of the fibre U_P. Or, equivalently, an étale neighbourhood of P is an étale morphism

$$U \to \mathrm{Sym}^\infty(X/S)^+$$

over S such that the point

$$P : \mathrm{Spec}(K) \to \mathrm{Sym}^{\infty}(X/S)^{+}$$

factorizes through U.

As in Section 2, all étale neighbourhoods form the category of étale neighbourhoods of P on $\mathrm{Sym}^{\infty}(X/S)^{+}$ denoted by $\mathscr{N}_P$.

Now, Lemma 7.31.7 [23] gives us that in order to show that the corresponding stalk functor

$$\mathrm{st}_P : \mathsf{Shv}(\mathrm{Sym}^{\infty}(X/S)^{+}_{\text{Nis-ét}}) \to \mathsf{Set}$$

induces a point of the topos $\mathsf{Shv}(\mathrm{Sym}^{\infty}(X/S)^{+}_{\text{Nis-ét}})$, we need to show that the functor u_P satisfies all the three items of Definition 7.31.2 in loc.cit. The items (1) and (2) are satisfied in general, see Section 2. The last item (3) of Definition 7.31.2 in [23] is satisfied when the category $\mathscr{N}_P$ is cofiltered. Therefore, our aim is now to show that, in case of the space of 0-cycles $\mathrm{Sym}^{\infty}(X/S)^{+}$ the category $\mathscr{N}_P$ is cofiltered.

6. Étale neigbourhoods of a point on $\mathrm{Sym}^{\infty}(X/S)^{+}$

We start with the following representability lemma, which will be necessary for the study of the category $\mathscr{N}_P$.

Lemma 5. *For any nonnegative integer d and for any two morphisms*

$$U \to S^{\infty}(X/S)^{+} \quad \textit{and} \quad V \to S^{\infty}(X/S)^{+}\ ,$$

where U and V are locally Noetherian schemes over S, the fibred product

$$U \times_{S^{\infty}(F/S)^{+}} V\ ,$$

in the category of presheaves $\mathsf{PShv}(X/S)$, is represented by a locally Noetherian scheme over S.

Proof. We need to find a locally Noetherian scheme over S representing the fibred product

$$U \times_{S^{\infty}(X/S)^{+}} V$$

in the category $\mathsf{PShv}(\mathsf{Noe}/S)$.

Denote the morphism from U to $S^{\infty}(X/S)^{+}$ by F, and the morphism from V to $S^{\infty}(X/S)^{+}$ by G. Clearly, the object $U \times_{S^{\infty}(F/S)^{+}} V$ is the coequalizer of the compositions of the projections from $U \times_S V$ on to U and V with the morphisms F and G respectively. Since $S^{\infty}(X/S)^{+}$ is an abelian group object, one can consider the difference

$$H = F \circ \mathrm{pr}_U - G \circ \mathrm{pr}_V : U \times_S V \to S^{\infty}(X/S)^{+}$$

between these two compositions in the category $\mathsf{PShv}(\mathsf{Noe}/S)$. Then the coequalizer $U \times_{\mathrm{Sym}^{\infty}(F/S)^{+}} V$ fits in to the Cartesian square

$$\begin{tikzcd}
U \times_{\mathrm{Sym}^{\infty}(F/S)^{+}} V \arrow[r] \arrow[d] & U \times_S V \arrow[d, "H"] \\
S \arrow[r] & S^{\infty}(X/S)^{+}
\end{tikzcd}$$

and the lemma reduces to the case when $U = S$, and F is a section of the structural morphism from $S^\infty(X/S)^+$ to S.

Next, the morphism of presheaves

$$G : V \to S^\infty(X/S)^+$$

is uniquely determined by sending the identity morphism id_V to some element in the abelian group $S^\infty(X/S)^+(V)$, which is the equivalence class

$$[(g_1, g_2)]$$

of two morphisms

$$g_1 : V \to S^\infty(X/S) \quad \text{and} \quad g_2 : V \to S^\infty(X/S)$$

of presheaves over S. In particular, the morphism G factorized through the product

$$S^{\infty,\infty}(X/S) = S^\infty(X/S) \times_S S^\infty(X/S) \ .$$

As we mentioned already, since $S^\infty(X/S)$ is a cancellation monoid, the commutative square (21) is a Cartesian square in $\mathsf{PShv}(\mathsf{Noe}/S)$. Let

$$V_S = S^\infty(X/S) \times_{S^{\infty,\infty}(X/S)} V$$

be the fibred product of $S^\infty(X/S)$ and V over $S^{\infty,\infty}(X/S)$. The composition of the two Cartesian squares

$$\begin{tikzcd} V_S \arrow[r] \arrow[d] & V \arrow[d] \\ S^\infty(X/S) \arrow[r, "\Delta"] & S^{\infty,\infty}(X/S) \end{tikzcd}$$

and

$$\begin{tikzcd} S^\infty(X/S) \arrow[r] \arrow[d] & S^{\infty,\infty}(X/S) \arrow[d] \\ S \arrow[r] & S^\infty(X/S)^+ \end{tikzcd}$$

shows that the object V_S fits in to the Cartesian square

$$\begin{tikzcd} V_S \arrow[r] \arrow[d] & V \arrow[d, "G"] \\ S \arrow[r] & S^\infty(X/S)^+ \end{tikzcd}$$

In other words,

$$V_S = S \times_{S^\infty(X/S)^+} V$$

is, at the same time, the fibred product of S and V over $S^\infty(X/S)^+$.

Choose d such that the image of the morphism

$$V \to S^{\infty,\infty}(X/S)$$

is in

$$S^{d,d}(X/S) = S^d(X/S) \times_S S^d(X/S) \ .$$

Since the morphism

$$S^{d,d}(X/S) \to S^{\infty,\infty}(X/S)$$

is a monomorphism in $\mathsf{PShv}(\mathsf{Noe}/S)$, it follows that the object V_S fits also in to the Cartesian square

$$\begin{tikzcd} V_S \arrow[r] \arrow[d] & V \arrow[d] \\ S^d(X/S) \arrow[r] & S^{d,d}(X/S) \end{tikzcd}$$

In other words, V_S is the fibred product of the schemes $S^d(X/S)$ and V over the scheme $S^{d,d}(X/S)$. In particular, V_S is a scheme itself. $\square$

We need one easy but useful technical notion. Suppose we are given with a locally Noetherian scheme U over S and a morphism

$$F : U \to \mathrm{Sym}^{\infty}(X/S)^{+}$$

in the category of sheaves $\mathsf{Shv}((\mathsf{Noe}/S)_{\mathrm{Nis}})$. Any such a morphism is uniquely determined by sending $\mathrm{id}_U : U \to U$ to a section

$$s_F \in \mathrm{Sym}^{\infty}(X/S)^{+}(U) \ .$$

Since $\mathrm{Sym}^{\infty}(X/S)^{+}$ is the Nisnevich sheafification of the presheaf $S^{\infty}(X/S)^{+}$ and the morphism

$$S^{\infty}(X/S)^{+} \to \mathrm{Sym}^{\infty}(X/S)^{+}$$

is a monomorphism in $\mathsf{PShv}(\mathsf{Noe}/S)$ by Lemma 3, we obtain that the section s_F is the equivalence class of pairs, each of which consists of a Nisnevich cover

$$\{U_i \to U\}_{i \in I}$$

and a collection of sections

$$s_i \in S^{\infty}(X/S)^{+}(U_i) \ ,$$

such that the restrictions of s_i and s_j on $U_i \times_U U_j$ coincide for all indices i and j in I. Therefore, if

$$\hat{U} = \coprod_{i \in I} U_i \ ,$$

we obtain two morphisms

$$\hat{U} \to U$$

and

$$\hat{F} : \hat{U} \to S^{\infty}(X/S)^{+}$$

such that the square

$$\begin{tikzcd} \hat{U} \arrow[r, "\hat{F}"] \arrow[d] & S^{\infty}(X/S)^+ \arrow[d] \\ U \arrow[r, "F"] & \mathrm{Sym}^{\infty}(X/S)^+ \end{tikzcd}$$

is commutative. For short, we will say that $\hat{U}$ (respectively, $\hat{F}$) is an extension of U (resp., F) by the pair $(\{U_i \to U\}_{i\in I}, \{s_i\}_{i\in I})$ representing the section s_F.

Theorem 6. *Let P be a point of the space $\mathrm{Sym}^{\infty}(X/S)^+$, and let $\mathscr{N}_P$ be the category of étale neighbourhoods of the point P on $\mathrm{Sym}^{\infty}(X/S)^+$. Then $\mathscr{N}_P$ is cofiltered.*

Proof. The proof follows a pretty standard way of argumentation, see, for example, Lemma 57.18.3. First of all, Lemma 5 gives us that the category $\mathscr{N}_P$ is nonempty, so that the first axiom of a cofiltered category is satisfied.

Let

$$F : U \to \mathrm{Sym}^{\infty}(X/S)^+ \quad \text{and} \quad G : V \to \mathrm{Sym}^{\infty}(X/S)^+$$

be two étale neighbourhoods of the point P, and look at the fibred product

$$\begin{tikzcd} U \times_{\mathrm{Sym}^{\infty}(X/S)^+} V \arrow[r] \arrow[d] & V \arrow[d, "G"] \\ U \arrow[r, "F"] & \mathrm{Sym}^{\infty}(X/S)^+ \end{tikzcd}$$

Let s_F be the section of the sheaf $\mathrm{Sym}^{\infty}(X/S)^+$ on U which determines the morphism F, and let

$$\hat{F} : \hat{U} \to S^{\infty}(X/S)^+$$

be the extension of the morphism F given by a pair $(\{U_i \to U\}_{i\in I}, \{s_i\}_{i\in I})$ representing s_F. Similarly, one can construct an extension $\hat{G}$ of the morphism G induced by a pair representing the section s_G.

By Lemma 5, the fibred product $\hat{U} \times_{S^{\infty}(X/S)^+} \hat{V}$ is a locally Noetherian scheme over S. Consider the universal morphism

$$\hat{U} \times_{S^{\infty}(X/S)^+} \hat{V} \to U \times_{\mathrm{Sym}^{\infty}(X/S)^+} V \ ,$$

commuting with the extensions of U and V. Let us show that the composition

$$H : \hat{U} \times_{S^{\infty}(X/S)^+} \hat{V} \to U \times_{\mathrm{Sym}^{\infty}(X/S)^+} V \to \mathrm{Sym}^{\infty}(X/S)^+$$

is étale, with regard to the Chow atlas on $\mathrm{Sym}^{\infty}(X/S)^+$.

Indeed, since the morphism

$$S^{\infty}(X/S)^+ \to \mathrm{Sym}^{\infty}(X/S)^+$$

is a monomorphism in $\mathsf{PShv}(\mathsf{Noe}/S)$ by Lemma 3, the square

$$\begin{tikzcd} \hat{U} \times_{S^\infty(X/S)^+} \hat{V} \arrow[r] \arrow[d, "\mathrm{id}"'] & S^\infty(X/S)^+ \arrow[d] \\ \hat{U} \times_{S^\infty(X/S)^+} \hat{V} \arrow[r] & \mathrm{Sym}^\infty(X/S)^+ \end{tikzcd}$$

is Cartesian, so that the obvious morphism

$$h : \hat{U} \times_{S^\infty(X/S)^+} \hat{V} \to S^\infty(X/S)^+$$

is the pullback of the morphism H.

For short, let

$$\hat{U}_{d,d} = \hat{U} \times_{S^\infty(X/S)^+} S^{d,d}(X/S) ,$$

and

$$\hat{V}_{d,d} = \hat{V} \times_{S^\infty(X/S)^+} S^{d,d}(X/S) .$$

Then

$$h_0 : \hat{U}_{d,d} \times_{S^{d,d}(X/S)} \hat{V}_{d,d} \to S^{d,d}(X/S)$$

is the pullback of the morphism h, and since h is the pullback of H, we obtain the Cartesian square

$$\begin{tikzcd} \hat{U}_{d,d} \times_{S^{d,d}(X/S)} \hat{V}_{d,d} \arrow[r, "h_0"] \arrow[d] & S^{d,d}(X/S) \arrow[d] \\ \hat{U} \times_{S^\infty(X/S)^+} \hat{V} \arrow[r, "H"] & \mathrm{Sym}^\infty(X/S)^+ \end{tikzcd}$$

Therefore, in order to prove that H is étale, we need only to show that h_0 is étale.

Now again, since the morphism from $S^\infty(X/S)^+$ to $\mathrm{Sym}^\infty(X/S)^+$ is a monomorphism in $\mathsf{PShv}(\mathsf{Noe}/S)$ by Lemma 3, we see that the commutative square

$$\begin{tikzcd} \hat{U} \arrow[r] \arrow[d, "\mathrm{id}"'] & S^\infty(X/S)^+ \arrow[d] \\ \hat{U} \arrow[r] & \mathrm{Sym}^\infty(X/S)^+ \end{tikzcd}$$

is Cartesian. Composing it with the Cartesian square

$$\begin{tikzcd} \hat{U}_{d,d} \arrow[r] \arrow[d] & S^{d,d}(X/S) \arrow[d] \\ \hat{U} \arrow[r] & S^{\infty}(X/S)^{+} \end{tikzcd}$$

we obtain the Cartesian square

$$\begin{tikzcd} \hat{U}_{d,d} \arrow[r] \arrow[d] & S^{d,d}(X/S) \arrow[d] \\ \hat{U} \arrow[r] & \mathrm{Sym}^{\infty}(X/S)^{+} \end{tikzcd}$$

The bottom horizontal morphism is the composition of two étale morphisms, and hence it is étale. Since étale morphisms are stable under pullbacks, the top horizontal morphism

$$\hat{U}_{d,d} \to S^{d,d}(X/S)$$

in the latter square is étale as well.

Similarly, the morphism

$$\hat{V}_{d,d} \to S^{d,d}(X/S)$$

is étale.

Thus, the bottom horizontal and the right vertical morphisms in then Cartesian square

$$\begin{tikzcd} \hat{U}_{d,d} \times_{S^{d,d}(X/S)} \hat{V}_{d,d} \arrow[r] \arrow[d] & \hat{V}_{d,d} \arrow[d] \\ \hat{U}_{d,d} \arrow[r] & S^{d,d}(X/S)^{+} \end{tikzcd}$$

are étale. Since étale morphisms are stable under pullbacks and compositions, the diagonal composition h_0 of this square is étale as well.

As this is true for any d, we see that the morphism

$$\hat{U} \times_{S^{\infty}(X/S)^{+}} \hat{V} \to \mathrm{Sym}^{\infty}(X/S)^{+}$$

is étale.

The fact that the point $P : \mathrm{Spec}(K) \to \mathrm{Sym}^{\infty}(X/S)^{+}$ factorizes through $\hat{U} \times_{S^{\infty}(X/S)^{+}} \hat{V}$ is obvious.

Now we need to prove the last axiom of a cofiltered category. Assume again that we have two étale neighbourhoods U and V of P as above, and assume also that we have two morphisms

$$a, b : U \rightrightarrows V$$

between these neighbourhoods.

Let
$$s_G \in \mathrm{Sym}^\infty(X/S)^+(V)$$
be the section determined by the morphism G, and choose a representative in s_G. Such a representative consists of a Nisnevich covering
$$\{V_i \to V\}_{i\in I}$$
and a collection of sections
$$s_i \in S^\infty(X/S)^+(V_i) \ ,$$
such that the restrictions of s_i and s_j on $V_i \times_V V_j$ coincide for all indices i and j in I. Construct the corresponding extension
$$\hat{G} : \hat{V} \to S^\infty(X/S)^+$$
of the morphism G getting the commutative square
$$\begin{tikzcd}
\hat{V} \arrow[r, "\hat{G}"] \arrow[d] & S^\infty(X/S) \arrow[d] \\
V \arrow[r, "G"] & \mathrm{Sym}^\infty(X/S)^+
\end{tikzcd}$$
Pulling back the étale covering $\{V_i \to V\}_{i\in I}$ along the morphisms a and b, and taking the unification
$$\{U_{ij} \to U\}_{(i,j)\in I\times I}$$
of these two pullback coverings in to one, one can construct the extension
$$\hat{F} : \hat{U} \to S^\infty(X/S)^+ \ ,$$
such that the diagram

(26)
$$\begin{tikzcd}
\mathrm{Spec}(K)\hat{} \arrow[r] \arrow[d] & \hat{V} \arrow[d, "\hat{G}"] \\
\hat{U} \arrow[ur, shift left, "\hat{a}"] \arrow[ur, shift right, "\hat{b}"'] \arrow[r, "\hat{F}"] & S^\infty(X/S)^+
\end{tikzcd}$$
is commutative, where $\mathrm{Spec}(K)\hat{}$ is an extension over $\mathrm{Spec}(K)$. Moreover, the squares
$$\begin{tikzcd}
\hat{U} \arrow[r, "\hat{a}"] \arrow[d] & \hat{V} \arrow[d] \\
U \arrow[r, "a"] & V
\end{tikzcd}
\qquad
\begin{tikzcd}
\hat{U} \arrow[r, "\hat{b}"] \arrow[d] & \hat{V} \arrow[d] \\
U \arrow[r, "b"] & V
\end{tikzcd}$$

$$\begin{tikzcd} \mathrm{Spec}(K)\hat{} \arrow[r] \arrow[d] & \mathrm{Spec}(K) \arrow[d] \\ \hat{U} \arrow[r] & U \end{tikzcd} \qquad \begin{tikzcd} \mathrm{Spec}(K)\hat{} \arrow[r] \arrow[d] & \mathrm{Spec}(K) \arrow[d] \\ \hat{V} \arrow[r] & V \end{tikzcd}$$

are commutative.

Now, let W be the fibred product of U and V over $V \times_{\mathrm{Sym}^\infty(X/S)^+} V$, with regard to the morphisms (a,b) and Δ, and let h be the corresponding universal morphism, as it is shown in the commutative diagram

(27)
$$\begin{tikzcd} \mathrm{Spec}(K) \arrow[dr, dotted, "\exists ! h"] \arrow[drr, bend left] \arrow[ddr, bend right] & & \\ & W \arrow[r] \arrow[d] & V \arrow[d, "\Delta"] \\ & U \arrow[r, "{(a,b)}"] & V \times_{\mathrm{Sym}^\infty(X/S)^+} V \end{tikzcd}$$

Notice that the external commutativity is guaranteed by the fact that a and b are two morphisms from the neigbourhood U to the neigbourhood V of the same point P. The diagram (27) can be also extended by the commutative diagram

(28)
$$\begin{tikzcd} V \times_{\mathrm{Sym}^\infty(X/S)^+} V \arrow[r] \arrow[d] & V \arrow[d, "G"] \\ V \arrow[r, "G"] & \mathrm{Sym}^\infty(X/S)^+ \end{tikzcd}$$

Consider also the corresponding "underlying" commutative diagrams

(29)
$$\begin{tikzcd} \mathrm{Spec}(K)\hat{} \arrow[dr, dotted, "\exists ! \hat{h}"] \arrow[drr, bend left] \arrow[ddr, bend right] & & \\ & \hat{W} \arrow[r] \arrow[d] & \hat{V} \arrow[d, "\Delta"] \\ & \hat{U} \arrow[r, "{(\hat{a},\hat{b})}"] & \hat{V} \times_{S^\infty(X/S)^+} \hat{V} \end{tikzcd}$$

and

(30)
$$\begin{tikzcd}
\hat V \times_{S^\infty(X/S)^+} \hat V \arrow[r] \arrow[d] & \hat V \arrow[d, "\hat G"] \\
\hat V \arrow[r, "\hat G"] & S^\infty(X/S)^+
\end{tikzcd}$$

where $\hat h$ exists and unique due to the commutativities coming from the commutativities in the diagram (26).

Clearly, the commutative diagrams (27), (28), (29) and (30) can be joined in to one large commutative diagram by means of the morphisms

$$\hat U \to U \ , \ \ \hat V \to V \ , \ \ \text{etc}$$

One of the subdiagrams of that join is the commutative square

$$\begin{tikzcd}
\hat V \times_{S^\infty(X/S)^+} \hat V \arrow[r] \arrow[d] & V \arrow[d] \\
\hat V \arrow[r] & \mathrm{Sym}^\infty(X/S)^+
\end{tikzcd}$$

As we know from the first part of the proof, applied to the case when $U = V$, the diagonal composition

$$\hat V \times_{S^\infty(X/S)^+} \hat V \to \mathrm{Sym}^\infty(X/S)^+$$

is an étale neighbourhood of the point P.

Since the diagrams

$$\begin{tikzcd}
\hat U \arrow[r] \arrow[dr, "\hat b"'] & \hat V \times_{S^\infty(X/S)^+} \hat V \arrow[d] \\
& \hat V
\end{tikzcd}$$

and

$$\begin{tikzcd}
\hat U \arrow[r, "\hat b"] \arrow[d] & \hat V \arrow[d] \\
U \arrow[r, "b"] & V
\end{tikzcd}$$

are commutative, we see that the square

$$\begin{tikzcd} \hat{U} \arrow[r] \arrow[d] & \hat{V} \times_{S^\infty(X/S)^+} \hat{V} \arrow[d] \\ U \arrow[r, "F"] & \mathrm{Sym}^\infty(X/S)^+ \end{tikzcd}$$

is commutative.

The left vertical arrow in the latter square is étale, and the morphism F is étale by assumption. Therefore, their composition is étale, and we obtain the commutative diagram

$$\begin{tikzcd} \hat{U} \arrow[r] \arrow[dr] & \hat{V} \times_{S^\infty(X/S)^+} \hat{V} \arrow[d] \\ & \mathrm{Sym}^\infty(X/S)^+ \end{tikzcd} \tag{31}$$

in which the morphisms targeted at $\mathrm{Sym}^\infty(X/S)^+$ are étale.

Now, if $f : Y \to Y'$ is a morphism between locally Noetherian schemes over a space $\mathscr{X}$, if the structural morphisms $Y \to \mathscr{X}$ and $Y' \to \mathscr{X}$ are étale, with regard to the atlas on $\mathscr{X}$, then f is also étale. This is an obvious modification of Lemma 57.15.6 in [23]. Applying this property to the diagram (31), we obtain that the morphism

$$\hat{U} \to \hat{V} \times_{S^\infty(X/S)^+} \hat{V}$$

is étale.

As étale morphisms are stable under base change, the Cartesian square from the diagram (29) then shows that the morphism

$$\hat{W} \to \hat{V}$$

is étale. And since the morphisms $\hat{V} \to V$ is étale, the composition

$$\hat{W} \to \hat{V} \to V$$

is étale. Since G is étale by assumption, we see that the composition

$$\hat{W} \to \hat{V} \to V \xrightarrow{G} \mathrm{Sym}^\infty(X/S)^+ \tag{32}$$

is also étale.

Finally, analyzing the above join of the commutative diagrams (27), (28), (29) and (30) by means of the extension morphisms, we see that the composition (32) is the same as the composition

$$\hat{W} \to W \to U \xrightarrow{b} V \xrightarrow{G} \mathrm{Sym}^\infty(X/S)^+ \, .$$

Thus, we have obtained the commutative diagram

$$\begin{tikzcd} \hat{W} \arrow[r] \arrow[d] & V \arrow[d] \\ U \arrow[r] & \mathrm{Sym}^{\infty}(X/S)^{+} \end{tikzcd}$$

whose diagonal composition

$$\hat{W} \to \mathrm{Sym}^{\infty}(X/S)^{+} \tag{33}$$

is étale.

Analyzing the commutative diagrams above, it is easy to see that P factorizes through (33), so that the latter morphism is an étale neighbourhood of P. $\square$

7. Rational curves on the locally ringed site of 0-cycles

Theorem 6 has the following important implication. Namely, since all the items of Definition 7.31.2 in [23] are now satisfied, the stack functor

$$\mathrm{st}_P : \mathsf{Shv}(\mathrm{Sym}^{\infty}(X/S)^{+}_{\text{Nis-ét}}) \to \mathsf{Set}$$

induces a point of the topos $\mathsf{Shv}(\mathrm{Sym}^{\infty}(X/S)^{+}_{\text{Nis-ét}})$ by Lemma 7.31.7 in loc.cit. In particular, we obtain the full-fledged stalk

$$\mathscr{O}_{\mathrm{Sym}^{\infty}(X/S)^{+},\,P} = \mathrm{st}_P\,(\mathscr{O}_{\mathrm{Sym}^{\infty}(X/S)^{+}})$$

Moreover, the ringed site $\mathrm{Sym}^{\infty}(X/S)^{+}_{\text{Nis-ét}}$ is a locally ringed site in the sense of the definition appearing in Exercise 13.9 on page 512 in [1] (see page 313 in the newly typeset version), as well as in the sense of a sightly different Definition 18.39.4 in [23]. This is explained in Section 2.

For short of notation, let us write

$$\mathscr{O}_P = \mathscr{O}_{\mathrm{Sym}^{\infty}(X/S)^{+},\,P}\ .$$

This should not lead to a confusion, as the point P is a point on $\mathrm{Sym}^{\infty}(X/S)^{+}$. Since the site $\mathrm{Sym}^{\infty}(X/S)^{+}_{\text{Nis-ét}}$ is locally ringed, for each point P on this site the stalk $\mathscr{O}_P$ is a local ring by the same Lemma 18.39.2 in [23]. Then we also have the maximal ideal

$$\mathfrak{m}_P \subset \mathscr{O}_P$$

and the residue field

$$\kappa(P) = \mathscr{O}_P/\mathfrak{m}_P$$

at the point P.

The stalk functor also gives us the stalks

$$\Omega^1_{\mathrm{Sym}^{\infty}(X/S)^{+},\,P} = \mathrm{st}_P\,(\Omega^1_{\mathrm{Sym}^{\infty}(X/S)^{+}})$$

and

$$T_{\mathrm{Sym}^{\infty}(X/S)^{+},\,P} = \mathrm{st}_P\,(T_{\mathrm{Sym}^{\infty}(X/S)^{+}})$$

at P. Tensoring by $\kappa(P)$ we obtain the vector spaces

$$\Omega^1(P) = \Omega^1_{\mathrm{Sym}^{\infty}(X/S)^{+}}(P) = \Omega^1_{\mathrm{Sym}^{\infty}(X/S)^{+},\,P} \otimes_{\mathscr{O}_P} \kappa(P)$$

and

$$T(P) = T_{\mathrm{Sym}^\infty(X/S)^+}(P) = T_{\mathrm{Sym}^\infty(X/S)^+,\,P} \otimes_{\mathscr{O}_P} \kappa(P)$$

over the residue field $\kappa(P)$.

The second vector space $T(P)$ is then our *tangent space* to the space of 0-cycles $\mathrm{Sym}^\infty(X/S)^+$ at the point P. Notice that, since $\mathrm{Sym}^\infty(X/S)^+$ is an abelian group object in the category of Nisnevich sheaves on locally Noetherian schemes over S, whenever S is the spectrum of a field k, all tangent spaces $T(P)$ at k-rational points P are uniquely determined by the tangent space $T(0)$ at the zero point 0 on $\mathrm{Sym}^\infty(X/S)^+$ provided by the section of the structural morphism from X to S. In other words, one can develop a Lie theory on $\mathrm{Sym}^\infty(X/S)^+$.

Now we are fully equipped to promote the idea of understanding of rational equivalence of 0-cycles as rational connectivity on the space $\mathrm{Sym}^\infty(X/S)^+$. First of all, looking at any scheme U over S as a representable sheaf, we have the corresponding locally ringed site $U_{\text{Nis-ét}}$. Then a *regular morphism* from U to $\mathrm{Sym}^\infty(X/S)^+_{\text{Nis-ét}}$ is just a morphism of locally ringed sites

$$U_{\text{Nis-ét}} \to \mathrm{Sym}^\infty(X/S)^+_{\text{Nis-ét}}$$

in the sense of Definition 18.39.9 in [23]. Notice that since étale morphisms are stable under base change, if $U \to \mathrm{Sym}^\infty(X/S)^+$ is a morphism of sheaves, then it induces the corresponding morphism of locally ringed sites.

A rational curve on $\mathrm{Sym}^\infty(X/S)^+$ is a morphism of sheaves

$$f : \mathbb{P}^1 \to \mathrm{Sym}^\infty(X/S)^+ \ .$$

If

$$P : \mathrm{Spec}(K) \to \mathrm{Sym}^\infty(X/S)^+$$

is a point on the sheaf $\mathrm{Sym}^\infty(X/S)^+$, then we will be saying that f passes through the point P if P, as a morphism to $\mathrm{Sym}^\infty(X/S)^+$, factorizes through the morphism $f : \mathbb{P}^1 \to \mathrm{Sym}^\infty(X/S)^+$.

Now, two points P and Q on $\mathrm{Sym}^\infty(X/S)^+$ are *elementary rationally connected* if there exists a rational curve on $\mathrm{Sym}^\infty(X/S)^+$ passing through P and Q. The points P and Q are said to be *rationally connected* if there exists a finite set of points $R_1, \dots, R_n$ on $\mathrm{Sym}^\infty(X/S)^+$, such that $R_1 = P$, $R_n = Q$ and R_i is elementary rationally connected to R_{i+1} for each $i \in \{1, \dots, n-1\}$. If any two points on $\mathrm{Sym}^\infty(X/S)^+$ are rationally connected, then we will say that this space is rationally connected.

Let

$$P : \mathrm{Spec}(K) \to \mathrm{Sym}^\infty(X/S)^+ \quad \text{and} \quad Q : \mathrm{Spec}(L) \to \mathrm{Sym}^\infty(X/S)^+$$

be two points on $\mathrm{Sym}^\infty(X/S)^+$, represented by morphisms from the spectra of two fields K and L respectively. Suppose, in addition, that the fields K and L are embedded in to a common field, in which case we can replace both K and L by their composite KL. Then we can assume, without loss of generality, that $K = L$. In such a case, the points P and Q, as morphisms from the scheme $\mathrm{Spec}(K)$ to the sheaf $\mathrm{Sym}^\infty(X/S)^+$ induce two sections s_P and s_Q in

$$\mathrm{Sym}^\infty(X/S)^+(\mathrm{Spec}(K)) = \mathscr{Z}_0^\infty(X/S)(\mathrm{Spec}(K)) =$$
$$z(X/S,0)_\infty(\mathrm{Spec}(K)) \ .$$

Assume, in addition, that

$$S = \mathrm{Spec}(K) .$$

Then s_P and s_Q, as elements of the group

$$z(X/\mathrm{Spec}(K), 0)_\infty(\mathrm{Spec}(K)) ,$$

are two 0-cycles on the scheme X over $\mathrm{Spec}(K)$. And since relative 0-cycles are representable, see Section 4, rational connectivity of the points P and Q on $\mathrm{Sym}^\infty(X/S)^+$ is equivalent to rational equivalence of the 0-cycles s_P and s_Q on the scheme X. This all means that we can look at rational connectedness between points on $\mathrm{Sym}^\infty(X/S)^+$ as the generalized rational equivalence in the relative setting.

Let, for example, X be a smooth projective surface over an algebraically closed field k, and assume that X is of general type, i.e. the Kodaira dimension is 2, and that the transcendental part $H^2_{\mathrm{tr}}(X)$ in the second étale l-adic cohomology group $H^2_{\text{ét}}(X, \mathbb{Q}_l)$ is trivial, where l is different from the characteristic of k. Bloch's conjecture predicts that any two closed points P and Q on X are rationally equivalent as 0-cycles on X. This is equivalent to saying that the space $\mathrm{Sym}^\infty(X/k)^+$ s rationally connected in the sense above.

Let V be an arbitrary smooth projective variety over k. According to Kollár, [14], if we wish to show that V is rationally connected, we should to two steps. The first one is that we need to find a rational curve

$$f : \mathbb{P}^1 \to V$$

on V. If the first step is done, then we need to show that the rational curve f is free on V, i.e that the numbers

$$a_1 \geq \ldots \geq a_n$$

in the decomposition

$$f^*T_V = \mathscr{O}_{\mathbb{P}^1}(a_1) \oplus \ldots \oplus \mathscr{O}_{\mathbb{P}^1}(a_n)$$

have appropriate positivity, where T_V is the tangent sheaf on the variety V, see Section II.3 in the canonical book [14], or many other sources about free curves on varieties.

Now, since we have the tangent sheaf $T_{\mathrm{Sym}^\infty(X/k)^+}$ for our surface X over k, we can try to do the same on the space $\mathrm{Sym}^\infty(X/k)^+$. Namely, we should first find a rational curve

$$f : \mathbb{P}^1 \to \mathrm{Sym}^\infty(X/k)^+$$

on the space of 0-cycles. Of course, we do not know (at the moment) whether the tangent sheaf $T_{\mathrm{Sym}^\infty(X/k)^+}$ is locally free on the site $\mathrm{Sym}^\infty(X/S)^+_{\text{Nis-ét}}$, and, accordingly, we do not know whether the pullback $f^*T_{\mathrm{Sym}^\infty(X/k)^+}$ decomposes in to the direct sum of Serre twists. But it is not hard to show that $f^*T_{\mathrm{Sym}^\infty(X/k)^+}$ is a coherent sheaf on the projective line $\mathbb{P}^1$ over k. Being a coherent sheaf, it decomposes uniquely in to a direct sum of a torsion sheaf and a locally free sheaf, see, for example, Proposition 5.4.2. in [3]. Then

$$f^*T_{\mathrm{Sym}^\infty(X/k)^+} = \mathscr{O}_{\mathbb{P}^1}(a_1) \oplus \ldots \oplus \mathscr{O}_{\mathbb{P}^1}(a_n) \oplus \mathscr{T} ,$$

where $\mathscr{T}$ is a torsion sheaf on $\mathbb{P}^1$. Though the sheaf $\mathscr{T}$ is possibly non-zero, we still can apply the same line of arguments as in the proof of Theorem 3.7 in [14] or Proposition 4.8 in [4].

8. Appendix: representability of 0-cycles

It is important to understand the action of the isomorphism obtained by composing the isomorphisms (15) and (16) after the restriction on semi-normal schemes. The aim of the appendix is to describe this action in detail. Actually all we need is to slightly extend the arguments from [20].

Recall that symmetric powers can be also defined for objects in an abitrary symmetric monoidal category with finite colimits. Let, for example, R be a commutative ring, and let M a module over R. The d-th symmetric power $\mathrm{Sym}^d(M)$ of the module M in the category of modules over R can be defined as the quotient of $M^{\otimes d}$ by the submodule generated over R by the differences

$$m_1 \otimes \cdots \otimes m_d - m_{\sigma(1)} \otimes \cdots \otimes m_{\sigma(d)} \ ,$$

where $\sigma \in \Sigma_d$. For any collection $\{m_1, \ldots, m_d\}$ in M let $(m_1, \ldots, m_d)$ be the same collection as an element of the d-th symmetric power $\mathrm{Sym}^d(M)$ of the module M, i.e. the image of the tensor $m_1 \otimes \cdots \otimes m_d$ under the quotient homomorphism

$$M^{\otimes d} \to \mathrm{Sym}^d(M) \ .$$

The image of the injective homomorphism

$$\mathrm{Sym}^d(M) \to M^{\otimes d} \ ,$$

sending $(m_1, \ldots, m_d)$ to the sum

$$\sum_{\sigma \in \Sigma_d} m_{\sigma(1)} \otimes \cdots \otimes m_{\sigma(d)} \ ,$$

coincides with submodule of invariants $(M^{\otimes d})^{\Sigma_d}$ of the action of Σ_d on $M^{\otimes d}$. Therefore, one can identify $\mathrm{Sym}^d(M)$ with submodule of invariants $(M^{\otimes d})^{\Sigma_d}$.

A similar but Koszul dual theory applies to wedge powers, where the wedge power $\wedge^d M$ can be initially constructed as the quotient of $M^{\otimes d}$ by the submodule $E(M^{\otimes d})$ in $M^{\otimes d}$ generated by the tensors $v_1 \otimes \cdots \otimes v_d$ in which at least two vectors v_i and v_j are equal. This all is a folklore and can be found in, for example, §B.2 in [6].

In schematic terms, let B be an algebra over a ring A, i.e. one has a ring homomorphism

$$\phi : A \to B \ ,$$

and let

$$f : X = \mathrm{Spec}(B) \to \mathrm{Spec}(A) = Y$$

be the affine morphism induced by the homomorphism ϕ. Then one has the diagonal ring homomorphism

$$\phi_d : A \to B^{\otimes d} \ ,$$

where the d-fold tensor product $B^{\otimes d}$ is taken over A. The homomorphism ϕ_d gives the structural morphism

$$(X/Y)^d \to Y ,$$

where $(X/Y)^d$ is the d-fold product of X over Y. Let $(B^{\otimes d})^{\Sigma_d}$ be the subring of invariants of the action of the symmetric group. Since the image of ϕ_d is obviously in $(B^{\otimes d})^{\Sigma_d}$, we obtain the surjective homomorphism

$$\phi'_d : A \to (B^{\otimes d})^{\Sigma_d}$$

induced by ϕ_d. This gives us the decomposition

$$(X/Y)^d \to \mathrm{Sym}^d(X/Y) \to Y ,$$

where the second morphism is $\mathrm{Spec}(\phi'_d)$.

The multiplication in the A-algebra B induces the multiplication in $B^{\otimes d}$ by the formula

$$(b_1 \otimes \cdots \otimes b_d) \cdot (b'_1 \otimes \cdots \otimes b'_d) = (b_1 b'_1 \otimes \cdots \otimes b_1 b'_d) .$$

It is easy to see that if $(b_1 \otimes \cdots \otimes b_d)$ is in $(B^{\otimes d})^{\Sigma_d}$ and $(b'_1 \otimes \cdots \otimes b'_d)$ is in $E(B^{\otimes d})$, then the product $(b_1 b'_1 \otimes \cdots \otimes b_1 b'_d)$ is again in $E(B^{\otimes d})$. This is why the above product induces the product

$$(B^{\otimes d})^{\Sigma_d} \otimes \wedge^d B \to \wedge^d B .$$

If B is, moreover, is freely generated of dimension d, as an A-module, then the determinant

$$\det : \wedge^d B \xrightarrow{\sim} A$$

is an isomorphism, and we obtain a homomorphism

$$\psi_d : (B^{\otimes d})^{\Sigma_d} \longrightarrow A ,$$

such that ϕ_d is the section for ψ_d, thus bringing the section

$$s_{X/Y,d} : Y \to \mathrm{Sym}^d(X/Y)$$

of the above morphism $\mathrm{Sym}^d(X/Y) \to Y$.

Let us now explore the same situation globally. Let

$$f : X \to Y$$

be a morphism of schemes over a field k. Recall that f is said to be affine if and only if Y can be covered by affine open subsets

$$V_i = \mathrm{Spec}(A_i) ,$$

such that

$$U_i = f^{-1}(V_i)$$

is affine for each i, so

$$U_i = \mathrm{Spec}(B_i) ,$$

and

$$f|_{U_i} : U_i \to V_i$$

is induced by the homomorphism

$$A_i \to B_i ,$$

see page 128 in [12]. If f is affine, then

$$\mathscr{B} = f_* \mathscr{O}_X$$

is a quasi-coherent sheaf of $\mathscr{O}_Y$-algebras on Y, and

$$X = \mathbf{Spec}(\mathscr{B})$$

in the sense of loc.cit. The d-fold fibred product of X over Y is

$$\mathbf{Spec}(\mathscr{B}^{\otimes d}) ,$$

and the structural morphism from $(X/Y)^{\times d}$ to Y is induced by the homomorphism

$$\phi_d : \mathscr{O}_Y \to \mathscr{B}^{\otimes d} ,$$

where the tensor product $\mathscr{B}^{\otimes d}$ is over $\mathscr{O}_Y$. The image of ϕ_d is Σ_d-invariant, so that we obtain the homomorphism

$$\phi_d : \mathscr{O}_Y \to (\mathscr{B}^{\otimes d})^{\Sigma_d} .$$

Then the relative d-th symmetric power $\mathrm{Sym}^d(X/Y)$ exists and in fact

$$\mathrm{Sym}^d(X/Y) = \mathbf{Spec}((\mathscr{B}^{\otimes d})^{\Sigma_d}) .$$

The structural morphism

$$\mathrm{Sym}^d(X/Y) \to Y$$

is induced by the homomorphism ϕ_d above.

Following [20], let us now show that there exists also a section of the structural morphism $\mathrm{Sym}^d(X/Y) \to Y$, provided X is finite surjective of degree d over Y.

Assume first that f is finite and flat. The finiteness of f means, by definition, that f is affine and B_i is a finitely generated A_i-module for each i, see page 84 in [12]. Then $\mathscr{B}$ is a coherent flat $\mathscr{O}_Y$-module, with respect to the morphism

$$\mathscr{O}_Y \to \mathscr{B} = f_* \mathscr{O}_X ,$$

and so $\mathscr{B}$ is a locally free $\mathscr{O}_Y$-module by Proposition 9.2 (e) on page 254 in [12].

Let W be an irreducible component of the scheme X, and let V be the closure of $f(W)$ in Y. Since f is flat, V is an irreducible component of Y. Moreover, if ξ is the generic point of W in X, then $f(\xi)$ is the generic point of V in Y. Let d_ξ be the degree $[R(W) : R(V)]$, where $R(W)$ and $R(V)$ stay for the fields of rational functions on W and V respectively, endowed with the induced reduced closed subscheme structures on them.

We will say that $f : X \to Y$ is of constant degree d if the degrees d_ξ are equal to d for all irreducible components of the scheme X. If f is finite flat of constant degree d, then $\mathscr{B}$ is a locally free sheaf of rank d on $\mathscr{O}_Y$, so that one has the determinantal isomorphism

$$\det : \wedge^d \mathscr{B} \xrightarrow{\sim} \mathscr{O}_Y .$$

Applying the sheaf-theoretical version of the above local construction, we get the morphism of $\mathscr{O}_Y$-modules

$$(\mathscr{B}^{\otimes d})^{\Sigma_d} \otimes_{\mathscr{O}_Y} \wedge^d \mathscr{B} \to \wedge^d \mathscr{B} ,$$

where the tensor power $\mathscr{B}^{\otimes d}$ is taken over $\mathscr{O}_Y$. For one's turn, this gives the morphism

$$(\mathscr{B}^{\otimes d})^{\Sigma_d} \to \mathrm{End}_{\mathscr{O}_Y}(\wedge^d \mathscr{B}) \ .$$

Composing it with the above determinantal isomorphism we get the homomorphism of $\mathscr{O}_Y$-algebras

$$\psi_d : (\mathscr{B}^{\otimes d})^{\Sigma_d} \to \mathscr{O}_Y \ .$$

Since $\psi_d \circ \phi_d = \mathrm{id}_{\mathscr{O}_Y}$ we see that ψ_d induces the canonical section

$$s_{X/Y,d} : Y \to \mathrm{Sym}^d(X/Y)$$

of the structural morphism

$$\mathrm{Sym}^d(X/Y) \to Y \ .$$

Following [20], assume now that f is a finite and surjective (but maybe not flat) morphism of schemes over k. For our interests in this paper, it is sufficient to assume that the scheme X is integral and the scheme Y is normal and connected. Since X is integral, it is irreducible. As f is surjective, Y is irreducible too. Moreover, since f is finite, it is affine. As f is surjective, locally f is a collection of morphisms

$$\phi^* : \mathrm{Spec}(B) \to \mathrm{Spec}(A) \ ,$$

such that $\phi : A \to B$ is injective. Since X is integral, it is reduced, so that there is no nilpotens in B. Then there is also no nilpotens in A. Therefore, Y is reduced as well. Collecting these small observations we conclude that Y is integral.

Now, take any affine open

$$V = \mathrm{Spec}(A)$$

in Y with the preimage

$$f^{-1}(V) = \mathrm{Spec}(B)$$

in X, so that A is a subring in B, as $f|_U$ is surjective and both A and B are integral domains. Since B is a finitely generated A-module, it follows that B is integral over A by Proposition 5.1 in [2]. Then, for any non-zero element b in B there exists a monic polynomial

$$x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$$

with coefficients in A, such that b is a root of it. Without loss of generality one can assume that $a_0 \neq 0$. Then

$$1/b = 1/a_0 \cdot (-b^{n-1} - a_{n-1}b^{n-2} - \cdots - a_1) \ .$$

It means that the localization $B_{(0)}$ is a finitely generated $A_{(0)}$-module, i.e. $R(X)$ is a finite field extension of $R(Y)$. Let d be the degree $[R(X) : R(Y)]$.

Let U be the set of points $x \in X$, such that f is flat at x. Then U is open in X, see 9.4 on page 266 in [12]. Since both X and Y are integral, f is flat at the generic point of X. Therefore, the set U is non-empty.

Next, shrink U if necessary and assume that it is affine,

$$U = \mathrm{Spec}(B) \ ,$$

which is surjectively mapped onto the affine set $V = \mathrm{Spec}(A)$ in Y. Then

$$f|_U : U \to V$$

is a finite surjective flat morphism of schemes over the ground field k. Since $R(X)$ is a flat algebra over $R(Y)$, by the above local construction, we get the homomorphism

$$\psi_d : (R(X)^{\otimes d})^{\Sigma_d} \longrightarrow R(Y) .$$

Let now again $\mathscr{B}$ be the quasi-coherent sheaf $f_*\mathscr{O}_X$ of $\mathscr{O}_Y$-algebras on Y, so that

$$X = \mathbf{Spec}(\mathscr{B}) .$$

Let y be a point on Y. Locally,

$$y \in V \subset Y ,$$

where

$$V = \mathrm{Spec}(A)$$

and y is a prime ideal $\mathfrak{p}$ in A. Let

$$U = f^{-1}(V) = \mathrm{Spec}(B) .$$

By Propositions 5.1 and 5.2 on pages 110 - 111 in [12], we have that the stalk $\mathscr{B}_y$ is

$$((f|_U)_*\mathscr{O}_U)_y = ((f|_U)_*B)_{\mathfrak{p}} = B_{\mathfrak{p}}$$

and

$$B_{\mathfrak{p}} \subset B_{(0)} ,$$

i.e. $\mathscr{B}_y$ is canonically embedded into $R(X)$. Respectively, $\mathscr{B}_y^{\otimes d}$ is canonically embedded into $R(X)^{\otimes d}$. The homomorphism ψ_d is nothing but the homomorphism

$$\psi_{(0),d} : (B_{(0)}^{\otimes d})^{\Sigma_d} \to A_{(0)} ,$$

where the tensor product is taken over $A_{(0)}$. As above, $\psi_{(0),d}$ has the section

$$\phi_{(0),d} : A_{(0)} \to (B_{(0)}^{\otimes d})^{\Sigma_d} ,$$

induced by the canonical homomorphism $A_{(0)} \to B_{(0)}^{\otimes d}$.

Since $A_{\mathfrak{p}}$ is embedded into $A_{(0)}$ and $B_{\mathfrak{p}}$ is embedded into $B_{(0)}$, we have the homomorphism from $(B_{\mathfrak{p}}^{\otimes d})^{\Sigma_d}$, where the tensor product is taken over $A_{\mathfrak{p}}$, to $(B_{(0)}^{\otimes d})^{\Sigma_d}$. Certainly, the canonical homomorphism $\phi_{\mathfrak{p},d} : A_{\mathfrak{p}} \to B_{\mathfrak{p}}^{\otimes d}$ induces the homomorphism $\phi_{\mathfrak{p},d} : A_{\mathfrak{p}} \to (B_{\mathfrak{p}}^{\otimes d})^{\Sigma_d}$, so that we have the obvious commutative diagram

$$\begin{tikzcd}
(B_{\mathfrak{p}}^{\otimes d})^{\Sigma_d} \arrow[d] & A_{\mathfrak{p}} \arrow[l, "\phi_{\mathfrak{p},d}"'] \arrow[d] \\
(B_{(0)}^{\otimes d})^{\Sigma_d} & A_{(0)} \arrow[l, "\phi_{(0),d}"']
\end{tikzcd}$$

The bottom horizontal homomorphism is the canonical section of the homomorphism $\psi_{(0),d}$. One can construct a suitable homomorphism $\psi_{\mathfrak{p},d}$ from $(B_{\mathfrak{p}}^{\otimes d})^{\Sigma_d}$ to

$A_\mathfrak{p}$, such that $\phi_{\mathfrak{p},d}$ would be a section for it, and the diagram

$$\begin{tikzcd}
(B_\mathfrak{p}^{\otimes d})^{\Sigma_d} \arrow[r, "\psi_{\mathfrak{p},d}"] \arrow[d] & A_\mathfrak{p} \arrow[d] \\
(B_{(0)}^{\otimes d})^{\Sigma_d} \arrow[r, "\psi_{(0),d}"] & A_{(0)}
\end{tikzcd}$$

would be commutative. This is due to the normality of Y and the finiteness of the morphism f.

Indeed, let α be an element in $(B_\mathfrak{p}^{\otimes d})^{\Sigma_d}$. Considering it as an element in $(B_{(0)}^{\otimes d})^{\Sigma_d}$ and applying $\psi_{(0),d}$ we get the element $\beta = \psi_{(0),d}(\alpha)$ in $A_{(0)}$. Since f is finite, so that B is a finitely generated module over A, the algebra B is integral over A. Then $B_\mathfrak{p}$ is integrals over $A_\mathfrak{p}$. Hence, $(B_\mathfrak{p}^{\otimes d})^{\Sigma_d}$ is integral over $A_\mathfrak{p}$, see Exercise 3 on page 67 in [2]. Then α is integral element over $A_\mathfrak{p}$. Since the bottom horizontal homomorphism $\phi_{(0),d}$ is the canonical section of the homomorphism $\psi_{(0),d}$, we see that the integrality of α implies integrality of β over $A_\mathfrak{p}$. Since Y is a normal scheme, it means that $A_\mathfrak{p}$ is integrally closed in the fraction field $A_{(0)}$. Therefore, β belongs to $A_\mathfrak{p}$. Thus, we obtain the desired homomorphism $\psi_{\mathfrak{p},d}$ from $(B_\mathfrak{p}^{\otimes d})^{\Sigma_d}$ to $A_\mathfrak{p}$.

The local homomorphism $\psi_{\mathfrak{p},d}$ can be also denoted as

$$\psi_{y,d} : (\mathscr{B}_y^{\otimes d})^{\Sigma_d} \to \mathscr{O}_{Y,y} \ .$$

Using the fact that $(\mathscr{B}^{\otimes d})^{\Sigma_d}$ and $\mathscr{O}_Y$ are sheaves, we can patch all the local homomorphisms $\psi_{y,d}$ into the global one,

$$\psi_d : (\mathscr{B}^{\otimes d})^{\Sigma_d} \to \mathscr{O}_Y \ .$$

Since locally $\phi_{y,d}$ is a section of $\psi_{y,d}$, the same holds globally. Likewise in the case of finite flat morphisms, since ϕ_d is a section of ψ_d globally, the homomorphism ψ_d gives the induced section

$$s_{X/Y,d} : Y \to \mathrm{Sym}^d(X/Y)$$

of the structural morphism

$$\mathrm{Sym}^d(X/Y) \to Y \ .$$

Remark 7. The section $s_{X/Y,d}$ has been achieved specifically for the d-th symmetric power of X over Y, where d is the degree of the morphism from X onto Y. In other circumstances the existence of the section section $s_{X/Y,d}$ is not guaranteed at all.

Example 8. Let X be the affine plane $\mathbb{A}^2$ and Y be the cone. The morphism from $\mathbb{A}^2$ onto Y is given by the embedding of the ring of symmetric polynomials

$$k[x^2, xy, y^2]$$

into the ring $k[x, y]$. In other words, the morphism $X \to Y$ glues any two antipodal points into one. Then $s_{X/Y,d}$ doesn't exists for $d = 1$, as there is no way to send the vertex of the cone to the plane. But $s_{X/Y,2}$ does exist as we can send the vertex to the doubled origin of coordinates as a point of the symmetric square.

Now, let S be a scheme of finite type over a field k, let X be a scheme projective over S, and fix a closed embedding

$$i : X \to \mathbb{P}^n_S$$

over S. In particular, X is AF over S and all relative symmetric powers $\mathrm{Sym}^d(X/S)$ exist in Noe/S. Notice that since X is projective over S, so is the scheme $\mathrm{Sym}^d(X/S)$, for every nonnegative integer d.

Let U be a noetherian scheme of finite type over S and let Z be a prime cycle in $z_d^{\mathrm{eff}}(X/S, 0)(U)$, considered with the induced reduced close subscheme structure on it. Let

$$f_Z : Z \to X \times_S U \to U$$

be the composition of the closed embedding of Z in to $X \times_S U$ with the projection onto U.

Since the morphism f_Z is finite, f_Z is affine, and hence the relative symmetric powers of Z/U exist. Then, as above, we have the canonical section

$$s_{Z/U,d} : U \to \mathrm{Sym}^d(Z/U)$$

of the structural morphism

$$\mathrm{Sym}^d(Z/U) \to U \ .$$

The closed embedding

$$Z \to X \times_S U$$

induces the morphism

$$\mathrm{Sym}^d(Z/U) \to \mathrm{Sym}^d(X \times_S U/U) \ ,$$

and we also have the obvious morphism

$$\mathrm{Sym}^d(X \times_S U/U) \to \mathrm{Sym}^d(X/S) \ .$$

Composing all these morphisms, we obtain the morphism

$$\theta_{X/S}(U, Z) : U \to \mathrm{Sym}^d(X/S)$$

over S.

The morphisms $\theta_{X/S}(U, Z)$ for degrees $d' \leq d$ extend by linearity and induce a map

$$\theta_{X/S,d}(U) : z_d^{\mathrm{eff}}((X, i)/S, 0)(U) \to \mathrm{Hom}_S(U, \mathrm{Sym}^d(X/S)) \ .$$

The latter maps for all schemes U yield a morphism of set valued presheaves

$$\theta_{X/S,d} : z_d^{\mathrm{eff}}((X, i)/S, 0) \to \mathrm{Hom}_S(-, \mathrm{Sym}^d(X/S))$$

on Noe/S.

Assume now that S is semi-normal over $\mathbb{Q}$. We claim that the restriction of the morphism $\theta_{X/S,d}$ on seminormal schemes is exactly the isomorphism obtained by composing the isomorphisms (15) and (16) considered in Section 4.

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